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Differential Equation

Exact finite difference and nonstandard finite difference schemes for solving the Fitzhugh-Nagumo equation

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Abstract. In this paper, we introduce an exact finite difference scheme for the Fitzhugh-Nagumo (FN) equation. Moreover, we develop a nonstandard finite difference scheme (NSFD) scheme for numerical solution of the FN model. The accuracy and computational efficiency of the proposed NSFD scheme is evaluated through some numerical simulations.

Keywords: The Fitzhugh-Nagumo equation, Exact finite difference scheme, Nonstandard finite difference scheme.

AMS Mathematics Subject Classification [2020]: 35K57, 65M06, 65M12.

1. Introduction

Partial differential equations (PDEs) are widely employed as fundamental models to describe complex physical phenomena, particularly in fluid mechanics, plasma physics, wave propagation, and chemical physics. Consider the FN equation [1]:

$$u_t = u_{xx} - u(1-u)(b-u), \tag{1}$$

where b is a constant and $u(x,t)$ denotes the unknown function dependent on the temporal variable t and spatial variable x. The FN equation (1) has various applications in the fields of flame propagation, logistic population growth, neurophysiology, and nuclear reactor theory. Among various techniques for solving PDEs, the NSFD schemes have been proved to be one of the most efficient approaches in recent years [2, 4–6]. By applying a NSFD scheme to a PDE, it becomes a discrete space-time system. While applying a NSFD scheme, it is necessary that the new difference equation system preserves the positivity and boundedness conditions and demonstrates the same qualitative behaviors as the continuous model. The organization of this paper is as follows: In Section 2, we review fundamental principles NSFD schemes construction for ordinary differential equations. Afterwards, we present an explicit exact finite difference scheme of the FN equation. In Section 3, we

*Speaker.

introduce a NSFD scheme for the FN equation, and the positivity and boundedness of the scheme are discussed. Section 4, presents some numerical simulations. Finally, the paper is closed by some including remarks in the last section.

2. Nonstandard finite difference schemes

In this part, we review the fundamentals of NSFD schemes. Mickens introduced the NSFD schemes for ordinary differential equations [3]. These schemes were developed to eliminate numerical instability that arise when using Standard finite difference, Rung-Kutta, and other classical numerical methods. By selecting an appropriate denominator function instead of the stepsize and employing nonlocal discretization, one can achieve the desired properties of positivity, boundedness, and stability in NSFD schemes, even larger stepsizes. To illustrate the construction of an NSFD scheme, we consider the following nonlinear initial value problem

$$X'(t) = f(X(t)), \qquad X(0) = X_0, \qquad t \in [0, T^*].$$

The formulation of an NSFD scheme relies on implementing at least one of two fundamental procedures outlined below:

1. The derivative is discretized using the approximation $X'(t_k) \approx \frac{X_{k+1}-X_k}{\psi(\Delta t)}$ where X_k denotes the numerical approximation of $X(t_k)$, and the nontrivial denominator function $\psi(\Delta t)$ is required to be continuous and strictly increasing, satisfying $\psi(\Delta t) = \Delta t + O(\Delta t^2)$, $0 < \psi(\Delta t) < 1$ as $\Delta t \to 0$

2. For both linear and nonlinear terms, we apply nonlocal approximations of the form

$$x(t)y(t)|_{t=t_k} \approx x_k y_{k+1}, \qquad x^2(t)|_{t=t_k} \approx x_k x_{k+1}.$$

In [1], it is shown that the anaylical solution to the FN equation is given by

$$u(x,t) = \frac{1}{2}(1 + \tanh(\frac{1}{2\sqrt{2}}(-x - \frac{2b-1}{\sqrt{2}}t))) = \frac{1}{1 + e^{\frac{\sqrt{2}}{2}x - \frac{(1-2b)}{2}t}}. \tag{2}$$

Based on the exact solution (2) for the FN equation, we can establish the following theorem.

Theorem 2.1. *An explicit exact NSFD scheme for the FN equation is given by*

$$\frac{U_j^{n+1} - U_j^n}{\phi} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\psi_1\psi_2} - \frac{(U_j^n)^2}{2}(U_{j+1}^n + U_{j-1}^n) + U_j^n(\frac{U_j^n + U_{j+1}^n}{2}) + bU_j^n U_{j-1}^n - bU_j^n,$$

where $h = \frac{1-2b}{\sqrt{2}}\Delta t$ *with* $1 - 2b \geq 0$, *and*

$$\psi_1(h) = \frac{e^{\frac{\sqrt{2}}{2}h} - 1}{\frac{\sqrt{2}}{2}}, \qquad \psi_2(h) = \frac{1 - e^{\frac{-\sqrt{2}}{2}h}}{\frac{\sqrt{2}}{2}}, \qquad \phi(\Delta t) = \frac{e^{\frac{(1-2b)\Delta t}{2}} - 1}{\frac{1-2b}{2}}.$$

3. Analysis of an NSFD scheme for the FN equation

In this section, we construct an NSFD scheme for the FN model (1) and investigate its fundamental properties. We consider the following NSFD scheme for the FN model (1).

$$\frac{U_j^{n+1} - U_j^n}{\phi} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\psi} - \frac{(3U_j^{n+1} - U_{j-1}^n)(U_{j-1}^n)^2}{2} + (b+1)(U_{j-1}^n)^2 - bU_j^{n+1}, \tag{3}$$

where U_j^n is an approximation to $u(x_j,t)$, and the denominator functions ϕ and ψ are defined as

$$\phi = \phi(\Delta t) = \frac{1-e^{-\frac{(1-2b)\Delta t}{2}}}{\frac{1-2b}{2}}, \qquad \psi = \psi(h) = (\frac{e^{\frac{\sqrt{2}}{2}h}-1}{\frac{\sqrt{2}}{2}})^2.$$

It can be verified that $\phi(\Delta t) \to \Delta t$ and $\psi(h) = h^2$ as h and Δt approach to zero. Comparing the difference equation (3) with the FN equation (1), one observes that the linear and nonlinear terms on the right-hand side of equation (1) are approximated as

$$u(x_j,t_n) \approx U_j^{n+1}, \qquad (u(x_j,t_n))^2 \approx (U_{j-1}^n)^2, \qquad ((u(x_j,t_n))^3 \approx (\frac{3U_j^{n+1}-U_{j-1}^n}{2})(U_{j-1}^n)^2.$$

This nonlocal discretization is essential for preserving the positivity and boundedness properties of the continuous model (1). By rearranging the equation (3) and introducing $R = \frac{\psi}{\phi}$, the explicit form for U_j^{n+1} can be written as

$$U_j^{n+1} = \frac{(1-2R)U_j^n + R(U_{j+1}^n + U_{j-1}^n) + \phi\left((b+1)(U_{j-1}^n)^2 + \frac{1}{2}(U_{j-1}^n)^3\right)}{1+b\phi+\frac{3}{2}\phi(U_{j-1}^n)^2}. \tag{4}$$

THEOREM 3.1. *If $1-2R \geq 0$ and $b \geq 0$, then the numerical solution (4) satisfies*

$$0 \leq U_j^n \leq 1 \implies 0 \leq U_j^{n+1} \leq 1,$$

for all relevant values of n and j.

PROOF. By subtracting the denominator from the numerator of (4) and using the condition $1-2R \geq 0$, one obtains

$$\begin{aligned} &(1-2R)U_j^n + R(U_{j+1}^n + U_{j-1}^n) + \phi\left((1+b)(U_{j-1}^n)^2 + \frac{1}{2}(U_{j-1}^n)^3\right) - b\phi - \frac{3}{2}\phi(U_{j-1}^n)^2 \\ &\leq (1-2R) + 2R + \phi(U_{j-1}^n)^2 + b\phi + \frac{1}{2}\phi(U_{j-1}^n)^2 - b\phi - \frac{3}{2}\phi(U_{j-1}^n)^2 = 1. \end{aligned} \tag{5}$$

Since

$$(1-2R)U_j^n + R(U_{j+1}^n + U_{j-1}^n) + \phi\left((1+b)(U_{j-1}^n)^2 + \frac{1}{2}(U_{j-1}^n)^3\right) \geq 0,$$

and

$$1 + b\phi + \frac{3}{2}\phi(U_{j-1}^n)^2 \geq 0.$$

Consequently, from inequality (5), one deduces that

$$0 \leq U_j^{n+1} = \frac{(1-2R)U_j^n + R(U_{j+1}^n + U_{j-1}^n) + \phi((1+b)(U_{j-1}^n)^2 + \frac{1}{2}(U_{j-1}^n)^3)}{1+b\phi+\frac{3}{2}\phi(U_{j-1}^n)^2} \leq 1.$$

This completes the proof. □

4. Numerical simulations

To verify the effectiveness of the NSFD scheme proposed in section 3, we simulate the initial-boundary problem

$$\begin{aligned} &u_t - u_{xx} = -u(1-u)(b-u), \quad x \in [0,1] \quad t \geq 0, \\ &u(x,0) = \frac{1}{1+e^{\frac{\sqrt{2}}{2}x}}, \quad u(0,t) = \frac{1}{1+e^{-\frac{(1-2b)}{2}t}}, \quad u(1,t) = \frac{1}{1+e^{(\frac{\sqrt{2}}{2}-\frac{1-2b}{2})t}}. \end{aligned} \tag{6}$$

Numerical simulations of problem (6) are presented in Figures 1 and 2 for the values $b = 2$ and $b = 1$, respectively. These figures demonstrate excellent agreement between the NSFD approximation and the analytical solution.

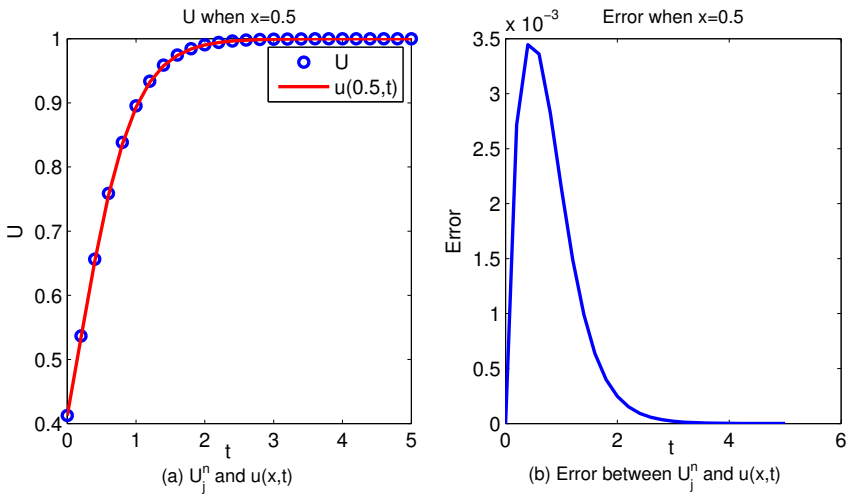

FIGURE 1. Comparison between the NSFD scheme and the exact solution at $x = 0.5$ for $b = 2$ with $\Delta t = 0.001$ and $h = 0.1$.

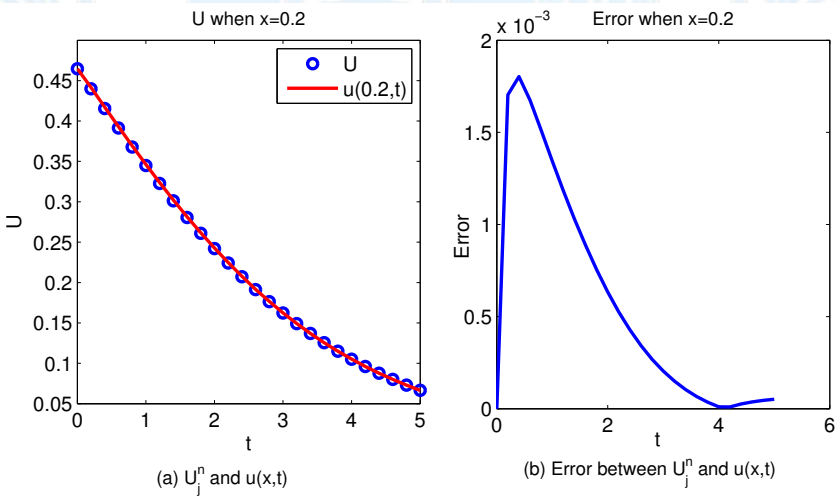

FIGURE 2. Comparison of the NSFD scheme and the exact solution at $x = 0.2$ for $b = 1$ with $\Delta t = 0.001$ and $h = 0.1$.

5. Concluding remarks

In this study, we presented an exact NSFD scheme for the FN equation based on its analytical solution. Then we proposed an efficient NSFD scheme to implement the model and analyzed its positivity and boundedness properties. Our numerical results showed that the solutions obtained from the NSFD scheme displayed excellent agreement with the solutions of the continuous model. As a future research direction, we plan to focus

on developing exact NSFD schemes and constructing effective NSFD schemes for various classes of deterministic and stochastic partial differential equations.

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Dynamics analysis of a novel 3D chaotic system

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Abstract. In this paper, the behavior of a novel three-dimensional chaotic system is investigated. To analyze the dynamical characteristics in detail and identify the existing oscillations for this system, the geometric method and the statistical method are employed. The geometric method calculates the curvature of phase space. And the statistical method enables quantitative analysis of the oscillations and assessment of the chaos intensity. The results indicate that the discussion of these two geometric and statistical approaches can be highly effective in detecting oscillatory patterns and accurately characterizing the chaotic behavior of three-dimensional dynamical systems.

Keywords: Chaos, Curvature, Lyapunov exponents, Detrended Fluctuation Analysis

AMS Mathematics Subject Classification [2020]: 34H10, 37C70, 37M10

1. Introduction

In recent years, chaotic dynamical systems have emerged as a prominent area of applied mathematics due to their complex and unpredictable behavior. These systems can be modeled as differential equations (continuous-time) or iterative maps (discrete-time) and are typically analyzed using tools such as equilibrium points, stability, bifurcation, and Lyapunov exponents [1, 2].

However, many classical methods face limitations when applied to biological systems, which often lack precise mathematical models. This highlights the growing need for approaches that are independent of strict mathematical modelling. In this study, in addition to classical analytical tools, phase space curvature and data-driven analysis of time series as oscillatory signals, are employed as two new methods [5, 6]. These approaches can discuss the complex behaviors in dynamical systems.

2. Behavior of a chaos system

Recently, a new three-dimensional system was introduced in [4], as follow:

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = ax - y - cx^3 - bz^3, \end{cases} \tag{1}$$

*Speaker.

where x, y, z are the state variables. In thise paper, we analyzis it for new parameters. To discuss the effects of parameter changes on the system behavior, including the Lyaponuv exponent, bifurcations, and attractors. By changing the parameter values, it is possible to identify chaotic and stable regions of the system and better understand its dynamical characteristics. Also, we investigate the curvature and data-driven analysis to determine the behavior of the system.

In the first step, to analyze the behavior of the system, it is essential to determine its equilibrium points and the stability of the system at these points. To find the equilibrium points of system (1), the right-hand side expressions of system (1) are set equal to zero. So:

$$\begin{cases} y = 0 \\ z = 0 \\ ax - y - cx^3 - bz^3 = 0. \end{cases} \tag{2}$$

By solving (2), he system equilibrium points are $(0, 0, 0)$, $(1.095, 0, 0)$,$(-1.095, 0, 0)$. The eigenvalues of Jacobian matrix in the equilibrium points of the system with parameters $a = 0.6$, $b = 0.4$, $c = 0.5$ are as follows:

- Equilibrium point (0,0,0):
  $\lambda_1 \approx -0.243 + 1.085i$,$\lambda_2 \approx -0.243 - 1.085i$, $\lambda_3 \approx 0.486$ Since there is a positive eigenvalue, this point is unstable.

- Equilibrium points (1.095,0,0) and (-1.095,0,0):
  $\lambda_1 \approx 0.380 + 1.197i$, $\lambda_2 \approx 0.380 - 1.197i$, $\lambda_3 \approx -0.760$
  Since there are positive eigenvalues, both points are also unstable.

Therefore, all three equilibrium points of this system are unstable, indicating that the system may exhibit chaotic or oscillatory behavior.

For discuses the behavior of the system, attractor diagram of system (1) are shown in figure 1. For more details , we calculate the Lyapunov exponent, bifurcation diagram, and sensitivity of system for initial condition.

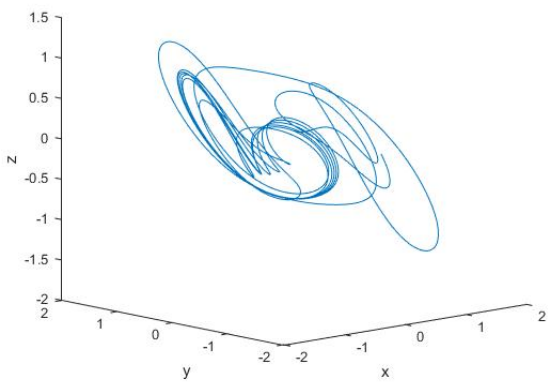

FIGURE 1. Attractors of system (1)

2.1. Lyapunov exponent. The Lyapunov exponent plays an essential role in investigating the behavior of dynamic systems. In fact, the Lyapunov exponent shows the average speed of divergence of the paths that start from the initial points close to each other. If a system has at least one positive Lyapunov exponent, it is chaos, and if a system has at least two, the Lyapunov exponent is hyper-chaos. The Lyapunov exponents are defined by [3]

$$L_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}. \tag{3}$$

Where $\delta u_i(t)$ is the changes of state variable. We, calculate the Lyapunov exponent of system (1). The diagram is shown in the figure 2.

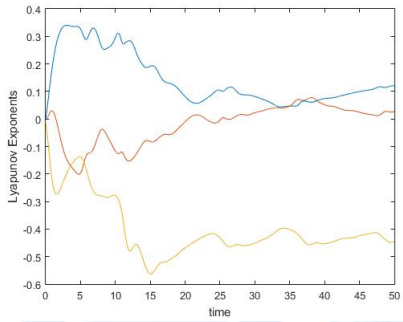

FIGURE 2. Lyapunov exponent of system (1)

2.2. Bifurcation. Bifurcation is a fundamental concept in the study of dynamical systems. Variation of system parameters can increase or decrease the number of equilibrium points and also affect their stability. In other words, a change in parameters can fundamentally alter the systems dynamic behavior. The bifurcation diagram of the function(1) for the range $-0.8 \leq a \leq 3$ is shown in figure 3.

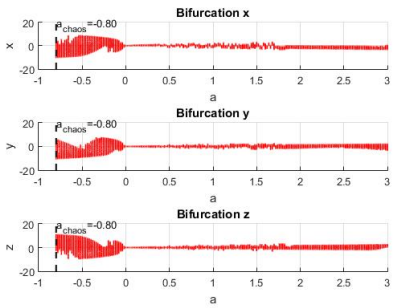

FIGURE 3. Bifurcation of system (1).

2.3. Sensitivity to initial conditions. Chaotic systems are sensitive to partial changes in initial conditions. The graphs of the time series of system (1) for the initial conditions $(0.1, 0.1, 0.1)$ and $(0.5, 0.5, 0.5)$ are given in figure 4.

The lyapunov exponent, phase space, sensitivity, and bifurcation show the system (1) is chaotic.

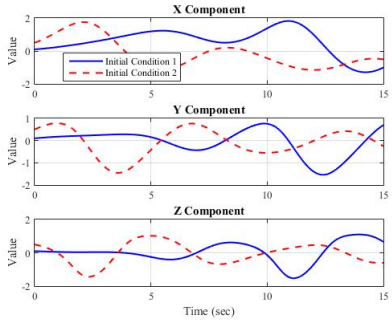

FIGURE 4. System sensitivity to initial conditions of system (1).

3. Curvature Method and the Statistical Method

In previous studies, the existence of chaotic behavior in dynamical systems was usually analyzed by calculating the Lyapunov exponent and bifurcation. In this study, we investigate the subsystem using the curvature method and statistical analysis of the time series. The results obtained from the curvature diagram and statistical method of the system are shown in figure 5

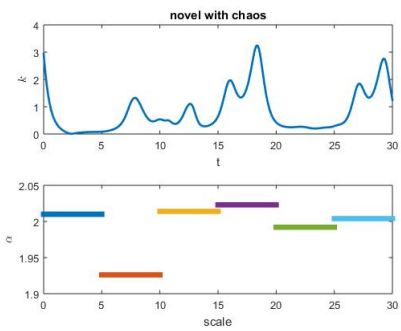

FIGURE 5. Curvature of system (1)

4. Conclusion

In this study, we applied a method for investigating the behavior of chaotic systems based on geometric analysis of the phase space combined with statistical evaluation. Numerical simulations conducted on a chaotic system demonstrate that the proposed approach is capable of effectively detecting chaos in the system's dynamics.

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Spectrum of a Sturm-liouville operator with discontinuities at two points

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Abstract. In this paper, we study a Sturm-liouville operator with eigenparameter dependent boundary condition and transmission conditions at two interior points. By modifying the inner product in a suitable krein space associated with the problem, we generate a new self-adjoint operator such that the eigenvalues of such a problem coincide with those of this operator and investigate some properties of its spectrum.

Keywords: Sturm-Liouville operator, Transmission condition, Krein space, eigenvalue, spectrum.

AMS Mathematics Subject Classification [2020]: 34L20, 47E05

1. Introduction

The Sturm–Liouville theory plays an important role in solving many problems of mathematical physics. Since many topics in mathematical physics require the investigation of the eigenvalues and eigenfunctions of Sturm–Liouville type boundary value problems. It is an active area of research in pure and applied mathematics. In recent years more and more researchers are interested in the spectral analysis of boundary value problems with eigenparameter-dependent boundary conditions. The literature on this subject is voluminous and we refer to [4–6] and corresponding references therein. In particular [6, 9] contains many references to problems in physics and mechanics. Moreover, some boundary value problems with discontinuities at one or two points have been also studied in [1,7]. Various physical application of discontinuous boundary value problems are found in the theory of heat and mass transfer, in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics, and in geophysical models for oscillations of the Earth(see [3, 8]).

In this paper, we will study the discontinuous Sturm-Liouville boundary value problem L consisting of the differential equation

$$\ell y := -\big(p(x)y'(x)\big)' + q(x)y(x) = \lambda y(x), \quad x \in J, \tag{1.1}$$

where $J := [a, \xi_1)\bigcup(\xi_1, \xi_2)\bigcup(\xi_m, b]$, $a < \xi_1 < \xi_2 < \xi_2 < b$; $p(x) = p_1^2$ for $x \in [a, \xi_1)$, $p(x) = p_1^2$ for $x \in [a, \xi_1)$, $p(x) = p_2^2$ for $x \in (\xi_1, \xi_2)$, and $p(x) = p_3^2$ for $x \in (\xi_2, b]$, $p_i > 0$

*Speaker.

$(i = 1, 2, 3)$ are given real numbers; $q(x) \in L_1[J, \mathbb{R}]$; λ is a complex eigenparameter; with eigenparameter-dependent boundary conditions at $x = a$ and $x = b$,

$$U(y) := \lambda(\alpha_1' y(a) - \alpha_2' y'(a)) - (\alpha_1 y(a) - \alpha_2 y'(a)) = 0, \tag{1.2}$$

$$V(y) := \lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = 0, \tag{1.3}$$

and four transmission conditions at the points of discontinuities ξ_1, ξ_2,

$$l_1(y) := y(\xi_1 + 0) - \gamma_1 y(\xi_1 - 0) - \gamma_1' y'(\xi_1 - 0) = 0, \tag{1.4}$$

$$l_2(y) := y'(\xi_1 + 0) - \gamma_2 y(\xi_1 - 0) - \gamma_2' y'(\xi_1 - 0) = 0, \tag{1.5}$$

$$l_3(y) := y(\xi_2 + 0) - \gamma_3 y(\xi_2 - 0) - \gamma_3' y'(\xi_2 - 0) = 0, \tag{1.6}$$

$$l_4(y) := y'(\xi_2 + 0) - \gamma_4 y(\xi_2 - 0) - \gamma_4' y'(\xi_2 - 0) = 0, \tag{1.7}$$

where α_j, α_j', β_j, β_j' $(j = 1, 2)$, γ_k and γ_k' $(k = 1, 2, 3, 4$ are real numbers. Throughout this paper, we assume that

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} \neq 0, \ \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} \neq 0, \ \theta_i := \begin{vmatrix} \gamma_{2i-1} & \gamma_{2i-1}' \\ \gamma_{2i} & \gamma_{2i}' \end{vmatrix} \neq 0, \quad i = 1, 2.$$

2. The operator equation formulation

In order to consider the problem L, we define the inner product $[\cdot, \cdot]_1$ in $L_2(J)$ as

$$[f, g]_1 = \frac{\theta_1 \theta_2}{p_1^2} \int_a^{\xi_1} f \overline{g} dx + \frac{\theta_2}{p_2^2} \int_{\xi_1}^{\xi_2} f \overline{g} dx + \frac{1}{p_3^2} \int_{\xi_2}^{b} f \overline{g} dx. \tag{2.1}$$

Obviously, the linear space $(L_2(J), [\cdot, \cdot]_1)$ is a Krein space.

Now, define the special inner product in the direct sum of linear spaces $L_2(J) \oplus \mathbb{C} \oplus \mathbb{C}$ by

$$[F, G] = [f, g]_1 + \frac{\theta_1 \theta_2}{\rho_1} f_1 \overline{g}_1 + \frac{1}{\rho_2} f_2 \overline{g}_2$$

For $F = (f(x), f_1, g_1)$, $G = (g(x), g_1, g_2) \in L_2(J) \oplus \mathbb{C} \oplus \mathbb{C}$. Then, $K := (L_2(J) \oplus \mathbb{C} \oplus \mathbb{C}, [\cdot, \cdot])$ is a Krein space.

A fundamental symmetry on this Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 & 0 \\ 0 & \operatorname{sgn}(\theta_1 \theta_2) \cdot \operatorname{sgn} \rho_1 & 0 \\ 0 & 0 & \operatorname{sgn} \rho_2 \end{bmatrix},$$

where $J_0 : L_2(J) \to L_2(J)$is defined by

$$J_0(x) = \begin{cases} f(x) \operatorname{sgn}(\theta_1 \theta_2), & x \in [a, \xi_1), \\ f(x) \operatorname{sgn} \theta_2, & x \in (\xi_1, \xi_2), \\ f(x), & x \in (\xi_2, b], \end{cases}$$

Let us define the inner product $\langle \cdot, \cdot \rangle := [J \cdot, \cdot]$. Then, $\langle \cdot, \cdot \rangle$ is a positive definite inner product which turns K into the Hilbert space $H := (L_2(J) \oplus \mathbb{C} \oplus \mathbb{C}, \langle \cdot, \cdot \rangle)$. For a short exposition, we shall use the following notations:

$$R_1(y) := \alpha_1 y(a) - \alpha_2 y'(a), \quad R_1'(y) := \alpha_1' y(a) - \alpha_2' y'(a),$$
$$R_2(y) := \beta_1 y(b) - \beta_2 y'(b), \quad R_2'(y) := \beta_1' y(b) - \beta_2' y'(b).$$

Let us define an operator A acting in K with domain of definition

$$D(A) := \{F = (f(x), f_1, f_2) \in K | f, f' \in AC_{loc}(J), \text{ and have}$$
$$\text{one-hand sided limits } f(\xi_i \pm 0), \ f'(\xi_i \pm 0), \ i = 1, 2,$$

$$\ell f \in L_2(J),\ l_k(f) = 0,\ k = 1,2,3,4,\ f_1 = R_1'(f),\ f_2 = R_2'(f)\}$$

and action law

$$AF := (\ell f, R_1(f), -R_2(f)) \quad \text{for } F = (f(x), R_1'(f), R_2'(f)) \in D(A).$$

Now we can rewrite the discontinuous boundary value problem L as

$$AY = \lambda Y, \quad Y := (y(x), R_1'(y), R_2'(y)) \in D(A).$$

Obviously, the eigenvalues of the boundary value problem L coincide with those of A, and its eigenfunctions are the first components of the corresponding eigenfunctions of the operator A.

Theorem 2.1. *The operator A is self-adjoint in K.*

Theorem 2.2. *The operator $B := JA$ is self-adjoint in H.*

The following lemma can be proved by applying theorems 2.2.1 and 2.5.2 of [**?**]:

Lemma 2.3. *Let $q(x) \in L_1(a_0, b_0)$, $a_0, b_0 \in \mathbb{R}$, be a real-valued function and $f(\lambda)$, $g(\lambda)$ be given entire functions. Then for any $\lambda \in \mathbb{C}$ the equation*

$$-y'' + q(x)y = \lambda y, \quad x \in [a_0, b_0]$$

has a unique solution $y = y(x, \lambda)$ satisfying the initial conditions

$$y(a_0) = f(\lambda), \quad y'(a_0) = g(\lambda) \quad (\text{or } y(b_0) = f(\lambda), \quad y'(b_0) = g(\lambda)).$$

For each fixed $x \in [a, b]$, $y(x, \lambda)$ is an entire function of λ.

We define two solutions We shall define two solutions

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda), & x \in [a, \xi_1), \\ \varphi_2(x,\lambda), & x \in (\xi_1, \xi_2), \\ \varphi_3(x,\lambda), & x \in (\xi_2, b] \end{cases} \quad \text{and } \psi(x,\lambda) = \begin{cases} \psi_1(x,\lambda), & x \in [a, \xi_1), \\ \psi_2(x,\lambda), & x \in (\xi_1, \xi_2), \\ \psi_3(x,\lambda), & x \in (\xi_2, b] \end{cases}$$

as follows:
Let $\varphi_1(x, \lambda)$ be the solution of equation (1.1) on the interval $[a, \xi_1)$ satisfying the initial conditions

$$\varphi_1(a, \lambda) = -\alpha_2 + \lambda\alpha_2', \quad \varphi_1'(a, \lambda) = -\alpha_1 + \lambda\alpha_1'.$$

Now by virtue of Lemma 2.3, we recursively define the solutions $\varphi_{i+1}(x, \lambda)$ $(i = 1, 2)$ of equation (1.1) on (ξ_i, ξ_{i+1}) ($(\xi_2, b]$ if $i = 2$) by the initial conditions

$$\begin{aligned} \varphi_{i+1}(\xi_i + 0, \lambda) &= \gamma_{2i-1}\varphi_i(\xi_i - 0, \lambda) + \gamma_{2i-1}'\varphi_i'(\xi_i - 0, \lambda), \\ \varphi_{i+1}'(\xi_i + 0, \lambda) &= \gamma_{2i}\varphi_i(\xi_i - 0, \lambda) + \gamma_{2i}'\varphi_i'(\xi_i - 0, \lambda). \end{aligned}$$

Analogously, we shall define the solutions $\psi_i(x, \lambda)$ $(i = 1, 2, 3)$ of equation (1.1) on (ξ_{i-1}, ξ_i) ($[a, \xi_1)$ if $i = 1$ and $(\xi_2, b]$ if $i = 3$) by the initial conditions

$$\psi_3(b, \lambda) = \beta_2 + \lambda\beta_2', \quad \psi_3'(b, \lambda) = \beta_1 + \lambda\beta_1',$$

$$\begin{aligned} \psi_i(\xi_i - 0, \lambda) &= \frac{\gamma_{2i}'\psi_{i+1}(\xi_i + 0, \lambda) - \gamma_{2i-1}'\psi_{i+1}'(\xi_i + 0, \lambda)}{\theta_i}, \\ \psi_i'(\xi_i - 0, \lambda) &= \frac{\gamma_{2i}\psi_{i+1}(\xi_i + 0, \lambda) - \gamma_{2i-1}\psi_{i+1}'(\xi_i + 0, \lambda)}{-\theta_i}. \end{aligned}$$

It is clear that $\varphi(x, \lambda)$ satisfies equation (1.1), the boundary condition (1.2) and the transmission conditions (1.4) and (1.5). Also $\psi(x, \lambda)$ satisfies equation (1.1), the boundary condition (1.3) and the transmission conditions (1.4) and (1.5).

For any solution $y(x,\lambda)$ of equation (1.1), we shall use the notation

$$y(x,\lambda) := y_\lambda(x).$$

Let us consider the Wronskians

$$\omega_i(\lambda) := W(\varphi_{i\lambda}, \psi_{i\lambda}; x), \quad x \in \Omega_i, \quad i = 1,2,3,$$

which are independent of x and are entire functions, where $\Omega_1 = [a, \xi_1)$, $\Omega_2 = (\xi_{i-1}, \xi_i)$ and $\Omega_3 = (\xi_2, b]$. A short calculation gives $\omega_{i+1}(\lambda) = \theta_i \omega_i(\lambda)$, $(i = 1,2)$.
Now we introduce the characteristic function $\omega(\lambda)$ as $\omega(\lambda) := \omega_3(\lambda)$.

Theorem 2.4. *The eigenvalues of the boundary value problem L coincide with he zeros of the characteristic function $\omega(\lambda)$.*

Theorem 2.5. *The residual spectrum of the operator A is empty, i.e., $\sigma_r(A) = \varnothing$.*

Theorem 2.6. *if $\theta_i > 0$ (i=1,2) and $\rho_j > 0$, then the operator A has only real point spectrum, i.e., $\sigma(A) = \sigma_p(A) \subseteq \mathbb{R}$.*

Theorem 2.7. *If $B > 0$, then the point spectrum of the operator A are real, i.e., $\sigma_p(A) \subseteq \mathbb{R}$.*

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Completeness of eigenfunctions of a discontinuous Sturm-Liouville problem with eigenparameter-dependent boundary conditions

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Abstract. In this paper, we investigate a class of Sturm-Liouville problems with eigenparameter dependent boundary conditions and transmission conditions at an interior point. A self-adjoint linear operator is defined in a suitable Hilbert space such that the eigenvalues of such a problem coincide with those of the operator. Then we study the resolvent operator and completeness of eigefunctions of the Sturm-Liouville problem and we derive the main result about expansion in series of eigenfunctions.

Keywords: Sturm-Liouville problem, Eigenfunction, Eigenvalue, series expansion, Resolvent operator.

AMS Mathematics Subject Classification [2020]: 34L10,47E05)

1. Introduction

The Sturm-Liouville problems have a wide range of applications in engineering, physics and mathematics. In recent years, many authers have devoted themselves to studing the Sturm-Liouville problems with internal discontinuous conditions, and have made many important achivements ([1–3]).

In this study, we consider the Sturm-Liouville equation

$$\ell y := \frac{1}{w(x)}\Big\{-\big(p(x)y'(x)\big)' + q(x)y(x)\Big\} = \lambda y(x), \quad x \in J := [a,\xi)\bigcup(\xi,b], \tag{1.1}$$

where $a < \xi < b$, $\frac{1}{p}, q, w \in L^1(J,\mathbb{R})$, $p(x) > 0$, $w(x) > 0$ on J, and the limits $p(\xi \pm 0)$, $q(\xi\pm)0$, $w(\xi \pm 0)$ are finite, $p(a)p(\xi+0) = p(b)p(\xi-0)$, λ is a complex eigenparameter, with eigenparameter-dependent boundary conditions at $x = a$ and $x = b$,

$$U(y) := \lambda(\alpha_1' y(a) - \alpha_2' y'(a)) - (\alpha_1 y(a) - \alpha_2 y'(a)) = 0, \tag{1.2}$$

$$V(y) := \lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = 0, \tag{1.3}$$

and the transmission conditions at the point of discontinuity $x = \xi$),

$$l_1(y) := y(\xi+0) - \gamma_1 y(\xi-0) - \gamma_1' y'(\xi-0) = 0, \tag{1.4}$$

*Speaker.

$$l_2(y) := y'(\xi+0) - \gamma_2 y(\xi-0) - \gamma_2' y'(\xi_i - 0) = 0, \tag{1.5}$$

where α_j, α_j', β_j, β_j', γ_j and γ_j' $(j = 1, 2)$, are real numbers, such that

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0,\ \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0,\ \theta_i := \begin{vmatrix} \gamma_1 & \gamma_1' \\ \gamma_2 & \gamma_2' \end{vmatrix} > 0.$$

2. An operator formulation in the modified Hilbert space

In order to study the Sturm-Liouville problem (1.1)–(1.5) we define the inner product in $L^2_w(J, \mathbb{C})$ as follows:

$$\langle f, g\rangle_1 := \frac{\theta}{p(a)} \int_a^\xi f(t)\overline{g(t)}w(t)dt + \frac{1}{p(b)} \int_\xi^b f(t)\overline{g(t)}w(t)dt, \quad f, g \in L^2_w(J, \mathbb{C}).$$

Obviously, $H_1 := \left(L^2_w(J, \mathbb{C}), \langle\cdot,\cdot\rangle_1\right)$ is a Hilbert space.

Then we define a new inner product in the Hilbert space $H := L^2_w(J, \mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ by

$$\langle F, G\rangle := \langle f, g\rangle_1 + \frac{\theta p(a)}{\rho_1} f_1\overline{g}_1 + \frac{p(b)}{\rho_2} f_2\overline{g}_2,$$

where where $F := (f(x), f_1, f_2)$, $G := (g(x), g_1, g_2) \in H$.

For convenience, we introduce the following notations:

$$\begin{aligned} R_1(y) &:= \alpha_1 y(a) - \alpha_2 y'(a), \quad & R_1'(y) &:= \alpha_1' y(a) - \alpha_2' y'(a), \\ R_2(y) &:= \beta_1 y(b) - \beta_2 y'(b), \quad & R_2'(y) &:= \beta_1' y(b) - \beta_2' y'(b). \end{aligned}$$

Then, we define a linear operator A acting in H with domain $D(A)$ as follows:

$$\begin{aligned} AF &:= (\ell f, R_1(f), -R_2(f)) \quad \text{for } F = (f(x), R_1'(f), R_2'(f)) \in D(A). \\ D(A) &:= \{F = (f(x), f_1, f_2) \in H | f, f' \in AC_{loc}(J), \text{ and have} \\ &\quad \text{one-hand sided limits } f(\xi_1 \pm 0),\ f'(\xi_1 \pm 0),\ i = \overline{1, m}, \\ &\quad \ell f \in L_2(J),\ l_k(f) = 0,\ k = 1, 2,\ f_1 = R_1'(f),\ f^2_w(J, \mathbb{C} = R_2'(f)\} \end{aligned}$$

The boundary value problem (1.1)–(1.5) can be expressed as the operator equation

$$AY = \lambda Y, \quad Y := (y(x), R_1'(y), R_2'(y)) \in D(A).$$

Obviously, we have

Lemma 2.1. *The eigenvalues of the boundary value problem (1.1)–(1.5) coincide with those of A, and its eigenfunctions are the first components of the corresponding eigenfunctions of the operator A.*

Lemma 2.2. *The domain $D(A)$ is dense in H.*

Theorem 2.3. *The operator A is symmetric in the Hilbert space H.*

3. Resolvent operator

In this section, we show that each number $\lambda \in \mathbb{C}$ which is not an eigenvalue of the operator A, is a regular value of this operator. We will investigate the resolvent

$$R(\lambda, A) := (\lambda I - A)^{-1},$$

where I is the unit operator. For an arbitrary element $F = (f(x), f_1, F_2) \in H$, let us write the operator equation

$$(\lambda I - A) = F \tag{3.1}$$

as a non-homogeneous boundry value problem

$$\ell y := \frac{1}{w(x)}\Big\{-\big(p(x)y'(x)\big)' + q(x)y(x)\Big\} = \lambda y(x) - f(x), \quad x \in J, \tag{3.2}$$

$$\lambda(\alpha_1' y(a) - \alpha_2' y'(a)) - (\alpha_1 y(a) - \alpha_2 y'(a)) = f_1, \tag{3.3}$$

$$\lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = f_2, \tag{3.4}$$

$$y(\xi+0) - \gamma_1 y(\xi-0) - \gamma_1' y'(\xi-0) = 0, \tag{3.5}$$

$$y'(\xi+0) - \gamma_2 y(\xi-0) - \gamma_2' y'(\xi_i-0) = 0, \tag{3.6}$$

which is equivalent to (3.1).

Now, we define two solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the differential equation (1.1). Let us denote by $\varphi_1(x,\lambda)$ the solution of the equation (1.1) on the interval $[a,\xi)$ satisfying the initial conditions

$$\varphi_1(a,\lambda) = -\alpha_2 + \lambda\alpha_2', \quad \varphi_1'(a,\lambda) = -\alpha_1 + \lambda\alpha_1'.$$

After defining the function $\varphi_1(x,\lambda)$, we define $\varphi_2(x,\lambda)$ the solution of the equation (1.1) on the interval $(\xi,b]$ by the initial conditions

$$\begin{aligned}\varphi_2(\xi+0,\lambda) &= \gamma_1\varphi_1(\xi-0,\lambda) + \gamma_1'\varphi_1'(\xi-0,\lambda),\\ \varphi_2'(\xi+0,\lambda) &= \gamma_2\varphi_1(\xi-0,\lambda) + \gamma_2'\varphi_1'(\xi-0,\lambda).\end{aligned}$$

Similarly, let us denote by $\psi_2(x,\lambda)$ the solution of the equaton (1.1) on the interval $(\xi,b]$ satisfying the initial condtions

$$\psi_2(b,\lambda) = \beta_2 + \lambda\beta_2', \quad \psi_2'(b,\lambda) = \beta_1 + \lambda\beta_1'.$$

After defining this solution, we denote by $\psi_1(x,\lambda)$ the solution of the equation (1.1) on the interval $[a,\xi$ satisfying the initial condtions

$$\begin{aligned}\psi_1(\xi-0,\lambda) &= \frac{\gamma_2'\psi_2(\xi+0,\lambda) - \gamma_1'\psi_2'(\xi+0,\lambda)}{\theta_i},\\ \psi_1'(\xi-0,\lambda) &= \frac{\gamma_2\psi_2(\xi+0,\lambda) - \gamma_1\psi_2'(\xi+0,\lambda)}{-\theta_i}.\end{aligned}$$

Let us consider the Wronskians $\omega_i(\lambda) = W(\varphi_i,\psi_i;\lambda)$, $x \in \Omega_i$, $i = 1,2$, where $\Omega_1 = [a,\xi)$, $\Omega_2 = (\xi,b]$, which independent of x.

Lemma 3.1. *For all $\lambda \in \mathbb{C}$ which is not an eigenvalue of the problem (1.1)–(1.5), we have $\omega_i(\lambda) \neq 0$ $(i = 1,2)$.*

Let us assume that λ is not an eigenvalue. Then, Lemma implies that the functions $\varphi_i(x,\lambda)$ and $\psi_i(x,\lambda)$ are linearly independent on Ω_i $(i = 1,2)$. Now, by the method of variation of constants we derive the following formula the resolvent of the boundary value problem (3.2)–(3.6):

$$\begin{aligned}Y(x,\lambda) = &\frac{\psi(x,\lambda)}{\omega(x,\lambda)}\int_a^x \frac{\varphi(t,\lambda)f(t)w(t)}{p(t)}dt + \frac{\psi(x,\lambda)}{\omega(x,\lambda)}\int_a^x \frac{\varphi(t,\lambda)f(t)w(t)}{p(t)}dt\\ &+ f_1\frac{\psi(x,\lambda)}{\omega_1(\lambda)} + f_2\frac{\varphi(x,\lambda)}{\omega_2(\lambda)},\end{aligned}$$

where

$$\omega(x,\lambda) := \begin{cases}\omega_1(\lambda), & x \in [a,\xi),\\ \omega_2(\lambda), & x \in (\xi,b].\end{cases}$$

Theorem 3.2. *Each $\lambda \in \mathbb{C}$ which is not an eigenvalue of the problem (1.1)–(1.5) is a regular value of the operator A, and the resolvent operator $R(\lambda, A) : H \to H$ is a compact operator.*

4. Expansion in series of system of eigenfunctions

Theorem 4.1. *The operator A is a self adjoint in the Hilbert space H.*

Corollary 4.2. *All eigenvalues of the boundary value problem (1.1)–(1.5) are real.*

Corollary 4.3. *If λ_1 and λ_2 are two different eigenvalues of the boundary value problem (1.1)–(1.5) and $y_1(x)$ and $y_2(x)$ are eigenfunctions corresponding to these eigenvalues , respectively, then*

$$\frac{\theta p(a)}{\rho_1}\int_a^{\xi} y_1(x)y_2(x)w(x)dx + \frac{p(b)}{\rho_2}\int_{\xi}^{b} y_1(x)y_2(x)w(x)dx + \frac{\theta p(a)}{\rho_{\ 1}} R_1'(y_1)R_1'(y_2) + \frac{p(b)}{\rho_2} R_2'(y_1)R_2'(y_2) = 0.$$

In the Hilbert space H, the operator A has countable number of real eigenvalues, the algebraic multiplicity of each eigenvalue is finite, the sequence of eigenvalues has lower bound and does not have a finite accumulation point. consequently, we can write the sequence of eigenvalues as $\lambda_1 \leq \lambda_2 \leq \cdots$.Let us denote the appropriate normed eigenelements as $\Phi_n := (\phi_n, R_1'(\phi_n), R_2'(\phi_n)$, $(\|\phi_n\|_H = 1,\ n = 1, 2, 3, \cdots)$. Then from Theorem 3.2 and 4.4 and the well-known Hilbert-schmidt Theorem (see [4]) we get the following theorem:

Theorem 4.4. *For each element $F \in H$ the Fourier series $\sum_{n=1}^{\infty}\langle F, \Phi_n\rangle\Phi_n$ will converge to F in the Hilbert space H.*

Corollary 4.5. *Each function $f \in L_w^2(J)$ can be written as a series expansion the eigenfunction system $\{\phi_n\}_{n=1}^{\infty}$ of the boundary value problem (1.1)–(1.5) as*

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{\theta p(a)}{\rho_1}\int_a^{\xi}\phi_n(t)f(t)w(t)dt + \frac{p(b)}{\rho_2}\int_{\xi}^{b}\phi_n(t)f(t)w(t)dt\right)\phi_n(x) \tag{4.1}$$

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On solutions of nonlinear Langevin equation featuring three fractional orders with anti-periodic boundary conditions

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Abstract. This paper investigates the existence and uniqueness of solutions for a nonlinear Langevin equation with three non-integer order derivatives and anti-periodic boundary conditions (APBCs). Using fixed point theorems such as Schauder and contraction mapping principles, the existence and uniqueness of solutions are examined.

Keywords: Langevin equation; Existence.

AMS Mathematics Subject Classification [2020]: 34A55, 34B99.

1. Introduction

In the present research work, the distinctive feature and the existence of the APBVPs of Langevin equation containing three distinct fractional orders are investigated:

$$
(1) \qquad \begin{cases} D^{\alpha_2}\big(D^{\alpha_1}+\lambda\big)x(t)=\sigma\big(t,D^{\alpha_3}x(t)\big), \\ x(0)+x(1)=0, \\ D^{\alpha_1}x(0)+D^{\alpha_1}x(1)=0, \\ \mathfrak{D}^{2\alpha_1}x(0)+\mathfrak{D}^{2\alpha_1}x(1)=0, \\ 0<\alpha_1\leq 1,\ 1<\alpha_2\leq 2,\ 0<\alpha_3\leq 1, \end{cases}
$$

where λ is a real number, D^{α_1}, D^{α_2} and D^{α_3} are the Caputo derivative, $\sigma:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

Moreover, $\mathfrak{D}^{\alpha_1}x$ is the sequential fractional derivative suggested by Miller and Ross [1]

$$
(2) \qquad \begin{cases} \mathfrak{D}^{\alpha_1}x=D^{\alpha_1}x, \\ \mathfrak{D}^{k\alpha_1}x=\mathfrak{D}^{\alpha_1}\mathfrak{D}^{(k-1)\alpha_1}x, \end{cases}
$$

The uniqueness and the existence of solutions for APBVPs (1) are examined by utilizing the SFP theorem and BCP.

*Speaker.

2. Main result

For the rest of the paper, the uniqueness and the existence of solutions for Langevin differential equation of non integer orders (1) will be examined. The space of continuous functions $C[0,1]$ will be explained. Also the space E is defined as [2]:

$$E = \{x|\ x \in C[0,1], D^{\alpha_3}x \in C[0,1]\}, \tag{3}$$

equipped with the norm

$$\|x\|_* = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |D^{\alpha_3}x(t)| = \|x\| + \|D^{\alpha_3}x\|. \tag{4}$$

Lemma 2.1. *The space $(E, \|.\|_*)$ is a Banach space.*

Proof. Suppose $\{u_n\}_{\infty}^{n=1}$ be a Cauchy sequence in the space $(E, \|.\|_*)$; Thus, it is evident $\{u_n\}_{\infty}^{n=1}$ and $\{D^{\alpha}u_n(t)\}_{\infty}^{n=1}$ convergence to some v, w on $[0,1]$ and $v, w \in [0,1]$. It is sufficient that we prove $w = D^{\alpha}v$.

Consider the following inequality

$$\begin{aligned} |I^{\alpha}D^{\alpha}u_n(t) - I^{\alpha}w(t)| &\leq \int_0^t \frac{|D^{\alpha}u_n(s) - w(s)|}{\Gamma(\alpha)}(t-s)^{\alpha-1}ds \\ &\leq \frac{\max|D^{\alpha}u_n(t) - w(t)|}{\Gamma(\alpha+1)}. \end{aligned} \tag{5}$$

□

Lemma 2.2. *For $\alpha_1 > 0$, the following relation holds:*

$$\begin{aligned} D^{\alpha_3}Tx(t) =& \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\sigma\big(s, D^{\alpha_3}x(s)\big)ds - \lambda x(t) \\ &- \frac{1}{2}\int_0^1 \frac{\sigma\big(s, D^{\alpha_3}x(s)\big)}{\Gamma(\alpha_2)}(1-s)^{\alpha_2-1}ds \\ &+ \left(\frac{\Gamma(2-\alpha_1)}{2} - \Gamma(2-\alpha_1)t\right)\int_0^1 \frac{\sigma\big(s, D^{\alpha_3}x(s)\big)}{\Gamma(\alpha_2-\alpha_1)}(1-s)^{\alpha_2-\alpha_1-1}ds. \end{aligned}$$

In subsequent section, the following presupposition are required in order to prove the major results of this study.

Hypotheses:

(H1): $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

(H2): There exist a non-negative function $\phi \in L^1[0,1]$ such that $|\sigma(t,x)| \leq \phi(t) + a|x|$ where $a \in \mathbb{R}^+$.

(H3): $|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$, where $K > 0$ is constant.

Theorem 2.3. *Supposing that hypotheses (H1) and (H2) are true, then there exists a solution for fractional boundary value problem (1). Provided $0 < K_3 < 1$, where K_3 is defined [3].*

Theorem 2.4. *Under assumptions (H1) and (H3) problem (1) have unique solution, provided that*

$$w = \max\{KK_1, K_2\} < 1.$$

Proof. By condition (H3), for any $x, y \in E$, $t \in [0,\ 1]$, we have

$$|Tx(t) - Ty(t)| \leq \int_0^t \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\left|\sigma\big(s, D^{\alpha_3}x(s)\big) - \sigma\big(s, D^{\alpha_3}y(s)\big)\right|ds$$

$$
\begin{aligned}
&+|\lambda|\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\,|x(s)-y(s)|\,ds\\
&+\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\left|\sigma\big(s,D^{\alpha_3}x(s)\big)-\sigma\big(s,D^{\alpha_3}y(s)\big)\right|ds\\
&+\frac{|\lambda|}{2}\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\,|x(s)-y(s)|\,ds\\
&+\left(\frac{1}{4\Gamma(\alpha_1+1)}-\frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left|\sigma\big(s,D^{\alpha_3}x(s)\big)\right.\\
&\qquad\qquad\left.-\sigma\big(s,D^{\alpha_3}y(s)\big)\right|ds\\
&+\frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4}+\frac{1+\alpha_1}{2}t^{\alpha_1}-t^{\alpha_1+1}\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\times\\
&\qquad\qquad\left|\sigma\big(s,D^{\alpha_3}x(s)\big)-\sigma\big(s,D^{\alpha_3}y(s)\big)\right|ds\\
\le& K\left\|D^{\alpha_3}x-D^{\alpha_3}y\right\|\frac{t^{\alpha_1+\alpha_2}}{\Gamma\left(\alpha_1+\alpha_2+1\right)}\\
&+|\lambda|\,\|x-y\|\,\frac{t^{\alpha_1}}{\Gamma\left(\alpha_1+1\right)}+\frac{K}{2\Gamma\left(\alpha_1+\alpha_2+1\right)}\left\|D^{\alpha_3}x-D^{\alpha_3}y\right\|\\
&+\frac{|\lambda|}{2\Gamma\left(\alpha_1+1\right)}\,\|x-y\|\\
&+\frac{K}{\Gamma\left(\alpha_2+1\right)}\left(\frac{1}{4\Gamma(\alpha_1+1)}-\frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\left\|D^{\alpha_3}x-D^{\alpha_3}y\right\|\\
&\left(\frac{1-\alpha_1}{4}+\frac{1+\alpha_1}{2}t^{\alpha_1}-t^{\alpha_1+1}\right)\left\|D^{\alpha_3}x-D^{\alpha_3}y\right\|\\
&\le\left(\frac{3}{2\Gamma\left(\alpha_1+\alpha_2+1\right)}+\frac{3}{4\Gamma\left(\alpha_1+1\right)\Gamma\left(\alpha_2+1\right)}\right)\times\\
&K\left\|D^{\alpha_3}x-D^{\alpha_3}y\right\|+\frac{3\,|\lambda|}{2\Gamma\left(\alpha_1+1\right)}\,\|x-y\|\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|D^{\alpha_3}Tx(t)-D^{\alpha_3}Ty(t)\right|=&\left|\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,D^{\alpha_3}x(s)\big)ds-\lambda x(t)\right.\\
&-\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,D^{\alpha_3}x(s)\big)ds\\
&+\left(\frac{\Gamma(2-\alpha_1)}{2}\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\sigma\big(s,D^{\alpha_3}x(s)\big)ds\\
&-\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,D^{\alpha_3}x(s)\big)ds+\lambda y(t)\\
&+\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,D^{\alpha_3}y(s)\big)ds\\
&\left.-\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\sigma\big(s,D^{\alpha_3}y(s)\big)ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left|\sigma\big(s, D^{\alpha_3}x(s)\big) - \sigma\big(s, D^{\alpha_3}y(s)\big)\right| ds \\
&\quad + |\lambda| \, |x(t) - y(t)| \\
&\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left|\sigma\big(s, D^{\alpha_3}x(s)\big) - \sigma\big(s, D^{\alpha_3}y(s)\big)\right| ds \\
&\quad + \left(\frac{\Gamma(2-\alpha_1)}{2}\right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \left|\sigma\big(s, D^{\alpha_3}x(s)\big)\right. \\
&\leq K \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| \frac{t_1^{\alpha}}{\Gamma(\alpha_2+1)} + |\lambda| \, \|x-y\| \\
&\quad + \frac{K}{2\Gamma(\alpha_2+1)} \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| \\
&\quad + \frac{\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)} + \frac{\Gamma(2-\alpha_1)t}{\Gamma(\alpha_2-\alpha_1+1)} K \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| \\
&\leq \left(\frac{3}{2\Gamma(\alpha_2+1)} + \right) K \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| \\
&\quad + |\lambda| \, \|x-y\| .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|Tx(t) - Ty(t)\|_* &= \max_{t\in[0,1]} |Tx(t) - Ty(t)| + \max_{t\in[0,1]} |D^{\alpha_3}Tx(t) - D^{\alpha_3}Ty(t)| \\
&\leq \left(\frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right) \times \\
&\quad K \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| + \frac{3|\lambda|}{2\Gamma(\alpha_1+1)} \|x-y\| \\
&\quad + \left(\frac{3}{2\Gamma(\alpha_2+1)} + \frac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}\right) K \left\|D^{\alpha_3}x - D^{\alpha_3}y\right\| . \\
&\leq K_1 K \left\|D^{\alpha_3}Tx(t) - D^{\alpha_3}Ty(t)\right\| + K_2 \|x-y\| \leq w \|x-y\|_* ,
\end{aligned}
$$

where $w = \max\{KK_1, K_2\}$. Since $w < 1$, then the operator T is a contraction operator.

Consequently, it is suggested by the contraction mapping principle that for problem (1), there is a **unique** solution. □

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Functional Analysis

Polynomial convexity of the spectrum of generators and its extensions on linearly complete regular algebras

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Abstract. **Let u be a generator for a commutative Banach algebra with unit. In this article, under the assumption that the boundedness radius β is sub-additive, we establish that the spectrum $Sp(u)$ is polynomially convex. As a consequence, we establish the polynomial convexity of the joint spectra in finitely generated algebras.**

Keywords: Strongly sequential algebras, Fundamental algebras, Spectrum, polynomial convexity, spectrum of generator.

AMS Mathematics Subject Classification [2020]: 46H05, 46H20.

1. Introduction

The study of non-normed topological algebras was initiated in 1940 by Arens, Gel'fand, Kaplansky, and others. Several other algebras belong to the class of Arens–Michael algebras and cannot be made into Banach algebras under their usual topology. The concept of an m-convex algebra appeared in the literature in 1964 under the name pseudo-normed algebra, and it is due to R. Arens. The first fundamental results on this type of algebra were published independently in 1952 by R. Arens and E. A. Michael. Around the same time, the Russian mathematician M. A. Naimark recognized the need to develop a theory of non-normed topological algebras, analogous to that of Banach algebras. Michael conducted a thorough investigation of locally multiplicatively convex (LMC) algebras and highlighted the issue of identifying which algebras possess continuous multiplicative linear functionals. This is commonly known as Michael's problem. In the same paper, Michael presented algebras in which the property that every multiplicative linear functional is continuous holds. This problem plays a crucial role in the development of research in topological and even normed algebras. In 1979, Husain introduced the concept of strongly

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sequential topological algebras. Moreover, in 2008, Honary and Najafi Tavani demonstrated that this idea holds in another class of topological algebras, known as Q-algebras.

In 1991, Ansari introduced the framework of fundamental algebras as a generalization of Cohen's factorization theorem. Subsequently, in 2001, he defined a new class—fundamental locally multiplicative (FLM) algebras by imposing an additional condition on fundamental algebras. Later, in 2016, Ansari and Sabet initiated the study of algebras that are simultaneously fundamental and strongly sequential. In 2016, Ansari and Sabet were the first to investigate algebras that are both fundamental and strongly sequential [3].

In this paper, we refer to such algebras as FSS-algebras, which properly contain FLM-algebras. We prove that, in a commutative Banach algebra A with unit and a generator u, the spectrum $Sp(u)$ remains polynomially convex when two additional conditions are imposed on FSS-algebras.

2. Preliminaries

First, we mention some of the basic concepts of topological algebras.
A topological algebra A is called *strongly sequential*, if there exists a neighborhood of zero U such that for every $x \in U$, the sequence $(x^n)_n$ converges to zero in its topology.
The *boundedness radius* of an element a in the topological algebra A, denoted by $\beta(a)$, is defined as follows

$$\beta(a) = \inf\{r > 0 : \frac{a^n}{r^n} \to 0\}.$$

If $\beta(a) < +\infty$, a is said to be a *bounded member*.

Proposition 2.1. *[3] A topological algebra A is strongly sequential if and only if β is continuous at zero.*

Topological algebra A is called *fundamental* if there exists $b > 1$ such that, for every sequence $(x_n)_n$ in A, the convergence of $b^n(x_{n+1} - x_n) \to 0$ in A implies the sequence $(x_n)_n$ is a Cauchy sequence [2]. In this paper, we will use the abbreviation FSS-algebra for the term fundamental strongly sequential topological algebra. Additionally, we denote the set of invertible elements in a topological algebra with unit A by $Inv(A)$, and the set of non-invertible elements by $Sing(A)$. Finally, a topological algebra with unit A is called a Q-algebra if $Inv(A)$ is an open set. An algebra with unit, in which $Inv(A) = A - \{0\}$, is called a *division algebra*.
The spectrum of $a \in A$, denoted by $Sp(A, a)$, is defined as follows

$$Sp(A, a) := \{\lambda : \lambda - a \in Sing(A)\}.$$

For brevity and without ambiguity, we use the notation $Sp(a)$ instead of $Sp(A, a)$. In this paper, we use the symbol A^* to denote the set of all continuous linear functionals on A.

A linear functional φ on A is called a *multiplicative linear functional* if it satisfies the following condition: For any two arbitrary elements $x, y \in A$

$$\varphi(xy) = \varphi(x)\varphi(y).$$

The set of nonzero multiplicative linear functionals on A, equipped with the A-topology, is called the *carrier space*. In this paper, we denote this set by Φ_A.

Proposition 2.2. *[3] Let A be a complete metrizable FSS-algebra. Then, the following results hold:*

(1) *If $f \in \Phi_A$, then $|f(x)| \leqslant \beta(x)$, for all $x \in A$.*
(2) *The carrier space of A is weak* compact.*

For every topological algebra A, A is called a linearly complete regular algebra if, for every closed subspace M of A and every element $a \in A$ such that $a \notin M$, there exists an element $\Lambda \in A^*$ satisfying $\Lambda a = 1$ and $\Lambda b = 0$ for all $b \in M$. For brevity, we use the abbreviation LCR instead of linearly complete regular.

Proposition 2.3. *[1] If A is a complete metrizable FSS and LCR-algebra, then the maximal modular ideals of A are the kernels of multiplicative linear functionals.*

3. Main results

Proposition 3.1. *Suppose A is a strongly sequential algebra, and let a be a generator for A. If $\lambda \in \mathbb{C} - Sp(a)$ and $P(\mathbb{C})$ denotes the set of all polynomials with complex coefficients, then there exists a polynomial $p \in P(\mathbb{C})$ such that*

$$|p(\lambda)| \geqslant \beta(p(a)).$$

Proof. According to the assumptions, there exists an element $b \in A$ such that $(\lambda - a)b = 1$. Since a is a generator for A, there exists a sequence of polynomials $p_k \in P(\mathbb{C})$ such that $p_k(a) - b \to 0$. By Proposition 2.1, the function β is continuous at zero. Hence, there exists a natural number N such that

$$\beta(b - p_N(a)) < 1/(\beta(\lambda - a)).$$

On the other hand, for all $x, y \in A$ the relation $\beta(xy) \leqslant \beta(x)\beta(y)$ holds. Thus, it follows that

$$\begin{aligned}\beta(1 - (\lambda - a)p_N(a)) &= \beta((\lambda - a)b - (\lambda - a)p_N(a)) \\ &= \beta((\lambda - a)(b - p_N(a))) \leqslant \beta(\lambda - a)\beta(b - p_N(a)) < 1.\end{aligned}$$

Now, if we put $p(z) = 1 - (\lambda - z)p_N(z)$, then it is clear that, $p \in P(\mathbb{C})$, $p(\lambda) = 1$, and $\beta(p(a)) < 1$. Thus, the proof is complete. □

Proposition 3.2. *Let A be a complete merizable FSS and LCR-algebra. Define the set*

$$J = \{\sum_{k=1}^{n}(\lambda_k - a_k)b_k \colon b = (b_1, b_2 \ldots, b_n) \in A^n\}$$

where $a = (a_1, a_2, \ldots, a_n) \in A^n$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$.
Then the following statements are equivalent:

(1) *$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in Sp(a)$.*
(2) *J is a proper ideal of A.*
(3) *$J \subset SingA$.*
(4) *$1 \notin J$.*

Proof. By applying Proposition 2.3, the proof proceeds along lines similar to those presented in Bonsall's book (see [5, p. 100]). □

Suppose A is a topological algebra in which every element is bounded. A real-valued function φ is defined on A is called a subadditive function if, for all $x, y \in A$, it satisfies the inequality

$$\varphi(x+y) \leqslant \varphi(x) + \varphi(y).$$

Kinani, Oubbi and Oudadess established that the boundedness radius function β is subadditive in commutative locally convex algebras [6].

Proposition 3.3. *Suppose that A is a complete metrizable FSS and LCR-algebra, and that β is a subadditive function on A. If the set $\{a_1, a_2, \ldots, a_n\}$ is a generator for A and $\lambda \in \mathbb{C}^n - Sp(a)$, then there exists a polynomial $p \in P(\mathbb{C}^n)$ such that*

$$|p(\lambda)| \geqslant \beta(p(a)).$$

Proof. See Proposition 9 in Section 19 of Bonsall's book for a detailed proof [5]. □

Theorem 3.4. *Assume that A is a complete metrizable FSS and LCR-algebra, and that β is subadditive. If $\{a_1, a_2, \ldots, a_n\}$ is a set of joint generators for A, then the joint spectrum $Sp(a)$, where $a = (a_1, \ldots, a_n)$, is a nonempty, compact, and polynomially convex subset of $\mathbb{C}^n$.*

Proof. We consider the function $G : \Phi_A \to \mathbb{C}^n$ defined by

$$G(\varphi) = \varphi(a) = (\varphi(a_1), \varphi(a_2), ..., \varphi(a_n)).$$

For each i, let

$$e_i = (0, 0, \ldots, 1, \ldots, 0).$$

Then we can express $G(\varphi)$ as

$$G(\varphi) = \widehat{a_1}(\varphi)e_1 + \widehat{a_2}(\varphi)e_2 + \cdots + \widehat{a_n}(\varphi)e_n$$

where each function $\widehat{a_i} : \Phi_A \to \mathbb{C}$ is defined by

$$\widehat{a_i}(\varphi) = \varphi(a_i), \quad i = 1, 2, \ldots, n.$$

Since each $\widehat{a_i}$ is continuous, it follows that G is also continuous. By Proposition 2.2, Φ_A is a compact set, and by Proposition 2.3, it is nonempty. Therefore, $G(\Phi_A)$ is a nonempty, compact subset of $\mathbb{C}^n$. To complete the proof, it remains to show that $Sp(a)$ is polynomially convex, i.e., that

$$hull(Sp(a)) \subseteq Sp(a).$$

To this end, we first prove that the inequality $|p|_{Sp(a)} \leqslant \beta(p(a))$ holds. For any $z \in Sp(a)$ and any polynomial $p \in P(\mathbb{C}^n)$, by the definition of $Sp(a)$, there exists $\varphi \in \Phi_A$ such that $z = \varphi(a)$. Then by Proposition 2.2, it follows

$$|p(z)| = |p(\varphi(a))| = |\varphi(p(a)| \leqslant \beta(p(a)).$$

Thus,

$$|p|_{Sp(a)} \leqslant \beta(p(a)).$$

Now, assume $\lambda \in hull(Sp(a))$. By the definition of $hull(Sp(a))$, for any polynomial $p \in P(\mathbb{C}^n)$, we have

$$|p(\lambda)| \leqslant |p|_{Sp(a)}.$$

Since $|p|_{Sp(a)} \leqslant \beta(p(a))$, it follows that

$$|p(\lambda)| \leqslant \beta(p(a)).$$

Finally, by Proposition 3.3, this implies that $\lambda \in Sp(a)$, completing the proof. □

4. Conclusion

In this paper, we show that, under suitable conditions, the spectrum of a generator in LCR-algeras is polynomially convex. These findings may be useful for understanding the spectral structure and uniqueness of topology in non-normable algebras. As a continuation of this research, future investigations may focus on extending spectral results in the framework of commutative FSS-algebras, particularly examining the spectra of generators in relation to topological properties such as convexity and compactness

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Babenko's attitude based on new sequential $\eth$-Caputo differential equations via measures of noncompactness

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Abstract. This study examines the existence and uniqueness (EU) of solutions for two nonlinear fractional IDEs that incorporate a $\eth$–based Caputo derivative and a $\eth$-based Riemann–Liouville integral. Our approach broadens the theoretical framework of the Mittag–Leffler function, adopts a strategy inspired by Babenko, and deploys topological degree theory (TDT) for condensing maps in conjunction with the Banach contraction principle.

Keywords: Babenko's attitude, noncompactness, $\eth$- Caputo derivative.

AMS Mathematics Subject Classification [2020]: 34A55, 34B99.

1. Introduction

Recently, the EU of a nonlinear IDE via a BC were examined by Li et al. [1]. This study employed several FP theorems:

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathfrak{p}}^{\varphi}\mathcal{Z}(\mathcal{Q}) + \mu\mathfrak{I}_{\mathfrak{p}}^{\kappa}\mathcal{Z}(\mathcal{Q}) = \mathcal{F}\left(\mathcal{Q}, \mathcal{Z}(\mathcal{Q})\right) \quad \mathcal{Q} \in [\mathfrak{p}, \mathfrak{q}], \quad l-1 < \varphi \leq l, \quad \kappa \geq 0 \\ \mathcal{Z}(\mathfrak{a}) = -f(\mathcal{Z}) \qquad \mathcal{Z}(\mathfrak{p}) = \mathcal{Z}'(\mathfrak{p}) = \ldots = \mathcal{Z}^{(l-1)}(\mathfrak{q}), \end{cases}$$

in which $0 \leq \mathfrak{p} < \mathfrak{q} < +\infty$ and μ is a constant.

Let $f : C[\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$, $\mathcal{F} : [\mathfrak{a}, \mathfrak{b}] \times \mathbb{R} \to \mathbb{R}$ and $A(\mathcal{Q}) \in C[\mathfrak{a}, \mathfrak{b}]$.

We will examine the EU of solutions for the subsequent nonlinear $\eth$- IDE with NBC and varying coefficients when $l < \varphi \leq l+1$ and $\beta \geq 0$

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathfrak{a}^+}^{\varphi;\eth}\mathcal{Z}(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\eth}\mathcal{Z}(\mathcal{Q}) = \mathcal{F}\left(\mathcal{Q}, \mathcal{Z}(\mathcal{Q})\right) \quad \mathcal{Q} \in [\mathfrak{a}, \mathfrak{b}], \\ \mathcal{Z}(\mathfrak{a}) = -f(\mathcal{Z}), \qquad \mathcal{Z}''(\mathfrak{a}) = \ldots = \mathcal{Z}^{(l)}(\mathfrak{a}) = 0, \\ \int_{\mathfrak{a}}^{\mathfrak{b}} \eth'(\mathcal{Q})\mathcal{Z}(\mathcal{Q})d\mathcal{Q} = \Lambda, \end{cases} \tag{1}$$

where Λ is constant, ${}^{c}\mathfrak{D}_{\mathfrak{a}^+}^{\varphi;\eth}$ and $\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\eth}$ are considered the $\eth$-R-L fractional integral operators and $\eth$-Caputo fractional, in the state order. Babenko's attitude [2] and topological degree theory for condensing map are powerful tools for solving differential and integro-differential

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equations with initial conditions (ICs) by treating bounded integral operators as normal variables. In particular, for $\eth(\mathcal{Q}) = \mathcal{Q}$, problem 1 arises as a result

$$(2)\qquad \begin{cases} {}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}^{\kappa}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) = \digamma\left(\mathcal{Q}, \mathcal{Z}(\mathcal{Q})\right) & \mathcal{Q} \in [\mathfrak{a}, \mathfrak{b}], \\ \mathcal{Z}(\mathfrak{a}) = -f(\mathcal{Z}), & \mathcal{Z}''(\mathfrak{a}) = ... = \mathcal{Z}^{(l)}(\mathfrak{a}) = 0, \\ \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{Z}(\mathcal{Q})d\mathcal{Q} = \Lambda, \end{cases}$$

and for $\eth(\mathcal{Q}) = \mathcal{Q},\ l = 1$, problem 1 turns out to be

$$(3)\qquad \begin{cases} {}^{c}\mathfrak{D}^{\varphi}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}^{\kappa}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) = \digamma\left(\mathcal{Q}, \mathcal{Z}(\mathcal{Q})\right) & \mathcal{Q} \in [\mathfrak{a}, \mathfrak{b}], \\ \mathcal{Z}(\mathfrak{a}) = -f(\mathcal{Z}) & \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{Z}(\mathcal{Q})d\mathcal{Q} = \Lambda, \end{cases}$$

2. Main results

Lemma 2.1. *Given a postulated function $\mathfrak{h} \in C(\mathfrak{H}, \mathbb{R})$, the solution to the fractional BVP*

$$(4)\qquad \begin{cases} {}^{c}\mathfrak{D}^{\varphi;\eth}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) + A(\mathcal{Q})\mathfrak{I}^{\kappa;\eth}_{\mathfrak{a}^+}\mathcal{Z}(\mathcal{Q}) = \mathfrak{h}(\mathcal{Q}), & \mathcal{Q} \in [\mathfrak{a}, \mathfrak{b}], \quad \mathfrak{k} > 0 \\ \mathcal{Z}(\mathfrak{a}) = -f(\mathcal{Z}), & \mathcal{Z}''(\mathfrak{a}) = ... = \mathcal{Z}^{(l)}(\mathfrak{a}) = 0 \\ \int_{\mathfrak{a}}^{\mathfrak{b}} \eth'(\mathcal{Q})\mathcal{Z}(\mathcal{Q})d\mathcal{Q} = \Lambda, \end{cases}$$

is determined by

(5)

$$\mathcal{Z}(\mathcal{Q}) = \int_{\mathfrak{a}}^{\mathcal{Q}} \eth'(\mathfrak{y})e^{-\mathfrak{k}(\eth(\mathcal{Q})-\eth(s))}\mathfrak{I}^{\varphi;\eth}_{\mathfrak{a}^+}\left(\mathfrak{h}(\mathfrak{y})\right)d\mathfrak{y} + \frac{1}{\mathfrak{k}\delta_1}\Phi^{\mathfrak{b},\varsigma}_{\eth}\mathfrak{h}(\mathcal{Q}) - \frac{1}{\mathfrak{k}\delta_3}\left(\frac{\mathfrak{k}\delta_2}{\delta_1} - e^{-\mathfrak{k}\eth(\mathcal{Q})}\right)\varkappa^{\mathfrak{b},\varsigma}_{\eth}\mathfrak{h}(\mathcal{Q}).$$

Lemma 2.2. *$\mathfrak{G}_1$ is a continuous function that satisfies the growth condition mentioned below:*

$$\|\mathfrak{G}_1\mathcal{Z}\| \leq \mathfrak{k}_1\left(\mu\|\mathcal{Z}\| + \nu\right).$$

Lemma 2.3. *$\mathfrak{G}_2$ is Lipschitz via constant $\mathfrak{k}_2 = \mathfrak{Z}E_{\varphi+\kappa,1}\left(M\left(\eth(\mathfrak{b}) - \eth(\mathfrak{a})\right)^{\varphi+\kappa}\right)$. Plus, $\mathfrak{G}_2$ fulfills the growth condition stated below*

$$\|\mathfrak{G}_2\mathcal{Z}(\mathcal{Q})\| \leq E_{\varphi+\kappa,1}\left(M\left(\eth(\mathfrak{b}) - \eth(\mathfrak{a})\right)^{\varphi+\kappa}\right)\left(\mu^*\|\mathcal{Z}\| + \nu^*\right).$$

Lemma 2.4. *The operator $\mathfrak{G}_1$ is compact, regarding $\mathfrak{G}_1 : C(\mathfrak{H}, \mathbb{R}) \to C(\mathfrak{H}, \mathbb{R})$. As a consequence, $mathfrakG_1$ is ell-Lipschitz through zero consistent.*

Theorem 2.5. *Presuming that conditions* (H1)-(H3) *are satisfied, it follows that the BVP* (1) *possesses at least one solution denoted by $\mathcal{Z}$ and belonging to the set of continuous functions from $\mathfrak{H}$ to $\mathbb{R}$, provided that the constant $\mathfrak{k}_2 < 1$. Moreover, the set of solutions is encompassed within the space $C\left(\mathfrak{H}, \mathbb{R}\right)$.*

Theorem 2.6. *In the context of the assumption labeled as* (H1) - (H2)*, the BVP cited as* (1) *has a unique solution given that the prescribed condition is satisfied:*

$$\mathfrak{Q} = \mathfrak{E}\mathfrak{k}_1 + \mathfrak{k}_2 < 1.$$

Proof. Assuming that both $\mathcal{Z}$ and $\mathcal{Z}^*$ belong to $C(\mathfrak{H}, \mathbb{R})$, and that $\mathcal{Q}$ is an element of $\mathfrak{H}$, we will encounter

$$|\mathcal{K}\mathcal{Z}(\mathcal{Q}) - \mathcal{K}\mathcal{Z}^*(\mathcal{Q})| \leq \sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}^{\varphi;\eth}_{\mathfrak{a}^+}A(\mathcal{Q})\mathfrak{I}^{\kappa;\eth}_{\mathfrak{a}^+}\right)^j\mathfrak{I}^{\varphi;\eth}_{\mathfrak{a}^+}\right\|\|\digamma\left(\mathcal{Q}, \mathcal{Z}(\mathcal{Q})\right) - \digamma\left(\mathcal{Q}, \mathcal{Z}^*(\mathcal{Q})\right)\|$$
$$\times \int_{\mathfrak{a}}^{\mathcal{Q}} \eth'(\mathfrak{y})\left(\eth(\mathcal{Q}) - \eth(\mathfrak{y})\right)^{\varphi-1}d\mathfrak{y}$$

$$
\begin{aligned}
&+\|f(\mathcal{Z})-f(\mathcal{Z}^*)\|\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{\mathfrak{a}^+}^{\varphi;\partial}A(\mathcal{Q})\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\partial}\right)^j\right\| \\
&+\frac{2\|f(\mathcal{Z})-f(\mathcal{Z}^*)\|}{\partial(\mathfrak{b})-\partial(\mathfrak{a})}\left|\partial(\mathcal{Q})-\partial(\mathfrak{a})\right|\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{\mathfrak{a}^+}^{\varphi;\partial}A(\mathcal{Q})\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\partial}\right)^j\right\| \\
&+\frac{2\left|\partial(\mathcal{Q})-\partial(\mathfrak{a})\right|}{(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^2\,\Gamma(\varphi+1)} \\
&\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{\mathfrak{a}^+}^{\varphi;\partial}A(\mathcal{Q})\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\partial}\right)^j\right\| \\
&\qquad\times\int_{\mathfrak{a}}^{\mathfrak{b}}\partial'(\mathfrak{y})\,(\partial(\mathfrak{b})-\partial(\mathfrak{y}))^{\varphi}\left|\digamma(\mathfrak{y},\mathcal{Z}(\mathfrak{y}))-\digamma(\mathfrak{y},\mathcal{Z}^*(\mathfrak{y}))\right|d\mathfrak{y} \\
&+\frac{2\left|\partial(\mathcal{Q})-\partial(\mathfrak{a})\right|}{(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^2\,\Gamma(\varphi+1)\Gamma(\kappa)}\sum_{j=0}^{\infty}\left\|\left(\mathfrak{I}_{\mathfrak{a}^+}^{\varphi;\partial}A(\mathcal{Q})\mathfrak{I}_{\mathfrak{a}^+}^{\kappa;\partial}\right)^j\right\| \\
&\qquad\times\int_{\mathfrak{a}}^{\mathfrak{b}}\partial'(\mathfrak{x}_1)\,(\partial(\mathfrak{b})-\partial(\mathfrak{x}_1))^{\varphi}\,A(\mathfrak{x}_1) \\
&\qquad\qquad\left(\int_{\mathfrak{a}}^{\mathfrak{x}_1}\partial'(\mathfrak{y})\,(\partial(\mathfrak{x}_1)-\partial(\mathfrak{y}))^{\kappa-1}\left|\mathcal{Z}(\mathfrak{y}))-\mathcal{Z}^*(\mathfrak{y})\right|d\mathfrak{y}\right)d\mathfrak{x}_1 \\
&\leq\|\digamma(\mathcal{Q},\mathcal{Z}(\mathcal{Q}))-\digamma(\mathcal{Q},\mathcal{Z}^*(\mathcal{Q}))\|\sum_{j=0}^{\infty}\frac{M^j\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{(\varphi+\kappa)j+\varphi}}{\Gamma\left((\varphi+\kappa)j+\varphi+1\right)} \\
&\quad+3\|f(\mathcal{Z})-f(\mathcal{Z}^*)\|\sum_{j=0}^{\infty}\frac{M^j\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&\quad+\frac{2\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi}}{\Gamma(\varphi+2)}\left|\digamma(\mathfrak{y},\mathcal{Z}(\mathfrak{y}))-\digamma(\mathfrak{y},\mathcal{Z}^*(\mathfrak{y}))\right| \\
&\qquad\times\sum_{j=0}^{\infty}\frac{M^j\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{(\varphi+\kappa)j}}{\Gamma\left((\varphi+\kappa)j+1\right)} \\
&\quad+\frac{2M\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\|\mathcal{Z}-\mathcal{Z}^*\|\sum_{j=0}^{\infty}\frac{M^j\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{(\varphi+\kappa)j+\varphi}}{\Gamma\left(\varphi+\kappa)j+\varphi+1\right)} \\
&\leq\mathfrak{E}\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi}\,E_{\varphi+\kappa,\varphi+1}\left(M\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi+\kappa}\right) \\
&\quad+\left(3\mathfrak{E}_1+\frac{2\mathfrak{E}\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi}}{\Gamma(\varphi+2)}+\frac{2M\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi+\kappa}}{\Gamma(\varphi+2)\Gamma(\kappa+1)}\right) \\
&\qquad\times E_{\varphi+\kappa,1}\left(M\,(\partial(\mathfrak{b})-\partial(\mathfrak{a}))^{\varphi+\kappa}\right)\Big]\,\|\mathcal{Z}-\mathcal{Z}^*\| \\
&=(\mathfrak{E}\mathfrak{k}_1+\mathfrak{k}_2)\,\|\mathcal{Z}-\mathcal{Z}^*\| \\
&=\mathfrak{Q}\|\mathcal{Z}-\mathcal{Z}^*\|
\end{aligned}
$$

In light of the aforementioned concession $\mathfrak{Q}<1$, it can be comprehended that the mapping denoted by $\mathcal{K}$ exhibits the property of contraction. Consequently, by virtue of the Banach FP theorem, $\mathcal{K}$ possesses a distinct FP which can be regarded as the unique solution to problem (1). □

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Generalization of Darbo's fixed point theorem via the measure of noncompactness for solving a system of implicit integral equations

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Abstract:

In this paper, we study the existence of solutions to a system of implicit integral functional equations involving a nonlinear operator in the space $BC(\mathbb{R}_+^3)$. By employing the concept of the measure of noncompactness together with a generalized version of Darbo's fixed point theorem adapted to the three-dimensional case, we establish sufficient conditions for boundedness, continuity, and weak contraction of the associated operators. Using these results, we prove that the considered system admits at least one solution in the space $BC(\mathbb{R}_+^3)$.

Keywords: Measure of noncompactness, implicit integral equations, Darbo's fixed point theorem, system of nonlinear functional integral equations.

1. Introduction

The measure of noncompactness, first introduced and studied by Kuratowski in 1930, serves as a powerful tool in Banach spaces. In 1955, Darbo employed this concept to formulate a fixed- point theorem, guaranteeing the existence of fixed points for specific categories of weakly contractive operators. Various generalizations of Darbo's theorem, such as those due to Schauder and Sadowski, have provided a robust analytical framework for investigating the existence of solutions to nonlinear and implicit systems of integral equations. The two-dimensional cases in the space $BC(\mathbb{R}_+^2)$ have been discussed in earlier works. The present study aims to generalize those results to three-dimensional systems and to establish sufficient conditions for the existence and uniqueness of solutions in $BC(\mathbb{R}_+^3)$.

2. Main results

We consider the system of three implicit integral equations of the form:

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$$\begin{cases} \varrho(\iota,\mathcal{J},t) = \begin{array}{l} a_1(\iota,\mathcal{J},t)+h_1\big(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\big)\int_0^{\iota} g_1(\iota,\mathcal{J},t,\upsilon)f_1\big(\upsilon,\mathcal{J},t,\varrho(\upsilon,\mathcal{J},t),\wp(\upsilon,\mathcal{J},t),w(\upsilon,\mathcal{J},t)\big)d\upsilon \\ +\, p_1(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\,)\int_0^{\mathcal{J}}\int_0^{\iota} q_1(\iota,\mathcal{J},t,\upsilon,\nu)r_1(\upsilon,\nu,t,\varrho(\upsilon,\nu,t),\wp(\upsilon,\nu,t),w(\upsilon,\nu,t))d\upsilon d\nu \end{array} \\ \wp(\iota,\mathcal{J},t) = \begin{array}{l} a_2(\iota,\mathcal{J},t)+h_2(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\,)\int_0^{\iota} g_2(\iota,\mathcal{J},t,\upsilon)f_2\big(\upsilon,\mathcal{J},t,\varrho(\upsilon,\mathcal{J},t),\wp(\upsilon,\mathcal{J},t),w(\upsilon,\mathcal{J},t)\big)d\upsilon \\ +\, p_2(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\,)\int_0^{\mathcal{J}}\int_0^{\iota} q_2(\iota,\mathcal{J},t,\upsilon,\nu)r_2(\upsilon,\nu,t,\varrho(\upsilon,\nu,t),\wp(\upsilon,\nu,t),w(\upsilon,\nu,t))d\upsilon d\nu \end{array} \\ w(\iota,\mathcal{J},t) = \begin{array}{l} a_3(\iota,\mathcal{J},t)+h_3(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\,)\int_0^{\iota} g_3(\iota,\mathcal{J},t,\upsilon)f_3\big(\upsilon,\mathcal{J},t,\varrho(\upsilon,\mathcal{J},t),\wp(\upsilon,\mathcal{J},t),w(\upsilon,\mathcal{J},t)\big)d\upsilon \\ +\, p_3(\iota,\mathcal{J},t,\varrho(\iota,\mathcal{J},t),\wp(\iota,\mathcal{J},t),w(\iota,\mathcal{J},t)\,)\int_0^{\mathcal{J}}\int_0^{\iota} q_3(\iota,\mathcal{J},t,\upsilon,\nu)r_3(\upsilon,\nu,t,\varrho(\upsilon,\nu,t),\wp(\upsilon,\nu,t),w(\upsilon,\nu,t))d\upsilon d\nu \end{array} \end{cases} \quad (1)$$

Here, the functions $\varrho, \wp, w: \mathbb{R}_+^3 \to \mathbb{R}$ are continuous and bounded. The coefficient functions $a_i, h_i, g_i, f_i, p_i, q_i, r_i$ $(i = 1,2,3)$ satisfy suitable continuity and boundedness conditions.

Definition 2.1.([3]) The function $\mathfrak{I}: \mathfrak{M}_{\mathbb{B}} \to [0,\infty)$ is referred to as an MNC in $\mathbb{B}$ if it meets the criteria listed below:
(1) The family $Ker\ \mathfrak{I} = \{K \in \mathfrak{M}_{\mathbb{B}}: \mathfrak{I}(X) = 0\} \neq \emptyset$ and $\ker\mathfrak{I} = \mathfrak{N}_{\mathbb{B}}$.
(2) $K_1 \subset K_1 \Longrightarrow \mathfrak{I}(K_1) \leq \mathfrak{I}(K_2)$
$\mathfrak{I}(\overline{K}) = \mathfrak{I}(Conv\ K) = \mathfrak{I}(K)$(3) ،
(4) $\mathfrak{I}(JK_1 + (1-J)K_2) \leq J\,\mathfrak{I}(K_1) + (1-J)\mathfrak{I}(K_2)$ for $J \in [0,1]$
(5) If $K_k \subset \mathfrak{M}_{\mathbb{B}}$ be a sequence such that $\overline{K_k} = K_k$ and $K_{k+1} \subset K_k$, for $k \in N$, where $\lim_{n\to\infty} \mathfrak{I}(K_k) = 0$,
then $K_\infty = \bigcap_{k=1}^{\infty} K_k \neq \emptyset$.
The subset $\ker\mathfrak{I}$ stated in A1 stands for the kernel of $\mathfrak{I}$, and since ,

$$\mathfrak{I}(K_\infty) = \mathfrak{I}(\textstyle\bigcap_{k=1}^{\infty} K_k) \leq \mathfrak{I}(K_k)$$

we see that $\mathfrak{I}(K_\infty) = 0$. Therefore, $K_\infty \in \ker\mathfrak{I}$.
Moreover, it is also worth mentioning that if, in addition to the above properties,
(6) $\ker\mathfrak{I} = \mathfrak{N}_{\mathbb{B}}$, then we say that $\mathfrak{I}$ is complete.
(7) The mapping $\mathfrak{I}$ is said to be homogeneous if $\mathfrak{I}(Jk) = |J|\mathfrak{I}(\mathrm{k})$ for every $J \in \mathbb{R}$.
(8) The mapping $\mathfrak{I}$ is said to be sub additive if $\mathfrak{I}(k_1 + k_2) \leq \mathfrak{I}(k_1) + \mathfrak{I}(k_2)$,
for every $k_1, k_2 \in \mathfrak{M}_{\mathbb{B}}$. A measure $\mathfrak{I}$ that satisfies both the homogeneity and subadditivity properties is called a sublinear measure of noncompactness.
(9) The mapping $\mathfrak{I}$ has the maximum property if
$\mathfrak{I}(k_1 \cup k_2) = \max\{\mathfrak{I}(k_1), \mathfrak{I}(k_2)\}$, for every $k_1, k_2 \in \mathfrak{M}_{\mathbb{B}}$.
(10) A measure of noncompactness $\mathfrak{I}$ satisfying $\ker\mathfrak{I} = \mathfrak{N}_{\mathbb{B}}$ and having the maximum property is called a regular measure of noncompactness.

Theorem 2.2. ([2]) Assume $\mathfrak{I}_1, \mathfrak{I}_2, \dots, \mathfrak{I}_\tau$ **are** measure in Banach spaces $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_\tau$ respectively. Moreover, suppose that the function $\mathfrak{P}: [0,\infty)^\tau \to [0,\infty)$ is convex, and $\mathfrak{P}(\boldsymbol{k_1}, \boldsymbol{k_2}, \dots, \boldsymbol{k_\tau}) = 0$ if and only if $\boldsymbol{k_i} = 0$ for $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$. Then

$$\tilde{\mathfrak{I}}(\boldsymbol{K}) = \mathfrak{P}(\mathfrak{I}_1(\boldsymbol{k_1}), \mathfrak{I}_2(\boldsymbol{k_2}), \dots, \mathfrak{I}_\tau(\boldsymbol{k_\tau}))$$

defines a measure of noncompactness in $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_\tau$ where $\boldsymbol{k_i}$ represents the natural projection of K onto $\mathbb{B}_i$.

Example 2.3. Assume $\mathfrak{I}$ be a measure of noncompactness on a Banach space $\mathbb{B}$, and let the function $\mathfrak{P}: [0,\infty)^3 \to [0,\infty)$ be convex such that, $\mathfrak{P}(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}) = 0$, if and only if $\boldsymbol{k_i} = 0$ for $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$. Then

$$\tilde{\mathfrak{I}}(K) = \mathfrak{P}\big(\mathfrak{I}_1(K), \mathfrak{I}_2(K), \mathfrak{I}_3(K)\big)$$

defines a measure of noncompactness in $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$ where $\boldsymbol{k_i}$ represents the natural projection of K onto $\mathcal{D}$.

Theorem 2.4. (Schauder [1]) Assume $\mathcal{D}$ be a closed and convex subset of Banach space $\mathbb{B}$. Then, every compact and continuous mapping $\mathcal{O}: \mathcal{D} \to \mathcal{D}$ has at least one fixed point.

Theorem 2.5. (Darbo [5]) let $\mathcal{O}: \mathcal{D} \to \mathcal{D}$ be a continuous operator. Suppose that $\mathfrak{I}\left(\mathcal{O}(\mathcal{Q})\right) \leq J\mathfrak{I}(\mathcal{Q})$ for each non-empty $\mathcal{Q} \subseteq \mathcal{D}$ and for some $\mathrm{J} \in [0,1)$. Then the set of fixed points $\mathcal{O}$ is non-empty.

Definition 2.6. Let $\mathbb{B}$ be a Banach space and $\mathcal{D} \subset \mathbb{B}$ be a nonempty subset.
Consider the mapping $\mathcal{O}: \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to \mathcal{D}$. An element $(z_1^*, z_2^*, z_3^*) \in \mathcal{D} \times \mathcal{D} \times \mathcal{D}$ is called a tripled fixed point of the mapping $\mathcal{O}$ if the following conditions are satisfied:

$$\begin{cases} \mathcal{O}(z_1^*, z_2^*, z_3^*) = z_1^* \\ \mathcal{O}(z_2^*, z_1^*, z_3^*) = z_2^* \\ \mathcal{O}(z_3^*, z_1^*, z_2^*) = z_3^* \end{cases}$$

Theorem 2.7. Assume $\mathcal{D}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{B}$, and let $\mathfrak{I}$ be an arbitrary measure of noncompactness. Moreover, assume that
$\mathcal{O}: \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to \mathcal{D} \times \mathcal{D} \times \mathcal{D}$ is a continuous operator satisfying;

$$\varphi\left(\tilde{\mathfrak{I}}\left(\mathcal{O}(K)\right)\right) \leq \varphi\left(\mathfrak{I}(K)\right) - \psi\left(\tilde{\mathfrak{I}}(K), \tilde{\mathfrak{I}}(K), \tilde{\mathfrak{I}}(K)\right)$$

$K \subset \mathcal{D} \times \mathcal{D} \times \mathcal{D}$, for any nonempty subset
where $\tilde{\mathfrak{I}}$ is defined by Example (2.3) and $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $\psi \in \Psi$. Then, $\mathcal{O}$ has at least one fixed point.

proof: We construct the sequence $\{\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n\}$ such that $\mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0 = \mathcal{D} \times \mathcal{D} \times \mathcal{D}$ and for $conv\mathcal{O}(\mathcal{D}_{n-1} \times \mathcal{D}_{n-1} \times \mathcal{D}_{n-1}) = \mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n$ .n = 1,2, ...
we have

$\mathcal{O}(\mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0) = \mathcal{O}(\mathcal{D} \times \mathcal{D} \times \mathcal{D}) \subset \mathcal{D} \times \mathcal{D} \times \mathcal{D} = \mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0$
$\mathcal{D}_1 \times \mathcal{D}_1 \times \mathcal{D}_1 = \mathrm{Conv}\mathcal{O}(\mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0) \subseteq \mathrm{Conv}(\mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0) = \mathcal{D}_0 \times \mathcal{D}_0 \times \mathcal{D}_0$
Therefor by continuing this process we obtain

$$\ldots \subseteq \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \subseteq \cdots \subseteq \mathcal{D}_2 \times \mathcal{D}_2 \times \mathcal{D}_2 \subseteq \mathcal{D}_1 \times \mathcal{D}_1 \times \mathcal{D}_1.$$

If there exists an integer $\mathrm{N} \geq 0$ such that $\tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N}) = 0$, then $\mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N}$ is relatively compact. Since

$$\mathcal{O}(\mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N}) \subseteq \mathrm{Conv}\mathcal{O}(\mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N} \times \mathcal{D}_\mathrm{N}) = \mathcal{D}_{N+1} \times \mathcal{D}_{N+1} \times \mathcal{D}_{N+1} \subseteq \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n},$$

Therefore, theorem (2.4) ensures that $\mathcal{O}$ has a fixed point. Now, assume that $\tilde{\mathfrak{I}}(\mathrm{D}_\mathrm{n} \times \mathrm{D}_\mathrm{n} \times \mathrm{D}_\mathrm{n}) > 0$ for all $\mathrm{n} \geq 0$.
According to our assumption,

$$\varphi\left(\tilde{\mathfrak{I}}(\mathcal{D}_{\mathrm{n}+1} \times \mathcal{D}_{\mathrm{n}+1} \times \mathcal{D}_{\mathrm{n}+1})\right) = \varphi\left(\tilde{\mathfrak{I}}\left(\mathrm{Conv}\mathcal{O}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})\right)\right)$$
$$= \varphi\left(\tilde{\mathfrak{I}}\left(\mathcal{O}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})\right)\right) \leq \varphi\left(\tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})\right)$$
$$-\psi\left(\tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n}), \tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n}), \tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})\right). \qquad (2)$$

Since $\tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})$ is decreasing and nonnegative real numbers, thus, there is an $\mathrm{r} \geq 0$ so that $\lim_{\mathrm{n}\to\infty} \tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n}) = \mathrm{r}$. We show that $\mathrm{r} = 0$. From (2) we obtain

$$\varphi(\mathrm{r}) = \lim_{\mathrm{n}\to\infty} \sup \varphi\left(\tilde{\mathfrak{I}}(\mathcal{D}_{\mathrm{n}+1} \times \mathcal{D}_{\mathrm{n}+1} \times \mathcal{D}_{\mathrm{n}+1})\right)$$
$$\leq \lim_{\mathrm{n}\to\infty} \sup \varphi\left(\tilde{\mathfrak{I}}(\mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n} \times \mathcal{D}_\mathrm{n})\right)$$

$$-\limsup_{n\to\infty} \psi\left(\tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n), \tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n), \tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n)\right)$$
$$\leq \limsup_{n\to\infty} \varphi\left(\tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n)\right)$$
$$-\psi\left(\limsup_{n\to\infty} \tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n), \limsup_{n\to\infty} \tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n), \limsup_{n\to\infty} \tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n)\right)$$
$$= \varphi(r) - \psi(r, r, r).$$

Thus, $\psi(r, r, r) = 0$ and hence $r = 0$. This implies that $\tilde{\mathfrak{I}}(\mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n) \to 0$ as $n \to \infty$.
Therefore, $\mathcal{D}_{n+1} \times \mathcal{D}_{n+1} \times \mathcal{D}_{n+1} \subset \mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n$, so by axiom $(MNC5)$
Of Definition (2.1) we derive that the set $\mathcal{D}_\infty \times \mathcal{D}_\infty \times \mathcal{D}_\infty = \bigcap_{n=1}^{\infty} \mathcal{D}_n \times \mathcal{D}_n \times \mathcal{D}_n$ is a nonempty, convex, closed set, invariant under the operator $\mathcal{O}$ and belongs to $ker\mathfrak{I}$. Now by Theorem (2.4) has at least one fixed point in $\mathcal{D}_\infty \times \mathcal{D}_\infty \times \mathcal{D}_\infty$ and hence in $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$.

3. Conclusion

Using the boundedness and continuity estimates, it has been shown that the operator $\mathcal{O}$ is a contraction. Therefore, by applying Banach's or Schauder's fixed point theorem, it follows that the considered three-dimensional system in the Banach space admits at least one solution.
Furthermore, the obtained results can be applied to solve a class of implicit integral equations in Banach spaces.
In future research, one can extend this study to investigate the solvability of an infinite system of generalized fractional integral equations in sequence spaces.

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Majorization and Berezin symbol for bounded linear operators in Hilbert space

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Abstract. In this paper, we present several inequalities for majorization and strong majorization of a bounded linear operator on a reproducing kernel Hilbert space. As applications of Berezin norm, we deduce some new results in this field. Further, several applications of the newly obtained inequalities are also provided.

Keywords: majorization, strong majorization, positive operators, reproducing kernel, Berezin norm

AMS Mathematics Subject Classification [2020]: 47A12, 47A30, 47L30

1. Introduction

We denote $B(H)$ as the C^*-algebra of all bounded linear operators on a non trivial complex Hilbert space H with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. An operator $T \in B(H)$ is called positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$, and we then write $T \geq 0$. For $T \in B(H)$, we use the following notation: $|T|$ denotes the absolute value of T, that is $|T| = (T^*T)^{\frac{1}{2}}$, where T^*stands for the adjoint of T, $ran(T)$ the range of T and $ker(T)$ the null space of T.

Let Ω be a subset of a topological space X such that the boundary $\partial\Omega$is nonempty. Recall that a reproducing kernel Hilbert space is the Hilbert space $H = H(\Omega)$ of complex valued functions on set Ω such that:

(1) For any $x \in \Omega$, the evaluation functions $f \to f(x)$ are continuous on H.
(2) For any $x \in \Omega$, there exists a function $f_x \in H$ such that $f_x(x) \neq 0$.

Then, according to the Riesz representation theorem, the assumption implies that, for each $x \in \Omega$, there exists a unique function $K_x \in H$ such that

$$f(x) = \langle f, K_x\rangle, \quad f \in H.$$

The function $f_x(y)$ is called the reproducing kernel of H at point x and the family $\{K_x : x \in \Omega\}$ is called the reproducing kernels of the space H. The normalized reproducing kernel k_x is defined by $k_x := \frac{K_x}{\|K_x\|}$, since virtue of assumption (2), we surely have $k_x \neq 0$. It is well known that every reproducing kernel Hilbert space is separable. So, if $\{e_n\}_{n\geq 0}$

is an orthonormal basis for a functional Hilbert space $H(\Omega)$, then

$$k_x(y) = \sum_{n=0}^{\infty} \overline{e_n(x)} e_n(y).$$

For more details and references on reproducing kernel Hilbert spaces, see [3, 4]. For a bounded linear operator $T \in B(H)$, the Berezin transform (symbol) of T, which firstly has been introduced by Berezin [1, 2], denoted by $\mathfrak{T}$, is the complex-valued function on Ω defined by

$$\mathfrak{T}(x) := \langle Tk_x, k_x \rangle.$$

The Berezin set, Berezin number and the Berezin norm of the bounded linear operator $\mathfrak{T}$, denoted by $Ber(\mathfrak{T})$, $ber(\mathfrak{T})$ and $\|\mathfrak{T}\|_{ber}$, respectively, are defined as

$$Ber(\mathfrak{T}) := \{\langle \mathfrak{T}k_\lambda, k_\lambda \rangle : \quad \lambda \in \Omega\}, \;\; ber(\mathfrak{T}) := \sup\{|\langle \mathfrak{T}k_\lambda, k_\lambda \rangle| : \quad \lambda \in \Omega\}$$

and

$$\|\mathfrak{T}\|_{ber} := \sup\{|\langle \mathfrak{T}k_\lambda, k_\mu \rangle| : \; \lambda, \mu \in \Omega\},$$

where k_λ, k_μ are the normalized reproducing kernels for λ, μ, respectively.

For $\mathfrak{T}, \mathfrak{S} \in B(H)$, it is clear from the definition, Berezin norm satisfy the following results:

$$\begin{aligned}
&(1) ber(\mathfrak{T}) \leq \|\mathfrak{T}\|_{ber}, \quad (4) \|\lambda\mathfrak{T}\|_{ber} = |\lambda| \|\mathfrak{T}\|_{ber}, \quad \text{for all } \lambda \in \mathbb{C},\\
&(2) \|\mathfrak{T}\|_{ber} \leq \|\mathfrak{T}\|, \qquad (5) \|\mathfrak{T} + \mathfrak{S}\|_{ber} \leq \|\mathfrak{T}\|_{ber} + \|\mathfrak{S}\|_{ber},\\
&(3) \|\mathfrak{T}^*\|_{ber} = \|\mathfrak{T}\|_{ber}.
\end{aligned}$$

The Berezin symbol has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. Furthermore, Berezin set and Berezin number of operators are new numerical characteristics of operators on the reproducing kernel Hilbert space which are introduced by Karaev in [5]. For the basic properties and facts on these new concepts, see [6].

2. Majorization and strong Majorization

In this section, we derive some consequences of the property "$\mathfrak{T}$ majorizes $\mathfrak{S}$".

Definition 2.1. Assume that $\mathfrak{T}, \mathfrak{S} \in B(H)$. Then $\mathfrak{T}$ majorizes $\mathfrak{S}$ and denoted by $\mathfrak{T} \prec_m \mathfrak{S}$, if the exists $c > 0$ such that

$$\|\mathfrak{S}\|_{ber} \leq c\|\mathfrak{T}\|_{ber}.$$

Theorem 2.2. *Let $\mathfrak{T}, \mathfrak{S} \in B(H)$ and let $\mathfrak{T} \prec_m \mathfrak{S}$. Whenever $k_{\lambda_n} \subseteq H$ with $\|\mathfrak{T}k_{\lambda_n}\| \to 0$, then $\|\mathfrak{S}k_{\lambda_n}\| \to 0$.*

Proof. Let $\mathfrak{T} \prec_m \mathfrak{S}$, that is $\mathfrak{T}$ majorizes $\mathfrak{S}$. So, the exists $c > 0$ such that $\|\mathfrak{S}\|_{ber} \leq c\|\mathfrak{T}\|_{ber}$. Then,

$$\sup\{|\langle \mathfrak{S}k_\lambda, k_\mu \rangle| : \; \lambda, \mu \in \Omega\} \leq \sup_{\lambda \in \Omega} \|\mathfrak{S}k_\lambda\| \leq \sup_{\lambda \in \Omega} \|\mathfrak{T}k_\lambda\|. \tag{1}$$

Now, if the exists $k_{\lambda_n} \subseteq H$ such that $\|\mathfrak{T}k_{\lambda_n}\| \to 0$, then by equation (1) we get $\|\mathfrak{S}k_{\lambda_n}\| \to 0$. □

If $\mathfrak{T} \in B(H)$, then

$$|\langle \mathfrak{T}k_\lambda, k_\mu \rangle|^2 \leq \langle |\mathfrak{T}|^{2r} k_\lambda, k_\lambda \rangle \langle |\mathfrak{T}^*|^{2(1-r)} k_\mu, k_\mu \rangle, \tag{2}$$

where k_λ, k_μ are the normalized reproducing kernels in $H(\Omega)$ and $r \in [0, 1]$; see [6].

LEMMA 2.3. *If $\mathfrak{T} \in B(H)$, then $ber(\mathfrak{T}) = \|\mathfrak{T}\|_{ber}$.*

PROOF. Since $\mathfrak{T}$ is positive, so from equation (2) with $r = \frac{1}{2}$ we have

$$|\langle \mathfrak{T}k_\lambda, k_\mu\rangle|^2 \leq \langle \mathfrak{T}k_\lambda, k_\lambda\rangle\langle \mathfrak{T}k_\mu, k_\mu\rangle.$$

where k_λ, k_μ are the normalized reproducing kernels in $H(\Omega)$. This implies $ber(\mathfrak{T}) \geq \|\mathfrak{T}\|_{ber}$. Also, $ber(\mathfrak{T}) \leq \|\mathfrak{T}\|_{ber}$ is trivial. □

DEFINITION 2.4. Let $\mathfrak{T}, \mathfrak{S} \in B(H)$. we say that $\mathfrak{T}$ strong majorizes $\mathfrak{S}$ and denoted by $\mathfrak{T} \prec_s \mathfrak{S}$ if there exists $M > 0$ such that for all $k_\lambda \in H$,

$$|\langle \mathfrak{T}k_\lambda, k_\lambda\rangle| \leq M|\langle \mathfrak{T}k_\lambda, k_\lambda\rangle|. \tag{3}$$

LEMMA 2.5. *Let $\mathfrak{T}, \mathfrak{S} \in B(H)$ and $\mathfrak{S}$ is positive. If $\mathfrak{T} \prec_s \mathfrak{S}$, then $\mathfrak{T} \prec_m \mathfrak{S}$.*

PROOF. Let $\mathfrak{S}$ is positive. So by Lemma 2.3, we have $ber(\mathfrak{T}) = \|\mathfrak{T}\|_{ber}$. Now, if $\mathfrak{T} \prec_s \mathfrak{S}$ then, there exists $M > 0$ such that $|\langle \mathfrak{T}k_\lambda, k_\lambda\rangle| \leq M|\langle \mathfrak{T}k_\lambda, k_\lambda\rangle|$. So, $\sup_{\lambda\in\Omega} |\langle \mathfrak{T}k_\lambda, k_\lambda\rangle| \leq M \sup_{\lambda\in\Omega} |\langle \mathfrak{T}k_\lambda, k_\lambda\rangle|$. Therefore $ber(\mathfrak{S}) \leq M ber(\mathfrak{T}) \leq M\|\mathfrak{T}\|_{ber}$. Since $\mathfrak{S}$ is positive operator, then

$$\|\mathfrak{S}\|_{ber} \leq M\|\mathfrak{T}\|_{ber}.$$

□

EXAMPLE 2.6. Let $\{e_n : n \in \mathbb{Z}_+\}$ be a standard orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Consider $\ell^2(\mathbb{Z}_+)$ as a reproducing kernel Hilbert space on the set $\mathbb{Z}_+$. With the identification a sequence as a function, the kernel function e_n for $n \in \mathbb{Z}_+$ defined as

$$e_n(m) = \begin{cases} 1, & \text{if} \quad n = m; \\ 0, & \text{if} \quad n \neq m. \end{cases}$$

for $n, m \in \mathbb{Z}_+$. Suppose that $\mathfrak{S}, \mathfrak{T} \in B(\ell^2)$ are defined by
$\mathfrak{S}x = (0, x_1, x_2, x_3, \dots)$ and $\mathfrak{T}x = (x_1, x_2, x_3, \dots)$, for $x = (x_n) \in \ell^2$. Hence for $e = (e_n) \in \ell^2$, we have $\langle \mathfrak{S}e, e\rangle = 0$ and $\langle \mathfrak{T}e, e\rangle = |e_1|^2 + |e_2|^2 + |e_3|^2 + \dots$. Clearly $|\langle \mathfrak{S}e, e\rangle| \leq |\langle \mathfrak{T}e, e\rangle|$, and so $\mathfrak{T} \prec_s \mathfrak{S}$.

Let $H_k(\mathbb{D})$ be a reproducing kernel Hilbert space with kernel function k defined as $k_\lambda(z) = k(z, \lambda)$ for all $z \in \mathbb{D}$. We say that $H_k(\mathbb{D})$ is an analytic Hilbert space if the multiplication operator M_z by the coordinate function z on $H_k(\mathbb{D})$, defined by $M_z f = zf$ for all $f \in H_k(\mathbb{D})$ is a contraction. Also

$$M_z^* k_\lambda = \overline{\lambda} k_\lambda, \quad \lambda \in \mathbb{D}.$$

THEOREM 2.7. *Suppose that the $H_k(\mathbb{D})$ and M_z be defined as above. Then $\|M_z\|_{ber} \leq 1$.*

PROOF. Let k_λ for $\lambda \in \mathbb{D}$ be the normalized kernel function. Then

$$\begin{aligned}
\|M_z\|_{ber} &= \sup\{|\langle M_z k_\lambda, k_\mu\rangle| : \quad \lambda, \mu \in \mathbb{D}\} \\
&\leq \sup\{|\langle M_z k_\lambda, k_\lambda\rangle|^{\frac{1}{2}} |\langle M_z k_\mu, k_\mu\rangle|^{\frac{1}{2}} : \quad \lambda, \mu \in \mathbb{D}\} \\
&= \sup\{|\langle k_\lambda, M_z^* k_\lambda\rangle|^{\frac{1}{2}} |\langle k_\mu, M_z^* k_\mu\rangle|^{\frac{1}{2}} : \quad \lambda, \mu \in \mathbb{D}\} \\
&= \sup\{|\langle k_\lambda, \overline{\lambda} k_\lambda\rangle|^{\frac{1}{2}} |\langle k_\mu, \overline{\mu} k_\mu\rangle|^{\frac{1}{2}} : \quad \lambda, \mu \in \mathbb{D}\} \\
&= \sup\{|\lambda|^{\frac{1}{2}} |\mu|^{\frac{1}{2}} : \quad \lambda, \mu \in \mathbb{D}\} = 1.
\end{aligned}$$

□

THEOREM 2.8. *Let $\mathfrak{T}, \mathfrak{S}_1, \mathfrak{S}_2 \in B(H)$. If $\mathfrak{S}_1 \prec_m \mathfrak{T}$, $\mathfrak{S}_2 \prec_m \mathfrak{T}$ and $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ then $\alpha_1\mathfrak{S}_1 + \alpha_2\mathfrak{S}_2 \prec_m \mathfrak{T}$.*

Proof. Since $\mathfrak{S}_1 \prec_m \mathfrak{T}$ and $\mathfrak{S}_2 \prec_m \mathfrak{T}$, then there exists constants $M_1 > 0$, $M_2 > 0$ such that $\|\mathfrak{S}_1\|_{ber} \leq M_1\|\mathfrak{T}\|_{ber}$ and $\|\mathfrak{S}_2\|_{ber} \leq M_2\|\mathfrak{T}\|_{ber}$, respectively. Therefore, we have

$$\begin{aligned}
\|\alpha_1\mathfrak{S}_1 + \alpha_2\mathfrak{S}_2\|_{ber} &= \sup\{|\langle(\alpha_1\mathfrak{S}_1 + \alpha_2\mathfrak{S}_2)k_\lambda, k_\mu\rangle| : \quad \lambda, \mu \in \mathbb{D}\} \\
&\leq |\alpha_1| \sup\{|\langle\mathfrak{S}_1 k_\lambda, k_\mu\rangle| : \quad \lambda, \mu \in \mathbb{D}\} + |\alpha_2| \sup\{|\langle\mathfrak{S}_2 k_\lambda, k_\mu\rangle| : \quad \lambda, \mu \in \mathbb{D}\} \\
&\leq (|\alpha_1|M_1 + |\alpha_2|M_2) \sup\{|\langle\mathfrak{T} k_\lambda, k_\mu\rangle| : \quad \lambda, \mu \in \mathbb{D}\} \\
&= M\|\mathfrak{T}\|_{ber},
\end{aligned}$$

where $M := |\alpha_1|M_1 + |\alpha_2|M_2$. □

Theorem 2.9. *Let $H(\Omega_i)$ be a reproducing kernel Hilbert space on Ω_i and $\mathfrak{T}_i, \mathfrak{S}_i \in B(H(\Omega_i))$ for $i = 1, 2$. If $\mathfrak{S}_1 \prec_m \mathfrak{T}_1$ and $\mathfrak{S}_2 \prec_m \mathfrak{T}_2$ then $\mathfrak{S}_1 \otimes \mathfrak{S}_2 \prec_m \mathfrak{T}_1 \otimes \mathfrak{T}_2$.*

Proof. Suppose $H(\Omega_i)$ be a reproducing kernel Hilbert space on Ω_i and $\mathfrak{T}_i, \mathfrak{S}_i \in B(H(\Omega_i))$ for $i = 1, 2$ where $\mathfrak{S}_1 \prec_m \mathfrak{T}_1$ and $\mathfrak{S}_2 \prec_m \mathfrak{T}_2$. Then $\mathfrak{S}_1 \otimes \mathfrak{S}_2, \mathfrak{T}_1 \otimes \mathfrak{T}_2 \in B(H(\Omega_1) \otimes H(\Omega_2))$. Let k_{λ_1}, k_{μ_1} for $\lambda_1, \mu_1 \in \Omega_1$ and k_{λ_2}, k_{μ_2} for $\lambda_2, \mu_2 \in \Omega_2$ be the ormalized kernel functions for $H(\Omega_1)$ and $H(\Omega_2)$, respectively. The $k_{\lambda_1} \otimes k_{\lambda_2}$ and $k_{\mu_1} \otimes k_{\mu_2}$ are the normalized kernel functions for $H(\Omega_1) \otimes H(\Omega_2)$. Therefore

$$\begin{aligned}
\|\mathfrak{S}_1 \otimes \mathfrak{S}_2\|_{ber} &= \sup\{|\langle(\mathfrak{S}_1 \otimes \mathfrak{S}_2)(k_{\lambda_1} \otimes k_{\lambda_2}), k_{\mu_1} \otimes k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \sup\{|\langle\mathfrak{S}_1 k_{\lambda_1} \otimes \mathfrak{S}_2 k_{\lambda_2}, k_{\mu_1} \otimes k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \sup\{|\langle\mathfrak{S}_1 k_{\lambda_1}, k_{\mu_1}\rangle||\langle\mathfrak{S}_2 k_{\lambda_2}, k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&\leq \sup\{|\langle\mathfrak{T}_1 k_{\lambda_1}, k_{\mu_1}\rangle||\langle\mathfrak{T}_2 k_{\lambda_2}, k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \sup\{|\langle\mathfrak{T}_1 k_{\lambda_1}, k_{\mu_1}\rangle||\langle\mathfrak{T}_2 k_{\lambda_2}, k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \sup\{|\langle\mathfrak{T}_1 k_{\lambda_1} \otimes \mathfrak{T}_2 k_{\lambda_2}, k_{\mu_1} \otimes k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \sup\{|\langle(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(k_{\lambda_1} \otimes k_{\lambda_2}), k_{\mu_1} \otimes k_{\mu_2}\rangle| : \ \lambda_1, \mu_1 \in \Omega_1, \ \lambda_2, \mu_2 \in \Omega_2\} \\
&= \|\mathfrak{T}_1 \otimes \mathfrak{T}_2\|_{ber}.
\end{aligned}$$

Therefore $\mathfrak{S}_1 \otimes \mathfrak{S}_2 \prec_m \mathfrak{T}_1 \otimes \mathfrak{T}_2$. □

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Analysis via A-valued reproducing kernel Hilbert spaces and applications

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Abstract. This paper is devoted to the study of A-valued reproducing kernel Hilbert spaces. We focus on two aspects: A-valued kernel functions and the relative reproducing kernel. In particular, we characterize the structure operator theory of finite graphs with special emphasis on a natural Laplace operator associated with the graph and we relate it to the graph Laplacian.

Keywords: A-valued functions, finite graph, relative reproducing kernel, graph Laplacian

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1. Introduction

Reproducing kernel Hilbert spaces and vector-valued reproducing kernel Hilbert spaces have been by now established as among the most powerful paradigms in machine learning and statistics, with numerous practical applications. While most of the literature on kernel methods so far has focused on scalar-valued functions, Reproducing kernel Hilbert spaces of A-valued functions have received increasing research attention in machine learning recently, from both theoretical and practical perspectives [2, 4]. We recall the definitions of reproducing kernel Hilbert space and A-valued reproducing kernel Hilbert space for vector valued functions.

Definition 1.1. [3] Let $X \subseteq R^d$ and H be a Hilbert space consisting of function $f : X \to \mathbb{C}$. H is called a reproducing kernel Hilbert space and a kernel function $K : X \times X \to \mathbb{C}$ is called a reproducing kernel for H if

(i) $K(., y) \in H$, for all $y \in X$,
(ii) $f(y) = \langle f, K(., y)\rangle_H$, for all $f \in H$ and $y \in X$,

where $\langle ., .\rangle_H$ is used to denote the inner product of H.

*Speaker.

Definition 1.2. Consider the Hilbert space H of functions on the set $X \times A$. We say that it is a relative A-valued reproducing kernel Hilbert space if for every pair (x, f) and (y, g) in $X \times A$, there is a unique $M_{x,y,f,g} \in H$ such that

$$u(x, f) - u(y, g) = \langle u, M_{x,y,f,g} \rangle_H. \tag{1}$$

2. A-valued reproducing kernel Hilbert spaces

In this chapter,we define reproducing kernel Hilbert spaces and A-valued reproducing kernel Hilbert spaces, and we show the correspondence between the two. In [1], Alpay et al. define a conception as relative reproducing kernel. Here, we introduce the relative reproducing kernel in Hilbert space H of functions from the set X to linear space A. As we did earlier, we set H Hilbert space of functions on the set $X \times A$.

Let $G = (V, S)$ be a connected graph with a nonempty finite set V, which is called the vertex set, and a subset $S = \{(a, b) : a, b \in V and\ a \neq b\}$ which is called the edge set. We put $V = X \times A$ of finite dimension. It will be convenient to choose a fixed base-point $o = (x_0, f_0) \in V$. In this case, set $M_{x,f} := M_{x,x_0,f,f_0}$, so (1) takes the form

$$u(x, f) - u(x_0, f_0) = \langle u, M_{x,f} \rangle_H. \tag{2}$$

for all $u \in H$. The corresponding positive definite kernel is as follows:

$$K(x, y, f, g) = \langle M_{y,g}, M_{x,f} \rangle_H.$$

The functions $M_{x,f}$ are called dipoles.
Let C be the adjacency matrix of G and E be a linear space of finite dimension.The set of all maps from G in to A denoted by F. Define

$$\Gamma(u, v) = \frac{1}{2} \sum_{(x,f),(y,g) \in V} C_{x,y}(\overline{u(x, f)} - \overline{u(y, g)})(v(x, f) - v(y, g)),\ \forall u, v \in F. \tag{3}$$

Since $C_{x,y} = C_{y,x}$ and $C_{x,y} = 0$ for nonadjacent vertices, the initial factor $\frac{1}{2}$ implies that there is exactly one term in the sum for each edge in the equation (3). We summarize some critical details on the bilinear form Γ as a lemma.

Lemma 2.1. *For every $u, v \in F$*
(1) $\mid \Gamma(u, v) \mid^2 \leq \Gamma(u, u)\Gamma(v, v)$.
(2) $\Gamma(u, u) = 0$ *if and only if u is constant function on V.*

Proof. The first is Cauchy-Schwarz inequality and obtain from (3). For the next, let $(x_0, f_0) \in V$ fixed. Since G is connected, for any $(x, f) \in G$ we can find a path from (x_0, f_0) to (x, f) such $(x_0, f_0), (x_1, f_1), \cdots, (x_n, f_n) \in A$ that all (x_i, f_i) are distinct. Assume that $\Gamma(u, u) = 0$, we conclude that $u(x_i, f_i) = u(x_{i+1}, f_{i+1})$ which implies $u(x, f) = u(x_0, f_0)$ for every $(x, f) \in G$, so u is constant. □

Definition 2.2. Let G be a connected graph with adjacency matrix C. We denote by H_G the Hilbert space $(F, \langle \cdot, \cdot \rangle_H)$, where

$$H_G = \{u : V \longrightarrow \mathbb{C} \quad | \quad \|u\|^2_{H_G} = \mid \sum_{(x,f) \in V} u(x, f) \mid^2 + \Gamma(u, u) < \infty\}.$$

Therefore, we define another bilinear form on F as follows:

$$\langle u, v \rangle_{H_G} := (\sum_{(x,f) \in V} \overline{u(x, f)})(\sum_{(x,f) \in V} v(x, f)) + \Gamma(u, v),$$

where $\langle\cdot,\cdot\rangle_{H_G}$ is an inner product in F. Let $u \in F$, we set $M = \sum_{(x,f)\in V} u(x,f)$. If $\langle u,u\rangle_{H_G} = 0$ then $|M|^2 = 0$ and $\Gamma(u,u) = 0$. Then by lemma (2.1), u is constant.

Remark 2.3. Let $(x,f),(y,g) \in V$, since G is connected, as proof of lemma(2.1), we have

$$\begin{aligned}
\mid u(x,f) - u(y,g) \mid &= \mid \sum_{i=0}^{n-1} u(x_{i+1},f_{i+1}) - u(x_i,f_i) \mid \\
&= \mid \sum_{i=0}^{n-1} \frac{1}{\sqrt{C_{x_i,x_{i+1}}}} \sqrt{C_{x_i,x_{i+1}}}(u(x_{i+1},f_{i+1}) - u(x_i,f_i)) \mid \\
&\leq (\sum_{i=0}^{n-1} \frac{1}{C_{x_i,x_{i+1}}})^{\frac{1}{2}} (\sum_{i=0}^{n-1} C_{x_i,x_{i+1}} \mid u(x_{i+1},f_{i+1}) - u(x_i,f_i) \mid^2)^{\frac{1}{2}} \\
&\leq M\Gamma(u,u)^{\frac{1}{2}} \leq M\|u\|_{H_G}.
\end{aligned}$$

Remark 2.3 asserts continuity of the linear function $L_{x,y,f,g} : H_G \to \mathbb{C}$ defined by $L_{x,y,f,g} = u(x,f) - u(y,g)$, so Riesz's lemma gives a $M_{x,y,f,g} \in H_G$ satisfying (1), for every $(x,f),(y,g) \in V$. Henceforth, it will be convenient to fix a reference point $o = (x_0,f_0) \in V$ to act as an origin and consider the linear evaluation operator $L_{x,f}$ on H_G by $L_{x,f}u = u(x,f) - u(o)$, then by proceed similarly to the proof of Remark 2.3, we conclude $|L_{x,f}u| \leq k\|u\|_{H_G}$, where k depends only on (x,f). So Riesz's lemma gives $M_{x,f} \in H_G$ satisfying (2), such that M_{x_0,f_0} corresponds to a constant function, since $\langle M_{x_0,f_0}, u\rangle_{H_G} = 0$ for every $u \in H_G$.

Corollary 2.4. *Let H_G be as in definition (2.2) and $o = (x_0,f_0) \in V$ fixed, then there exists a subset $\{M_{x,f} \in H_G : (x,f) \in V \setminus o\}$ such that $M_{x,f}(x_0,f_0) = 0$ for all $(x,f) \in V \setminus o$ and*

$$\begin{aligned}
u(x,f) - u(x_0,f_0) = &(\sum_{(z,h)\in V} \overline{u(z,h)})(\sum_{(y,g)\in V} M_{x,f}(y,g)) \\
&+ \frac{1}{2}\sum_{(z,h)}\sum_{(y,g)\in V} C_{y,z}(\overline{u(y,g)} - \overline{u(z,h)})(M_{x,f}(y,g) - M_{x,f}(z,h)).
\end{aligned}$$

Now, we study the operator theory of finite graphs G, with special emphasis on a natural Laplace operator associated with the graph in question. This operator will depend not only on G but also on a positive real valued function $C_{y,x}$ defined on the edges in G. In electrical network models, the function $C_{y,x}$ will determine a conductance number for each edge e. If $e = (x,y)$ connects vertices x and y in G, the number $C_{y,x}$ is the reciprocal of the resistance between x and y. When the conductance function C is fixed, then the corresponding graph Laplacian will be realized as operator in H_G to get effective boundary representations for functions defined on the graph itself. Hence the graph Laplacian serves as a key tool in these considerations.

Definition 2.5. Define the G-Laplace operator Δ as

$$(\Delta u)(x,f) = \sum_{(z,h)\in V} (u(z,h)) + (\sum_{(y,g)\in V} C_{y,x}(u(x,f) - u(y,g))),$$

and

$$\delta_{x,f}(y,g) = \begin{cases} 1 & x = y \quad and \quad f = g, \\ 0 & x \neq y \quad or \quad f \neq g. \end{cases}$$

It can be said that H_G has a discrete mass property, if $\delta_{x,f} \in H_G$ for all $(x, f) \in V$.

THEOREM 2.6. *Let G be a connected graph and $o = (x_0, f_0) \in V$ fixed. Also, let Δ and $\delta_{x,f}$ be as definition 2.5 and let $\{M_{x,f} : (x, f) \in V \setminus o\}$ be the system of dipoles from corollary 2.4. Then the following two rules hold:*

$$\langle \delta_{x,f}, u \rangle_{H_G} = (\Delta u)(x, f), \quad \text{and} \quad \Delta M_{x,f} = \delta_{x,f} - \delta_{x_0,f_0}.$$

PROOF. We compute the inner product as follows:

$$\begin{aligned}
\langle \delta_{x,f}, u \rangle_{H_G} &= (\sum_{(y,g)\in V} \delta_{x,f}(y, g))(\sum_{(z,h)\in V} u(z, h)) \\
&+ \frac{1}{2} \sum_{(z,h)\in V} \sum_{(y,g)\in V} C_{y,z}(\overline{\delta_{x,f}(y, g)} - \overline{\delta_{x,f}(z, h)})(u(y, g) - u(z, h)) \\
&= (\sum_{(z,h)\in V} u(z, h)) + \frac{1}{2}(\sum_{(z,h)\in V} C_{x,z}(u(x, f) - u(z, h)) \\
&- \sum_{(y,g)\in V} C_{y,x}(u(y, g) - u(x, f))) \\
&= (\sum_{(z,h)\in V} u(z, h)) + (\sum_{(y,g)\in V} C_{y,x}(u(x, f) - u(y, g))) \\
&= (\Delta u)(x, f).
\end{aligned}$$

Finally, we show

$$\langle \Delta M_{x,f}, M_{y,g} \rangle = \langle \delta_{x,f} - \delta_{x_0,f_0}, M_{y,g} \rangle.$$

For this purpose, we have

$$\begin{aligned}
\Delta M_{x,f}(y, g) &= \langle \delta_{y,g}, M_{x,f} \rangle_{H_G} \\
&= \delta_{y,g}(x, f) - \delta_{y,g}(x_0, f_0) \\
&= \delta_{x,f}(y, g) - \delta_{x_0,y_0}(y, g) \\
&= (\delta_{x,f} - \delta_{x_0,f_0})(y, g).
\end{aligned}$$

□

COROLLARY 2.7. *Let $\{M_{x,f} \mid (x, f) \in V \setminus o\}$, G and Δ be above, then*

$$\langle M_{x,f}, \Delta M_{y,g} \rangle_{H_G} = \langle \Delta M_{x,f}, M_{y,g} \rangle_{H_G}.$$

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Fixed point Results for Φ-set contractions in Menger spaces

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Abstract. The concept of measure of noncompactness allows us to select an important class of mapping which are more general than compact operators. In this paper, we introduce the axiomatic definition of measure of noncompactness on a bounded subset of the Menger spaces and establish some new fixed point theorems for set contraction maps.

Keywords: KKM property, Set-valued mappings, Menger space, Measure of noncompactness, Darbo fixed point.

AMS Mathematics Subject Classification [2020]: 54H10, 54E20

1. Introduction

Chen and Cahng obtained some fixed point theorems for KKM type set contraction mappings in various spaces [2–4] . Amini et al. [1] introduced generalized set contraction on topological spaces. In this paper, we first introduce a measure of noncompactness in Menger spaces and establish some fixed point theorems for generalized set contraction maps and KKM type ones in Menger spaces. We now state some allied definitions and results which are needed for the development of the present topic. Let X be a nonempty set. We denote by $P(X)$ the family of all subsets of X and by $\langle X\rangle$ the family of all nonempty finite subsets of X. Let $(M,\mathcal{F},\mathcal{T})$ be a Menger space and $X\subset M$. We say X is precompact set if for each $0<r<1$ and $t>0$ there exists a finite subset A of X such that $M\subseteq\bigcup_{x\in A}B_x(r,t)$, and X is said to be probabilistically bounded($p-$bounded) if $\sup_{t>0}\inf_{x,y\in X}\mathcal{F}_{x,y}(t)=1$. Let $P(X)$ denote the class of all nonempty subsets of X. We use the notion:

(1) $P_{cl}(X)=\{Y\in P(X):\ Y\ is\ closed\}$;
(2) $P_{cl,bd}(X)=P_{cl}(X)\cap P_{bd}(X)$.

Let A be a nonempty p-bounded subset of Menger space $(M,\mathcal{F},\mathcal{T})$. Then

(1) $co(A)=\cap\{B\subset M,\ B$ is a closed ball in M such that $A\subset B\}$;
(2) A is called subadmissible, if for each $D\in\langle A\rangle$, $co(D)\subset A$.

Tatari et al. [6], introduced the family $KKM(X,Y)$ on a subadmissible subset of Menger space $(M,\mathcal{F},\mathcal{T})$ and applied it for getting some fixed point existence results for set-valued

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mappings. Following the concepts for the $KKM(X,Y)$ family (see [6]), we have the following lemma.

LEMMA 1.1. *Let $(M,\mathcal{F},\mathcal{T})$ be a Menger space and X be a nonempty subadmissible subset of M. Suppose that Y is a topological space, $S \in KKM(X,Y)$ and A is a nonempty subset of X, then $S|_A \in KKM(A,Y)$.*

2. Main results

This section is devoted to prove some fixed point theorems for generalize set contraction maps with the KKM property on Menger spaces.

We define the axiomatic definition for the measure of noncompactness on a p-bounded subset of the Menger space $(M,\mathcal{F},\mathcal{T})$.

DEFINITION 2.1. A function $\alpha : P_{bd}(M) \to \Re^+$ is called a measure of noncompactness if it satisfies

(1) $\alpha_A(t) = 1$ if and only if A is precompact;
(2) $\alpha_A(t) = \alpha_{\bar{A}}(t)$;
(3) $A_1 \subset A_2$ implies $\alpha_{A_2}(t) \leq \alpha_{A_1}(t)$;
(4) $\alpha_{A_1 \cup A_2}(t) = \min\{\alpha_{A_1}(t), \alpha_{A_2}(t)\}$;
(5) if $\{A_n\}$ is a decreasing sequence of sets in $P_{cl,bd}(M)$ such that for $t > 0$ $\lim_{n\to\infty} \alpha_{A_n}(t) = 1$, then the intersection $A_\infty = \cap_{n=1}^{\infty} A_n$ is nonempty.

So, we can define set contraction as follows.

DEFINITION 2.2. Let X be a nonempty subset of a menger space $(M,\mathcal{F},\mathcal{T})$. A mapping $F : X \multimap M$ is said to be a k-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and $\alpha_{F(A)}(t) \geq \alpha_A(\frac{t}{k})$ for $0 < k < 1$ and $t > 0$.

DEFINITION 2.3. Let X be a nonempty subset of a Menger space $(M,\mathcal{F},\mathcal{T})$ and $\phi : \Re^+ \to \Re^+$ is mapping such that ϕ is strictly increasing, left continuous with $\phi(0) = 0$. A mapping $F : X \multimap M$ is said to be a Φ-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and

$$\alpha_{F(A)}(\phi(t)) \geq \alpha_A(\phi(\frac{t}{k})) \tag{1}$$

where $0 < k < 1$.

THEOREM 2.4. *Let X be a nonempty p-bounded subadmissible subset of a Menger space $(M,\mathcal{F},\mathcal{T})$. Suppose that $F \in KKM(X,X)$ is a Φ-set contraction and closed with nonempty compact values and $\overline{F(X)} \subseteq X$. Then F has a fixed point.*

PROOF. Let $x_0 \in X$, $X_0 = X$ and $X_{n+1} = co(F(X_n) \cup \{x_0\})$ for all $n \in N \cup \{0\}$. Then

(1) X_n is a subadmissible subset of X for all $n \in N \bigcup \{0\}$;
(2) $X_{n+1} \subset X_n$ for all $n \in N \bigcup \{0\}$;

By definition ϕ , for $t > 0$ we can find a positive number s, such that $t \geq \phi(s)$. Then for $t > 0$, we have

$$\begin{aligned} \alpha_{X_{n+1}}(t) &= \alpha_{co(F(X_n)\cup\{x_0\})}(t) \\ &\geq \alpha_{F(X_n)}(\phi(s)) \\ &\geq \alpha_{X_n}(\phi(\frac{s}{k})) \end{aligned}$$

$$\begin{aligned} &\geq \quad ... \\ &\geq \quad \alpha_{X_0}(\phi(\frac{s}{k^n}))\,. \end{aligned}$$

Thus $\alpha_{X_n}(t) \geq \alpha_{X_0}(\phi(\frac{s}{k^{n-1}}))$. Taking $n \to \infty$, we have $\alpha_{X_n}(t) \to 1$ as $n \to \infty$. Then $\bigcap_{n=0}^{\infty} \overline{X_n}$ is compact. Moreover since $\overline{F(X)} \subseteq X$, then $\bigcap_{n=0}^{\infty} \overline{X_n} \subset X$. Hence $K = \bigcap_{n=0}^{\infty} X_n$ is a precompact subset of X and $\overline{K} \subseteq X$. Let $Y = \overline{F(K)}$. Then, for each $0 < r < 1$, there exists a finite subset A of K such that $Y \subseteq \bigcup_{x\in A} B_x(r,t)$ for $t > 0$. Now define a map $G : K \multimap K$ by $G(x) = Y \setminus B_x(r,t)$, for all $x \in K$; then $G(x)$ is closed for each $x \in K$ and $\bigcap_{x\in A} G(x) = \emptyset$. Thus G is not a generalized KKM map with respect to $F|_K$. Hence, there exists a finite subset $\{x_0, x_1, ..., x_n\}$ of K such that $F(co(\{x_0, x_1, ..., x_n\}) \cap K) \nsubseteq \bigcup_{i=0}^{n} G(x_i)$. Thus there exists an $x_r \in F(co(\{x_0, x_1, ..., x_n\}) \cap K)$ such that $x_r \notin \bigcup_{i=0}^{n} G(x_i)$. If $x_r \in F(z)$ for some $z \in co(\{x_0, x_1, ..., x_n\}) \cap K$, then $z \in B_{x_r}(r,t) \cap K$, and so $x_r \in F(z) \cap B_z(r,t)$. Since Y is compact subset of X, then x_r converges to x as $r \to 1$. Consequently, z also converges to x as $r \to 1$. Since F is closed, then $x \in F(x)$. □

We note that if $\phi(t) = t$, then Φ-set contraction reduce to that of k-set contraction. Now, by applying Theorem (2.4), we obtain a coincidence point theorem.

THEOREM 2.5. *Let X be a nonempty p-bounded subadmissible subset of a Menger space $(M, \mathcal{F}, \mathcal{T})$ and $G : X \multimap X$ has nonempty subadmissible values and for every compact subset D of X and any $w \in X$, $G^-(w) \cap D$ is open in D. Suppose that $F \in KKM(X, X)$ is closed map with nonempty compact values and $\overline{F(X)} \subset X$ such that*

$$\alpha_{F(A)}(\phi(t)) \geq \alpha_A(\phi(\frac{t}{k}))$$

for any nonempty subset A of X, where α is an arbitrary measure of noncompactness and $\phi : [0,1] \to [0,1]$ is given function such that ϕ is continuous and $\phi(t) > 0$ for all $t \in (0,1)$. Then, there exists $x_0, y_0 \in X$ such that $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$.

PROOF. By Theorem (2.4), there is a precompact subset K of X such that $F(K) \subseteq K$. Since $G(x) \neq \emptyset$, then $X = \bigcup_{x\in X} G^-(x)$ and so $\overline{K} = \bigcup_{x\in X}(G^-(x) \cap \overline{K}$. But $\overline{K}$ is compact and thus for som $x_1, ..., x_n \in X$ such that $\overline{K} = \bigcup_{i=1}^{n}(G^-(x_i) \cap K)$. If $S : X \multimap \overline{K}$ is defined by $S(x) = \overline{K} \setminus (G^-(x) \cap \overline{K})$ for each $x \in X$, then $\bigcap_{i=1}^{n} S(x_i) = \emptyset$. Therefore, S is not a generalized KKM map with respect to F. Hence there exists a finite subset $B = \{b_1, ..., b_m\}$ of X such that $F(co(B)) \nsubseteq S(B)$. Hence, there exist $x_0 \in co(B)$ and $y_0 \in F(x_0)$ such that $y_0 \notin S(A)$. Thus $y_0 \in G^-(b_i) \cap \overline{K}$ and so $b_i \in G(y_0)$ for $i = 1, ..., m$. Since $G(y_0)$ is subadmissible, then $co(B) \subseteq G(y_0)$ and so $x_0 \in G(y_0)$. □

In the following by applying Theorems (2.4) and the axiomatic definition of measure of noncompactness in metric space [5], prove fixed point theorems for the KKM type maps.

THEOREM 2.6. *Let X be a nonempty bounded subadmissible subset of a metric space (M, d) and let $F \in KKM(X, X)$ be closed map with nonempty compact values and $\overline{F(X)} \subset X$ such that*

$$\phi(\mu(F(A))) \leq k\phi(\mu(A)) \tag{2}$$

for any nonempty subset A of X, where μ is an arbitrary MNC and ϕ is Φ-function. Then F has a fixed point.

Proof. It is well known that any metric space can be made into Menger space if we assume $\mathcal{F}_{x,y}(t) = \mathcal{H}(t - d(x,y))$ where $\mathcal{T}$ is defined as $\mathcal{T}_M(a,b) = \min\{a,b\}$. Then for every bounded subset A of M $\alpha_{F(A)}(t) = \mathcal{H}(t - \mu(F(A)))$. Define

$$\varphi(t) = \sup\{a : \ \phi(a) < t\}.$$

Then we have

$$\mathcal{H}(\varphi(t) - \mu(F(A))) = \mathcal{H}(t - \phi(\mu(F(A)))).$$

Then we have for $t > 0$ and $0 < k < 1$,

$$\begin{aligned}
\alpha_{F(A)}(\varphi(t)) &= \mathcal{H}(\varphi(t) - \mu(F(A))) \\
&= \mathcal{H}(t - \phi(\mu(F(A)))) \\
&\geq \mathcal{H}(\frac{t}{k} - \phi(\mu(A))) \\
&= \mathcal{H}(\varphi(\frac{t}{k}) - \mu(A)) \\
&\geq \alpha_A(\varphi(\frac{t}{k}))\,.
\end{aligned}$$

Thus we see that the inequality (2) in a metric space (M,d) implies the inequality (1) in the corresponding Menger space. By an application of Theorem (2.4) , we can prove this theorem. □

3. Conclusion

As a generalization of set contraction of Menger space, we develope a new idea in this study that we named set contraction. We also obtained certain fixed point and coincidence point results in the frame work of such spaces. We make sure that the idea of our method, which is presented can be applied to this theory in the upcoming future .

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New set contractions in Menger spaces via KKM principle

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Abstract. Here, the generalized set contractions in Menger spaces with respect to the measure of noncompactness is introduced. Some fixed point theorems for the KKM type maps for set-valued mapping are given.

Keywords: KKM property, Set-valued mappings, Menger space, Measure of noncompactness, Darbo fixed point.

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1. Introduction

In 1942, to qualify the space between two points, the idea of probabilistic metric space was proposed by Menger. By applying this notion, it turns out in a metric space distribution functions can be seen in lieu of positive real numbers. Indeed, the idea of probabilistic metric space in applied in states which the distance between two points is not specified, but the probable distance between two points is determined.
In 2005, Amini et al. [1] introduced KKM-type mapping in metric spaces and got results about fixed poit theorems. Later Tatari et al. in [3] extended this work on KKM-type compact mappings on sub-admissible subset of Menger space. Chen [2] generalized the Amini results in the setting of nearly subadmissible subset in complete metric spaces for a k-set contraction and he established some fixed point theorems where the compactness is not needed. In the sequel, we recall some definitions and facts which will be used in this paper.
Let X be a nonempty set, we use the notions:

(1) $P_{cl}(X) = \{Y \in P(X) : \ Y \ is \ closed\}$;
(2) $P_{cl,bd}(X) = P_{cl}(X) \cap P_{bd}(X)$.

Let A be a nonempty p-bounded subset of PM space $(M, \mathcal{F}, \mathcal{T})$. Then

(1) $co(A) = \cap\{B \subset M, \ B$ is a closed ball in M such that $A \subset B\}$;
(2) A is called subadmissible, if for each $D \in \langle A \rangle$, $co(D) \subset A$.

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2. Main results

This section is devoted to prove some fixed point theorems for generalize set contraction maps with the KKM property on Menger space.

We now are going to give the axiomatic definition for the measure of noncompactness on a p-bounded subset of the Menger space $(M, \mathcal{F}, \mathcal{T})$.

Definition 2.1. A function $\alpha : P_{bd}(M) \to \Re^+$ is called a measure of noncompactness if it satisfies

(1) $\alpha_A(t) = 1$ if and only if A is precompact;
(2) $\alpha_A(t) = \alpha_{\bar{A}}(t)$;
(3) $A_1 \subset A_2$ implies $\alpha_{A_2}(t) \leq \alpha_{A_1}(t)$;
(4) $\alpha_{A_1 \cup A_2}(t) = \min\{\alpha_{A_1}(t), \alpha_{A_2}(t)\}$;
(5) if $\{A_n\}$ is a decreasing sequence of sets in $P_{cl,bd}(M)$ such that for $t > 0$ $\lim_{n\to\infty} \alpha_{A_n}(t) = 1$, then the intersection $A_\infty = \cap_{n=1}^{\infty} A_n$ is nonempty.

Definition 2.2. Let X be a nonempty subset of Menger space $(M, \mathcal{F}, \mathcal{T})$. A mapping $F : X \multimap M$ is said to be a k-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and $\alpha_{F(A)}(t) \geq \alpha_A(\frac{t}{k})$ for $0 < k < 1$ and $t > 0$.

In the following, we define another new set contraction in Menger space.

Definition 2.3. Let X be a nonempty subset of Menger space $(M, \mathcal{F}, \mathcal{T})$ and $\psi, \phi : \Re^+ \to \Re^+$ are two mappings such that ϕ is strictly increasing and left continuous with $\phi(0) = 0$, $\psi(x) < x$ for $x > 0$ and $\lim_{n\to\infty}\psi^n(x) = 0$. A mapping $F : X \multimap M$ is said to be a (ϕ, ψ_1, ψ_2)-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and for any $\varepsilon > 0$ and $0 < \delta < 1$,

$$\alpha_A(\phi(\varepsilon)) > 1 - \delta \implies \alpha_{F(A)}(\phi(\psi_1(\varepsilon))) > 1 - \psi_2(\delta). \tag{1}$$

Theorem 2.4. *Let X be a nonempty p-bounded subadmissible subset of Menger space $(M, \mathcal{F}, \mathcal{T})$. Suppose that $F \in KKM(X, X)$ is a (ϕ, ψ_1, ψ_2)-set contraction and closed with nonempty compact values and $\overline{F(X)} \subseteq X$. Then F has a fixed point.*

Proof. Let $x_0 \in X$ and $X_{n+1} = co(F(X_n) \cup \{x_0\})$ for all $n \in N \cup \{0\}$. Then

(1) X_n is a subadmissible subset of X for all $n \in N \bigcup \{0\}$;
(2) $X_{n+1} \subset X_n$ for all $n \in N \bigcup \{0\}$;

Let $0 < \delta < 1$ be given by the property that $\phi(t) \to \infty$ as $t \to \infty$ we can find $\varepsilon > 0$ such that

$$\alpha_{X_1}(\phi(\varepsilon)) > 1 - \delta.$$

Then from (1)

$$\alpha_{co(F(X_1)\cup\{x_0\})}(\phi(\psi_1(\varepsilon))) > \alpha_{F(X_1)}(\phi(\psi_1(\varepsilon))) > 1 - \psi_2(\delta),$$

That is, $\alpha_{X_2}(\phi(\psi_1(\varepsilon))) > 1 - \psi_2(\delta)$. Continuing this process we obtain $\alpha_{X_3}(\phi(\psi_1^2(\varepsilon))) > 1 - \psi_2^2(\delta)$, and in general for all $n \in N \bigcup \{0\}$,

$$\alpha_{X_{n+1}}(\phi(\psi_1^n(\varepsilon))) > 1 - \psi_2^n(\delta).$$

Let $t > 0$ be arbitrary. Following the properties of ϕ and ψ we can find an integer m such that for all $n > m$,

$$\phi(\psi_1^n(\varepsilon)) < t.$$

Consequently for all $n > m$,

$$\alpha_{X_{n+1}}(t) \geq \alpha_{F(X_{n+1})}(\phi(\psi_1^n(\varepsilon))) > 1 - \psi_2^n(\delta).$$

Making $n \to \infty$, we obtain, $\alpha_{X_{n+1}}(t) \to 1$ as $n \to \infty$. Then $K = \bigcap_{n=0}^{\infty} \overline{X_n}$ is compact. Moreover since $\overline{F(X)} \subset X$, then $\bigcap_{n=0}^{\infty} \overline{X_n} \subset X$. Hence $K = \bigcap_{n=0}^{\infty} X_n$ is a precompact subset of X and $\overline{K} \subseteq X$. Let $Y = \overline{F(K)}$. Then, for each $0 < r < 1$, there exists a finite subset A of K such that $Y \subseteq \bigcup_{x \in A} B_x(r,t)$ for $t > 0$. Now define a map $G : K \multimap K$ by $G(x) = Y \setminus B_x(r,t)$, for all $x \in K$; then $G(x)$ is closed for each $x \in K$ and $\bigcap_{x \in A} G(x) = \emptyset$. Since $Y \subseteq K \subseteq X$ and $F \in KKM(X,X)$, then, $F|_K \in KKM(K,Y)$. Thus G is not a generalized KKM map with respect to $F|_K$. Hence, there exists a finite subset $\{x_0, x_1, ..., x_n\}$ of K such that $F(co(\{x_0, x_1, ..., x_n\}) \cap K) \nsubseteq \bigcup_{i=0}^{n} G(x_i)$. Thus there exists an $x_r \in F(co(\{x_0, x_1, ..., x_n\}) \cap K)$ such that $x_r \notin \bigcup_{i=0}^{n} G(x_i)$, that is $x_r \in F(z)$ for some $z \in co(\{x_0, x_1, ..., x_n\})$ and $x_r \in B_{x_i}(r,t)$ for all $i \in \{0, 1, ..., n\}$, so $x_i \in B_{x_r}(r,t)$. If $x_r \in F(z)$ for some $z \in co(\{x_0, x_1, ..., x_n\}) \cap K$, then $z \in B_{x_r}(r,t) \cap K$, and so $x_r \in F(z) \cap B_z(r,t)$. Since Y is compact subset of X, then x_r converges to x as $r \to 1$. Consequently, z also converges to x as $r \to 1$. Since F is closed, then $x \in F(x)$. □

Definition 2.5. Let X be a nonempty subset of Menger space $(M, \mathcal{F}, \mathcal{T})$. A mapping $F : X \multimap M$ is said to be a (ε, δ)-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and for any $\varepsilon > 0$ and $0 < \delta < 1$,

$$\alpha_A(\varepsilon) > 1 - \delta \implies \alpha_{F(A)}(k_1 \varepsilon) > 1 - k_2 \delta, \tag{2}$$

where $0 < k_1, k_2 < 1$.

Theorem 2.6. *Let X be a nonempty p-bounded subadmissible subset of Menger space $(M, \mathcal{F}, \mathcal{T})$ and let $F \in KKM(X,X)$ is an (ε, δ)-set contraction andclosed with nonempty compact values and $\overline{F(X)} \subset X$. Then F has a fixed point.*

Proof. If in Theorem (2.4) put $\phi(t) = t$, and $\psi_1(t) = k_1 t$, $\psi_2(t) = k_2 t$ where $0 < k_1, k_2 < 1$. Then the proof is complete. □

In the following, we define another new set contraction in Menger space.

Definition 2.7. Let X be a nonempty subset of Menger space $(M, \mathcal{F}, \mathcal{T})$ and $\gamma : [0, \infty) \to [0, \infty)$ satisfies $\gamma(t) < t$, and $\liminf_{r \to t^+} \gamma(r) < t$ for all t and $\gamma(0) = 0$. A mapping $F : X \multimap M$ is said to be a γ-set contraction if, for each $A \subset X$ with A p-bounded, $F(A)$ is p-bounded and

$$\alpha_{F(A)}(\gamma(t)) \geq \alpha_A(t) \ ,$$

Theorem 2.8. *Let X be a nonempty p-bounded subadmissible subset of Menger space $(M, \mathcal{F}, \mathcal{T})$. Suppose that $F \in KKM(X,X)$ is a γ-set contraction and closed with nonempty compact values and $\overline{F(X)} \subseteq X$. Then F has a fixed point.*

Proof. Let $x_0 \in X$ and $X_{n+1} = co(F(X_n) \cup \{x_0\})$ for all $n \in N \cup \{0\}$. Then

(1) X_n is a subadmissible subset of X for all $n \in N \bigcup \{0\}$;
(2) $X_{n+1} \subset X_n$ for all $n \in N \bigcup \{0\}$;

According to induction, the relation is proved:

$$\alpha_{X_n}(\gamma^n(t)) \geq \alpha_{X_0}(t) \quad \forall\, t > 0 \ . \tag{3}$$

With the property of (γ) for all $t > 0$, we can prove

$$\lim_{n \to \infty} \gamma^n(t) = 0. \tag{4}$$

Now want to show, by making $n \to \infty$, we obtain, $\alpha_{X_n}(t) \to 1$. In fact. it is necessary to show for all $s > 0$ and $0 < r < 1$, there exist a positive integer $n_0 = n_0(s, r)$ such that

$$\alpha_{X_n}(s) > 1 - r \quad \forall\, n \geq n_0(s, r)\ . \tag{5}$$

When $t \to \infty$, then $\alpha_{X_0}(t) \to 1$, thus

$$\exists t_0 \ \ s.t \ \ \alpha_{X_0}(t_0) > 1 - r\ .$$

When $s > 0$, by (4), there exists a positive integer $n_1 = n_1(s)$ such that

$$\forall n \geq n_1(s) \quad \varphi^n(t_0) < s\ . \tag{6}$$

Now we show, (5) for $n_1(s) = n_0(s, r)$ is correct. By monotonocity of α and (6) we have

$$\alpha_{X_n}(\gamma^n(t)) \leq \alpha_{X_n}(s) \quad \forall\, n \geq n_1(s)\ .$$

By (5) we have:

$$\alpha_{X_n}(s) \geq \alpha_{X_n}(\gamma^n(t)) \geq \alpha_{X_0}(t_0).$$

Therefor we have:

$$\alpha_{X_n}(s) \geq \alpha_{X_n}(t) \quad \forall\, n \geq n_1(s).$$

Then

$$\alpha_{X_n}(s) \geq 1 - r \quad \forall\, n \geq n_1(s).$$

The remainder conclusion follows from theorem (2.4). □

3. Conclusion

In this research by using the techniques of noncompactness measure and KKM theory, sufficient conditions for the existence of some fixed points in Menger spaces, are presented.

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Iterative Methods for Equilibrium Problems in Banach Spaces

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Abstract. We introduce a novel iterative algorithm for approximating common solutions of equilibrium problems and fixed point problems involving Bregman relatively nonexpansive mappings in reflexive Banach spaces. Our approach combines Bregman distance methods with resolvent operators, and we prove strong convergence under appropriate conditions on the bifunctions. The results extend and unify various existing methods in the literature.

Keywords: Bregman relatively nonexpansive mappings, resolvent operators, Strong convergence.

AMS Mathematics Subject Classification [2020]: 47H09,47H10

1. Introduction

Let E be a Banach space with dual E^*, and let C be a nonempty, closed, convex subset of E. The equilibrium problem for a bifunction $g : C \times C \to \mathbb{R}$ satisfying the condition $g(x, x) = 0$ for every $x \in C$ is stated as follows:

$$\text{Find } y^* \in C \text{ such that } g(x, y^*) \leq 0, \tag{1}$$

for all $x \in C$. The set of solutions of (1) is denoted by $EP(g)$.

Let $f : E \to (-\infty, \infty]$ be an admissible function, i.e., a proper, convex and lower semicontinuous function. The domain of f is the set $\{x \in E : f(x) < \infty\}$ denoted by dom$f$. The set of minimizers of f is denoted by Argmin f and its unique element by $argmin_{x\in E} f(x)$, if Argmin f is a singleton. Let $x \in$ int dom$f$, for any $y \in E$, the directional derivative of f at x is defined by

$$f^\circ(x, y) = \lim_{t\to 0} \frac{f(x + ty) - f(x)}{t}. \tag{2}$$

If the limit (2) as $t \to 0$ exists for each y, then f is said to be Gateaux differentiable at x. The function f is said to be Gateaux differentiable if it is Gateaux differentiable for all $x \in$ int dom$f$. When the limit as $t \to 0$ in (2) is attained uniformly for any $y \in E$ with

*Speaker.

$\|y\| = 1$, we say that f is Fréchet differentiable at x. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly at each $x \in C$ and $y \in E$ with $\|y\| = 1$. In this case, the gradient of f at E is the linear function $\nabla f(x)$ which is defined by $\langle y, \nabla f(x)\rangle := f^\circ(x, y)$ for all $y \in E$.

Let $x \in \text{int dom} f$. The subdifferential of f at E is the convex set defined by

$$\partial f(x) = \{l \in X^* : f(x) + \langle y - x, l\rangle \leq f(y),\ \forall y \in E\}, \tag{3}$$

where the Fenchel conjugate of f is the convex function $f^* : E^* \to (-\infty, \infty]$ defined by

Let E be a reflexive Banach space. The function f is Legendre if and only if it satisfies the following two conditions:

(L1) int dom$f \neq \emptyset$, f is Gateaux differentiable and dom $\nabla f =$ int dom f.
(L2) int dom$f^* \neq \emptyset$, f^* is Gateaux differentiable and dom $\nabla f^* =$ int dom f^*.

Since E is a reflexive Banach space, $(\partial f)^{-1} = \partial f^*$ (see [2, p 83]). Also, we know that $\nabla f = (\nabla f^*)^{-1}$, and this together with conditions (L1) and (L2) imply that ran$\nabla f =$ dom $\nabla f^* =$ int dom f^* and ran$\nabla f^* =$ dom $\nabla f =$ int dom f. In addition, since E is reflexive, then f is Legendre if and only if f^* is Legendre (see [1, corollary 5.5]).

Let $f : E \to (-\infty, \infty]$ be a Gateaux differentiable function. The bifunction D_f : $dom f \times int\ dom f \to [0, +\infty]$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \tag{4}$$

is called the Bregman distance with respect to f (see [?]). The Bregman distance does not satisfy the well known properties of a metric. Clearly, $D_f(x, x) = 0$, but $D_f(y, x) = 0$ may not imply $x = y$, but when f is Legendre this indeed holds (see [1, Theorem 7.3(vi)]). Also, from the Bregman distance, we obtain

$$\begin{aligned} |D_f(v, x) - D_f(v, y)| \leq & |f(x) - f(y)| + \|v - x\|\|\nabla f(x) - \nabla f(y)\| \\ & + \|x - y\|\|\nabla f(y)\|, \end{aligned} \tag{5}$$

for all $x, y \in int\ dom f$ and $v \in dom f$. The modulus of total convexity at $x \in$ int dom$f$ is the function $\upsilon_f(x, .) : [0, \infty) \to [0, \infty]$, defined by

$$\upsilon_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f,\ \|y - x\| = t\}.$$

The function f is called totally convex at $x \in$ int dom$f$ if $\upsilon_f(x, t)$ is positive for any $t > 0$ [?]. Let C be a nonempty subset of E. The modulus of total convexity of f on C is defined by

$$\upsilon_f(C, t) = \inf\{\upsilon_f(x, t) : x \in C \cap \text{int dom} f\}.$$

The function f is called totally convex on bounded subsets if $\upsilon_f(C, t)$ is positive for any nonempty and bounded subset C and for any $t > 0$.

The following result establishes the characteristic continuity properties for the derivative of a lower semicontinuous convex function.

Let E be a Banach space and $f : E \to (-\infty, +\infty]$ be a convex function. The function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E, such that the first one is bounded:

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The function $V_f : E \times E^* \to [0, +\infty]$ is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{6}$$

It is known that V_f is convex in the second variable. Hence, for all $z \in E$, we have

$$D_f\Big(z, \nabla f^*\big(\sum_{i=1}^{N} t_i \nabla f(x_i)\big)\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{7}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\Sigma_{i=1}^{N} t_i = 1$.

Let E be a real reflexive Banach space and C a nonempty, closed and convex subset of E. Throughout this paper, we assume that $g : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

(A1) g is pseudomonotone, i.e., $g(x,y) \geq 0 \Rightarrow g(y,x) \leq 0$, for all $x, y \in C$,

(A2) g is Bregman-Lipschitz-type continuous, i.e., there exist two positive constants c_1, c_2, such that

$$g(x,y) + g(y,z) \geq g(x,z) - c_1 D_f(y,x) - c_2 D_f(z,y), \quad \forall x, y, z \in C,$$

where $f : E \to (-\infty, +\infty]$ is a Legendre function. The constants c_1, c_2 are called Bregman-Lipschitz coefficients with respect to f,

(A3) g is weakly continuous on $C \times C$,

(A4) $g(x, .)$ is convex, lower semicontinuous and subdifferentiable on C for every fixed $x \in C$,

(A5) $\limsup_{t \downarrow 0} g(tx + (1-t)y, z) \leq g(y,z)$, for each $x, y, z \in C$.

The mapping $T : C \to C$ is called Bregman relatively nonexpansive, if the following conditions are satisfied:

(1) $F(T)$ is nonempty,
(2) $D_f(u, Ty) \leq D_f(u, y)$, for all $y \in C$ and $u \in F(T)$,
(3) $\hat{F}(T) = F(T)$.

We assume that $\Theta : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

(B1) $\Theta(x,x) = 0$, for all $x \in C$.
(B2) Θ is monotone, i.e. $\Theta(x,y) + \Theta(y,x) \leq 0$, for all $x, y \in C$.
(B3) $\lim\limits_{t \downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x,y)$, for all $x, y, z \in C$.
(B4) for all $x \in C$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

The resolvent of Θ is the operator $Res_\Theta^f : E \to 2^C$ defined by [3]

$$Res_\Theta^f x = \{z \in C : \Theta(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \quad \forall y \in C\}. \tag{8}$$

Suppose $f : E \to (-\infty, \infty]$ is a Gateaux differentiable and strongly coercive function and Θ satisfies conditions B1 - B4. Then $dom\ Res_\Theta^f = E$ (see [3, Lemma 1]).

2. Main results

Assumptions

Assume the following conditions hold:

- C is a nonempty closed convex subset of a real reflexive Banach space E.
- $f : E \to \mathbb{R}$ is an admissible, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.
- $g : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions A1–A5.

- $\vartheta : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions B1–B4.

Algorithm

Algorithm 1 Iterative Algorithm

Initialization: Let $x_1 \in C$, $u \in E$ be initial points. Let $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\lambda_n\} \subset [a,b] \subset (0,p)$, where $p = \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}$, and c_1, c_2 are the Bregman-Lipschitz coefficients of g.

Step 1: Compute

$$y_n = \operatorname{argmin}\{\lambda_n g(x_n, y) + D_f(y, x_n) : y \in C\},$$

and

$$z_n = \operatorname{argmin}\{\lambda_n g(y_n, y) + D_f(y, x_n) : y \in C\}.$$

Step 2: Compute $u_n = \mathrm{Res}_{\vartheta}^{f} z_n$.

Step 3: Construct $C_n = \{v \in C : D_f(v, u_n) \leq D_f(v, x_n)\}$.

Step 4: Compute $x_{n+1} = \overleftarrow{\mathrm{Proj}}_{C_n}^{f} \left(\nabla f^* \left(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(u_n)\right)\right)$.

Step 6: Set n := n + 1 and go to step 1.

THEOREM 2.1. *Assume that* $\Gamma = EP(g) \cap F(\mathrm{Res}_{\vartheta}^{f}) \neq \emptyset$. *Then, under the assumptions stated in Section 2 and using the Algorithm 1, the sequence* $\{x_n\}$ *converges strongly to* $\overleftarrow{\mathrm{Proj}}_{\Gamma}^{f} u$.

3. Conclusion

In this paper, we have introduced a novel iterative algorithm that effectively approximates common solutions of equilibrium problems and fixed point problems involving Bregman relatively nonexpansive mappings in reflexive Banach spaces. By combining Bregman distance methods with resolvent operators, we have established strong convergence under appropriate conditions. Our results successfully extend and unify various existing methods in the literature, providing a comprehensive framework for solving these important classes of problems.

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Structure and Operators in a Class of Weighted Sequence Spaces

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Abstract. Hagler and Azimi introduced a class of hereditarily l_1 Banach spaces which fail the Schur property. Then Azimi extended these spaces to a class of hereditarily l_p Banach spaces for $1 \leq p < \infty$. Here a further geometric and topological investigation of the spaces is carried out. Moreover we observe that the subspace $[(e_{n_k})]$ generated by subsequence (e_{n_k}) of (e_n) is complemented and the identity operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ when $p > q$ is unbounded.

Keywords: Banach space, Schur property, hereditaliry l_p

AMS Mathematics Subject Classification [2020]: 46B20, 46E30

1. Introduction

A class of hereditarily l_1 Banach spaces has been introduced by Hagler and Azimi, which among the other interesting properties fails the Schur property [2] . Then these spaces has been extended to a new class of hereditarily l_p Banach spaces, the $X_{\alpha,p}$ [1]. In this paper, we investigate a significant class of such spaces whose construction involves a carefully chosen weighting sequence (α_i) and parameter $p \in [1, \infty)$. Here further geometric and topological investigation of the spaces is carried out. In the first result subclasses are constructed where each member has an unconditional basis (u_i) such that $u_i \to 0$ but not in norm. Among the other interesting properties, all constructed Azimi-Hagler spaces are dual spaces. In [8] Popov showed that the classical Pitt theorem on compactness of operators from l_p to l_q for $1 \leq q < p < \infty$ it fails in general setting of hereditarily l_p and l_q spaces.

By the Pitt theorem every bounded linear operator from l_p to l_q when $1 \leq q < p < \infty$ is compact. The proof of this theorem is based on the fact that any block basis of (e_n) in l_p is equivalent to (e_n) in l_p. But this is not the case for $X_{\alpha,p}$ spaces. In fact there are block basis sequences of(e_n) in $X_{\alpha,p}$ which are not equivalent to (e_n).

Before we define this new spaces let to recall the definition of the $X_{\alpha,p}$. Let (α_i) be a sequence of reals in $[0, 1]$ (whose terms are used as weighting factor in the definition of the norm) which satisfies the following properties:

(1) $1 = \alpha_1 \geq \alpha_2 \geq ...,$

*Speaker.

(2) $\lim_i \alpha_i = 0$,
and
(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

By a block F we mean an interval (finite or infinite) of integers. For a block F and $x = (t_1, t_2, ...)$ a sequence of scalars such that $\sum_j t_j$ converges, define $< x, F >= \sum_{j \in F} t_j$. A sequence $F_1, F_2, ..., F_n, ...$where each F_i is a finite block is admissible if

$$\max F_i < \min F_{i+1} \text{ for } i = 1, 2, 3, ...$$

For $x = (t_1, t_2, ...)$ a finitely nonzero sequence of scalars, define

$$\| x \| = \max \left(\sum_{i=1}^{n} \alpha_i \mid < x, F_i > \mid^p\right)^{\frac{1}{p}},$$

where the max is taken over all n, and admissible sequences F_1, $F_2, ..., F_n$ and $1 \leq p < \infty$. Then $X_{\alpha,p}$ is the completion of the finitely nonzero sequences of scalars $x = (t_1, t_2, ...)$ in this norm. For more information concerning these spaces, referred to [1] and [2].

Main Results

From the definition of the norm of $X_{\alpha,p}$, we can see that the unit vector basis is spreading (equivalent to each of its subsequence) and bi-monotone. That is for each $x = (x_1, x_2, x_3, ...) \in X_{\alpha,p}$ and $n < m$, $||(P_m - P_n)x|| \leq ||x||$. Observe each block F defines a functional which is bounded on $X_{\alpha,p}$. In fact $< x, F >= \sum_{i \in F} x_i = \sum_{i \in F} e_i^*(x)$.

Theorem 1.1. *If (e_{i_k}) is a subsequence of (e_k) in $X_{\alpha,p}$, then*
(1) *$[(e_{i_k})]$ is asymptotically isometric to ℓ_p.*
(2) *$[(e_{i_k})]$ complemented in $X_{\alpha,p}$.*

Proof. Part (1) is an immediate consequence of Theorem 1.1. in [1]. For the proof of (2) let (F_i) be a sequence of blocks without gaps $(maxF_i + 1 = minF_{i+1})$ such that $i_k \in F_k$, then $[(e_{i_k})]$ is complemented by the projection

$$Px = \sum_{i=1}^{\infty} < x, F_k > e_{i_k}$$

Since (F_i) has no gaps, any estimate of $||Px||$ is also an estimate of $||x||$, so $||P|| = 1$. □

Lemma 1.2. *For each non-increasing sequence of positive numbers (β_i) and*

$$v = (\beta_1, -\beta_1, \beta_2, -\beta_2, ...\beta_n, -\beta_n)$$

in the space of $X_{\alpha,p}$ we have

$$||v||^p = (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + ... + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p$$

Proof. Let each block F be a singleton with $F_i = \{i\}$. Then $| < v, F_{2i-1}| = | < v, F_{2i}| = \beta_i$. This implies

$$||v||^p \geq (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + ... + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p$$

We claim the sequence of (F_i) is the norming sequence for v, otherwise there is a sequence $(F_1, F_2, ...F_k)$ of consecutive block such that $k < 2n$ and $||v||^p = \sum_{i=1}^{n} \alpha_i| < v, F_i > |^p$, since for any block F, $< v, F >$ is β_i or 0, the number of block such that $< v, F > \neq 0$ at most is equal to k, and since $\{\beta_i\}$ is non-increasing and $k < 2n$,

$$||v||^p = \sum \alpha_i \beta_i^p \leq (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + ... + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p$$

So

$$||v||^p = (\alpha_1 + \alpha_2)\beta_1^p + (\alpha_3 + \alpha_4)\beta_2^p + ... + (\alpha_{2n-1} + \alpha_{2n})\beta_n^p$$

□

Corollary 1.3. *In the space of $X_{\alpha,p}$ and for integer n we have*

$$||\sum_{i=1}^{n} u_i|| = ||\sum_{i=1}^{n}(e_{2i} - e_{2i-1}|| = (\sum_{i=1}^{2n} \alpha_i)^{1/p} \qquad (2)$$

Proof. Put $\beta_i = 1$ in the Lemma 3.2 . □

We know that if $p > q$ the identity operator from ℓ_p to ℓ_q is unbounded. Here is a similar result for $X_{\alpha,p}$.

Theorem 1.4. *Identity operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ when $p > q$ is unbounded.*

Proof. Let I be bounded, then for any scalars a_i

$$||\sum_{i=1}^{n} a_i e_i||_{X_{\alpha,q}} = ||\sum_{i=1}^{n} I a_i e_i||_{X_{\alpha,q}} = ||I \sum_{i=1}^{n} a_i e_i||_{X_{\alpha,q}} \leq ||I|| ||\sum_{i=1}^{n} a_i e_i||_{X_{\alpha,p}}$$

with $a_i = (-1)^i$ and the Corollary3.3

$$(\sum_{i=1}^{n} \alpha_i)^{1/q} = ||\sum_{i=1}^{n}(-1)^i e_i||_{X_{\alpha,q}} \leq ||I|| ||\sum_{i=1}^{n}(-1)^n a_i e_i||_{X_{\alpha,p}} = ||I||(\sum_{i=1}^{n} \alpha_i)^{1/p}$$

therefore

$$(\sum_{i=1}^{n} \alpha_i)^{\frac{1}{q}-\frac{1}{p}} \leq ||I||$$

This is a contradiction, since $\sum_1^\infty \alpha_i$ is diverges. So I is unbounded. □

We use the following lemma from [2].

Lemma 1.5. *Let (u_i) be a sequence of norm one vectors in $X_{\alpha,p}$ and (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each i put $s_i = s(u_i)$. If $\lim s_i = 0$ then a subsequence (v_k) of (u_k) is equivalent to usual basis of ℓ_p.*

For $x \in X$, put $s(x) = \max| < x, G > |$ where the max is taken over all blocks G. We need the following lemma from [1] in proof of the next theorem.

Lemma 1.6. *Let (u_i) be a sequence of norm one vectors in $X_{\alpha,p}$ and (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each i put $s_i = s(u_i)$. If $\lim s_i = 0$ then a subsequence (v_k) of (u_k) is equivalent to usual basis of ℓ_p.*

Lemma 1.7. *Let (x_n) be a sequence of vectors in a Banach space X, such that for every increasing sequence, (n_k) of integers,*

$$\lim_{k\to\infty} \frac{||x_{n_1} + x_{n_2} + ... + x_{n_k}||}{k} = 0$$

then $x_n \to 0$ weakly.

Proof. If this is not true, then there exist $f \in X^*$ with $||f|| = 1$, $\delta > 0$ and a sequence (n_i) of integers such that $f(x_{n_i}) \geq \delta$. This implies that $\sum_{i=1}^k f(x_{n_i}) \geq k\delta$. Therefore,

$$\frac{||\sum_{i=1}^k x_{n_i}||}{k} \geq \frac{\sum_{i=1}^k f(x_{n_i})}{k} \geq \delta$$

which is a contradiction. □

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Fuzzy Mathematics

Some properties of BCH-algebras

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Abstract. In this paper, as using S-norms, we introduce the concept of anti fuzzy subalgebras and anti fuzzy closed ideals of BCH-algebras and lower level cuts of them will be defined and proved some results about them. Next we define direct sum, union and fuzzy relation of them and investigate them. Finally, we investigate the image and pre image of them under homomorphisms of BCH-algebras.

Keywords: Algebra and orders, theory of fuzzy sets, norms, direct sums and unions, homomorphisms.

AMS Mathematics Subject Classification [2020]: 11S45, 03E72 , 15A60

1. Introduction

Hu and Li [7, 8] introduced the notion of a BCH-algebra and subsequently gave examples of proper BCH-algebras and studied some properties. Certain other properties of BCH-algebras have been studied by Ahmad [1], Dudek and Thomys [6], Chaudhry [3], Roh et al. [10, 11], Chaudhry et al. [4], and Dar et al. [5], and Smarandache structure has been applied to BCH-algebra [2]. Fuzzy set theoriesprovide strict mathematical frameworks using vague phenomena and can be precisely and rigorously studied. The core theories in mathematics such as algebra, topology and analysis are subject to fuzzification. Many researchers have dealt with subalgebras, ideals, implicative ideals, hyper structure, homomorphism and relations, etc. and theirfuzzification.The concept of a fuzzy set, which was introduced by Zadeh in his definitive paper [12] of 1965, was applied by many researchers to generalize some of the basic concepts of algebras. In [9], Hong et al. applied the concept to BCH-algebras and studied fuzzy dot subalgebras of BCH-algebras. In this paper, we introduce the notion of anti fuzzy subalgebras and anti fuzzy closed ideals of BCH-algebras with respect to S-norms and investigate some results. Next we define direct sum, union and fuzzy relation of them and illustrated several ideas that evaluate their relationship in a BCH-algebras. Finally, we investigate the image and pre image of them under homomorphisms of BCH-algebras.

2. preliminaries

This section includes basic definitions and initial findings that will be useful later.

Definition 2.1. By a BCH-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ that satisfies the following axioms:
(1) $x * x = 0$,
(2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(3) $(x * y) * z = (x * z) * y$,
for all $x, y, z \in X$.
In a BCH-algebra, we can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

Definition 2.2. A non-empty subset I of a BCH-algebra X is called subalgebra of X if $x * y \in I$ for all $x, y \in I$.

Definition 2.3. A non-empty subset I of a BCH-algebra X is called a closed ideal of X if
(1) $0 * x \in I$,
(2) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Definition 2.4. Define a mapping $f : (X, *, 0) \to (X, \star, 0)$ of BCH-algebras a homomorphism if $f(x * y) = f(x) \star f(y)$, for all $x, y \in X$.

Definition 2.5. An S-norm S is a function $S : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(1) $S(x, 0) = x$,
(2) $S(x, y) \leq S(x, z)$ if $y \leq z$,
(3) $S(x, y) = S(y, x)$,
(4) $S(x, S(y, z)) = S(S(x, y), z)$,
for all $x, y, z \in [0, 1]$.

Definition 2.6. The functions $S_n : \prod_{i=1}[0, 1] \to [0, 1]$ are defined by

$$S_n(x_1, x_2, ..., x_n) = S(x_i, S_{n-1}(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n))$$

for all $1 \leq i \leq n$, where $n \geq 2$ such that $S_2 = T$ and $S_1 = id$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 2.7. *Let S be a s-norm. Then for every $x_i, y_i \in [0, 1]$, where $1 \leq i \leq n$, and $n \geq 2$, we have*

$$S_n(S(x_1, y_1), S(x_2, y_2), ..., S(x_n, y_n)) = S(S_n(x_1, x_2, ..., x_n), S_n(y_1, y_2, ..., y_n)).$$

Definition 2.8. Let $\mu, \nu : X \to [0, 1]$ define $\mu \cup \nu : X \to [0, 1]$ as $(\mu \cup \nu)(x) = S(\mu(x), \nu(x))$ for all $x \in X$.

Definition 2.9. Let $\mu : X \to [0, 1]$ and $\nu : Y \to [0, 1]$. The direct sum of μ and ν is denoted by $\mu \oplus \nu : X \times Y \to [0, 1]$ is defined as:

$$(\mu \oplus \nu)(x, y) = S(\mu(x), \nu(y)),$$

for all $(x, y) \in X \oplus Y$.

3. Main results

Definition 3.1. Let $\mu : X \to [0, 1]$ be a fuzzy subset of X. We define μ as an anti-fuzzy subalgebra of a BCH-algebra $(X, *, 0)$ under s-norm S if:

$$\mu(x * y) \leq S(\mu(x), \mu(y)),$$

for all $x, y \in X$.
The set of all anti-fuzzy subalgebras of the BCH-algebra X under the s-norm S is denoted by $AFSS(X)$.

Proposition 3.2. *Let S be idempotent. Then $\mu \in AFSS(X)$ if and only if for every $t \in (0, 1]$, the set*

$$\mu_t = \{x \in X : \mu(x) \leq t\}$$

is either empty or a subalgebra of X

Proposition 3.3. *Let $\mu \in AFSS(X)$ and S be idempotent. Then $\mu(0) \leq \mu(x)$ for all $x \in X$.*

Proposition 3.4. *Let $\mu \in AFSS(X)$ and $\nu \in AFSS(X)$. Then $\mu \cup \nu \in AFSS(X)$.*

Corollary 3.5. *Let $\mu_i \subseteq AFSS(X)$ for $i = 1, 2, 3, 4, ..., n$. Then $\cup_{i=1,2,3,...,n}\mu_i \in AFSS(X)$.*

Proposition 3.6. *Let $\mu \in AFSS(X)$ and $\nu \in AFSS(Y)$. Then $\mu \oplus \nu \in AFSS(X \oplus Y)$.*

Proposition 3.7. *If $\mu \in AFSS(X)$ and $\varphi : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of BCH-algebras, then $\varphi(\mu) \in AFSS(Y)$.*

Proposition 3.8. *If $\nu \in AFSS(Y)$ and $\varphi : (X, *, 0) \to (Y, \star, 0)$ be a homomorphism of BCH-algebras, then $\varphi^{-1}(\nu) \in AFSS(X)$.*

Definition 3.9. Define $\mu : X \to [0, 1]$ as an anti fuzzy closed ideal of BCH-algebra X under s-norm S if it satisfies the following inequalities:
(1) $\mu(0 * x) \leq \mu(x)$,
(2) $\mu(x) \leq S(\mu(x * y), \mu(y))$,
for all $x, y \in X$.
Denote by $AFCIS(X)$, the set of all anti fuzzy closed ideals of X under s-norm S.

Proposition 3.10. *Let $\mu \in AFSS(X)$ and S be idempotent. Then $\mu(0 * x) \leq \mu(x)$ for all $x \in X$.*

Proposition 3.11. *Let $\mu \in AFSS(X)$ and S be idempotent. If $\mu(x) \leq S(\mu(x*y), \mu(y))$ for all $x, y \in X$, then $\mu \in AFCIS(X)$.*

Proposition 3.12. *Assume $\mu \in AFCIS(X)$. Then $\mu \in AFSS(X)$.*

Proposition 3.13. *Let $\mu \in AFSS(X)$ and S be idempotent. If $\mu(y * x) \leq \mu(x * y)$ for all $x \in X$, then $\mu \in AFCIS(X)$.*

Proposition 3.14. *Let S be idempotent. Then $\mu \in AFCIS(X)$ if and only if $\mu_t = \{x \in X : \mu(x) \leq t\} \neq \emptyset$ be a closed ideal of BCH-algebra X for every $t \in (0, 1]$.*

Definition 3.15. Let $\mu : X \to [0, 1]$ bea fuzzy subset of X. Define $\mu^\lambda : X \to [0, 1]$ as $\mu^\lambda(x) = S(\lambda, \mu(x))$ for all $\lambda \in [0, 1]$ and $x \in X$. Also we can define $\mu^{\mu(a)} : X \to [0, 1]$ which $\mu^{\mu(a)}(x) = S(\mu(a), \mu(x))$ for all $a, x \in X$.

Proposition 3.16. *Let $\mu \in AFCIS(X)$ and S be idempotent. Then $\mu^\lambda \in AFCIS(X)$ for all $\lambda \in [0, 1]$.*

Proposition 3.17. *Let $\mu \in AFCIS(X)$ and S be idempotent. Then $\mu^{\mu(a)} \in AFCIS(X)$ for all $a \in [0, 1]$.*

Proposition 3.18. *Let $\mu \in AFCIS(X)$ then $\mu^f(x) = f(\mu(x)) \in AFCIS(X)$ for all $x \in X$ and increasing functions $f : [0,1] \to [0,1]$.*

Proposition 3.19. *Let $\mu \in AFCIS(X)$ of a non-negative BCH-algebra X. Then $\nu(x) = \sup\{\mu(x * y) : y \in X\} \in AFCIS(X)$ for all $x \in X$.*

Proposition 3.20. *Let $\mu \in AFCIS(X)$ and $\nu \in AFCIS(X)$. Then $\mu \cup \nu \in AFCIS(X)$.*

Corollary 3.21. *Let $\mu_i \subseteq AFCIS(X)$ for $i = 1, 2, 3, 4, ..., n$. Then $\cup_{i=1,2,3,...,n}\mu_i \in AFCIS(X)$.*

Proposition 3.22. *Let $\mu \in AFCIS(X)$ and $\nu \in AFCIS(Y)$. Then $\mu \oplus \nu \in AFCIS(X \oplus Y)$.*

Definition 3.23. Let $\mu, \nu : X \times X \to [0,1]$ be two fuzzy relations on a set X. Then $\mu \circ \nu : X \times X \to [0,1]$ is a fuzzy relation on a set X, where it defined as

$$(\mu \circ \nu)(x, y) = \inf\{S(\mu(x, z), \nu(z, y)) : z \in X\}$$

for all $x, y, z \in X$.

Proposition 3.24. *$\mu_1, \mu_1 \in AFCIS(X)$. Then $\mu_1 \circ \mu_2 \in AFCIS(X \times X)$.*

Proposition 3.25. *If $\mu \in AFCIS(X)$ and $\varphi : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of BCH-algebras, then $\varphi(\mu) \in AFCIS(Y)$.*

Proposition 3.26. *If $\nu \in AFCIS(Y)$ and $\varphi : (X, *, 0) \to (Y, *, 0)$ be a homomorphism of BCH-algebras, then $\varphi^{-1}(\nu) \in AFCIS(X)$.*

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On Fuzzy Small Submodules and Fuzzy Direct Sums of Modules

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Abstract. In this paper, we investigate fuzzy small submodules of a module and their properties. We introduce fuzzy small epimorphisms and fuzzy small covers, and explore the behavior of fuzzy small submodules under direct sums and quotient modules. Relations between fuzzy small submodules and classical small submodules, as well as the Jacobson radical in the fuzzy setting, are also studied. Several necessary and sufficient conditions for fuzzy small submodules are established, and we generalize well-known results on small submodules to the fuzzy context.

Keywords: Small submodules, Fuzzy submodules, Fuzzy direct sum

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1. Introduction

Since the introduction of fuzzy sets by Zadeh [1], numerous applications and generalizations have appeared in algebraic structures. Rosenfeld [2] was the first to define fuzzy subgroups of a group. Naegoita and Ralescu [3] extended this concept to modules, defining fuzzy submodules. Subsequently, fuzzy finitely generated submodules, fuzzy quotient modules, radicals of fuzzy submodules, and primary fuzzy submodules have been studied [4, 5]. Saikia and Kalita [6] introduced fuzzy essential submodules and investigated their characteristics. Fuzzy small submodules play a central role in understanding the fuzzy Goldie dimension and other structural properties of modules.

In this paper, we define fuzzy small submodules, explore their properties, introduce fuzzy small epimorphisms and covers, and investigate their behavior under fuzzy direct sums and quotients.

*Speaker.

2. Preliminaries

Definition 2.1 (Fuzzy Ideal). Let R be a ring. A fuzzy set μ of R is called a *fuzzy ideal* if, for all $x, y \in R$:

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$,
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Definition 2.2 (L-Submodule [3]). Let M be a module over a ring R, and let L be a complete Heyting algebra. An L-subset μ of M is called an *L-submodule* if for all $x, y \in M$ and $r \in R$:

(1) $\mu(\theta) = 1$,
(2) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$,
(3) $\mu(rx) \geq \mu(x)$.

The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by

$$\mu^* = \{\, x \in M \mid \mu(x) > 0 \,\}.$$

Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$\left(\frac{\nu}{\mu}\right)(\overline{x}) = \bigvee \{\, \nu(z) \mid z \in \overline{x} \,\}, \quad \forall x \in \nu^*, \tag{2.4}$$

where $\overline{x}$ denotes the coset $x + \mu^*$.

3. Main Results

Definition 3.1 (Small L-Submodule). Let M be a module over a ring R and $\mu \in L^M$. Then μ is a *small L-submodule* if for any $\nu \in L^M$,

$$\nu \neq \chi_M \implies \mu \vee \nu \neq \chi_M.$$

The notation $\mu \ll_L M$ indicates μ is small. For $L = [0,1]$, μ is called a *fuzzy small submodule*, denoted $\mu \ll_F M$.

Definition 3.2 (Fuzzy Small Cover). Let $\mu \in F^M$. If $\mu \ll_F M$, then χ_M (or M) is a *fuzzy small cover* of χ_M/μ^* (or M/μ).

Definition 3.3 (Fuzzy Small Epimorphism). Let $f : M \to L$ be an epimorphism. Then f is a *fuzzy small epimorphism* if $f^{-1}(\chi_{\{\theta\}}) \ll_F M$.

Definition 3.4 (Fuzzy Small Ideal). A fuzzy ideal μ of R with $\mu(0) = 1$ is called a *fuzzy small ideal* if it is a fuzzy small submodule of R_R.

Let μ and σ be any two fuzzy submodules of M such that $\mu \subseteq \sigma$. Then μ is called a *fuzzy submodule* of σ. Moreover, μ is called a *fuzzy small submodule* in σ, denoted by $\mu \ll_F \sigma$, if $\mu|_{\sigma^*} \ll_F \sigma^*$ in the sense that for every submodule γ of M satisfying $\gamma|_{\sigma^*} \neq \chi_{\sigma^*}$, it follows that $\mu|_{\sigma^*} + \gamma|_{\sigma^*} \neq \chi_{\sigma^*}$. Here, by $\mu|_{\sigma^*}$ and $\gamma|_{\sigma^*}$ we mean the restriction mappings of μ and γ on σ^*, respectively.

Theorem 3.5. *Let M be a module and $N \leq M$. Then $N \ll M$ if and only if $\chi_N \ll_F M$.*

Example 3.6. (1) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. For every $0 \neq n \in \mathbb{Z}$ the submodule $n\mathbb{Z}$ is not small in M. Hence $\chi_{n\mathbb{Z}} \not\ll_F M$ for every $0 \neq n \in \mathbb{Z}$, by above proposition.

(2) Consider the submodules $K = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}$ and $L = \{\bar{0}, \bar{6}\}$ of $M = \mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$ as $\mathbb{Z}$-module. It is easy to check that $K \not\ll M$ and $L \ll M$. So $\chi_K \not\ll_F M$ and $\chi_L \ll_F M$, by above proposition.

THEOREM 3.7. *Let $\mu \in F^M$. Then $\mu \ll_F M$ if and only if $\mu^* \ll M$.*

COROLLARY 3.8. *Let $\mu, \sigma \in F^M$. Then $\mu \ll_F \sigma$ if and only if $\mu^* \ll \sigma^*$.*

THEOREM 3.9. *Any finite sum of fuzzy small submodules of M is also a fuzzy small submodule of M.*

THEOREM 3.10. *Let $N \leq M$ and $\mu \in F^M$ with $\mu \subseteq \chi_N$. If $\mu|_N \ll_F N$, then $\mu \ll_F M$.*

COROLLARY 3.11. *Let $\mu, \nu \in F^M$ with $\mu \subseteq \nu$. If $\mu \ll_F \nu$, then $\mu \ll_F M$.*

DEFINITION 3.12 (Fuzzy Direct Sum). A fuzzy submodule σ in M is a *fuzzy direct sum* of μ and ν if $\sigma = \mu \vee \nu$ and $\mu \cap \nu = \chi_\theta$.

DEFINITION 3.13 (Fuzzy Direct Summand). $\mu \in F^M$ is a *fuzzy direct summand* of M if there exists $\nu \in F^M$ such that χ_M is a fuzzy direct sum of μ and ν.

THEOREM 3.14. *Let μ, ν be fuzzy submodules of M with $\mu \subseteq \nu$ and ν a direct summand of M. Then $\mu \ll_F M$ if and only if $\mu \ll_F \nu$.*

THEOREM 3.15. *Let $\sigma_1, \sigma_2 \in F^M$ with $\chi_M = \sigma_1 \oplus \sigma_2$. Let $\mu_1 \subseteq \sigma_1$ and $\mu_2 \subseteq \sigma_2$. Then $\mu_1 \ll_F \sigma_1$ and $\mu_2 \ll_F \sigma_2$ if and only if $\mu_1 \oplus \mu_2 \ll_F \sigma_1 \oplus \sigma_2$.*

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Triangular Fuzzy Solution of the Fuzzy Heat Equation

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Abstract. In this paper, we find the strong solution of the fuzzy heat equation by using the Maximum Principle. We use the Maximum Principle for solving a triangular fuzzy dirichlet problem for the heat equation with regular fuzzy nonhomogenuous part and regular fuzzy dirichlet conditions. Finally, by one example, we show that, this method gives the strong fuzzy solution.

Keywords: Fuzzy partial differential equation, Heat equation, Maximum Principle, Triangular fuzzy number.

1. Introduction

Many chemical, physical and biological phenomena modeled by partial differential equations (PDEs). One of the most important partial differential equations is the heat equation. Since dynamical systems are vague and incomplete thus these phenomena are described by fuzzy partial differential equations (FPDEs). In [1], FPDEs are studied first by Buchley and Feuring. A new technique using an adaptive fuzzy algorithm to obtain the solutions to a class of partial differential equations is presented, in [2]. Systems of linear differential equations with crisp real coefficients and with initial condition described by a vector of fuzzy numbers are studied by in [3]. A new method based on the geometric representations of linear transformations is proposed to find a solution, in [4]. Gasilov and et al. [5], investigate linear partial differential equations with fuzzy source functions and with fuzzy initial value and boundary conditions. Also, they investigated linear partial differential equations with boundary values expressed by fuzzy numbers in [3].

2. Triangular Fuzzy Numbers

A triangular fuzzy number is a special type of fuzzy numbers with membership function $\mu_{\widetilde{A}}(x)$, as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_m-a_1}, & a_1 \leq x \leq a_m, \\ \frac{a_2-x}{a_2-a_m}, & a_m \leq x \leq a_2, \\ 0, & otherwise, \end{cases}$$

that can show it in triple form of $\widetilde{A} =< a_1, a_m, a_2 >$.

*Speaker.

3. Triangular Fuzzy Functions

In this section, we use a fuzzy function as a fuzzy set of real functions. In this representation each real function has a certain membership degree. Value of $\widetilde{F}(x)$ is a triangular fuzzy number for given x. We have

$$\mu_{\widetilde{F}(t)}(x) = \alpha \iff \exists\, y(*)\ st\ (\mu_{\widetilde{F}}(y) = \alpha\ \wedge\ y(t) = \alpha)$$
$$\wedge\ \forall\, z(*)\ st\ (\mu_{\widetilde{F}}(z) > \alpha \implies z(t) \neq \alpha)$$

where " $\implies$ " and " $\wedge$ " are the logical implication and conjunction symbols, respectively. We have the triangular fuzzy number as follows.

Let $F_1(*)$, $F_m(*)$ and $F_2(*)$, be continuous functions on an interval I, then we define membership function $\mu_{\widetilde{F}}(y(.))$ as

$$\mu_{\widetilde{F}}(y(.)) = \begin{cases} \alpha, & y = F_1 + \alpha(F_m - F_1), 0 \leq \alpha \leq 1, \\ \alpha, & y = F_2 + \alpha(F_m - F_2), 0 \leq \alpha \leq 1, \\ 0, & otherwise. \end{cases} \tag{1}$$

The fuzzy function $\widetilde{F}$, determined by the membership function (3.1), is called as triangular fuzzy function and denoted by $\widetilde{F} =< F_1, F_m, F_2 >$. According to definition a triangular fuzzy function is a fuzzy set of real functions. There are two functions of membership α, and there exist one function with membership degree 1.

A triangular fuzzy function is called ***regular triangular fuzzy function*** if

$$\forall x \in I,\ F_1(x) \leq F_m(x) \leq F_2(x).$$

We assume that our used triangular fuzzy functions, be regular.

Example 3.1. Fig 2, shows the triangular fuzzy function $\widetilde{F} =< F_1, F_m, F_2 >$, where $F_1(x) = x^3 - x - 2$, is the lower brunch (boundary) with membership degree of zero, $F_m(x) = x^3 - x - \frac{1}{10}\cos x$, is with membership degree of one and $F_2(x) = x^3 - x + 1$, is the upper brunch (boundary) with membership degree of zero.

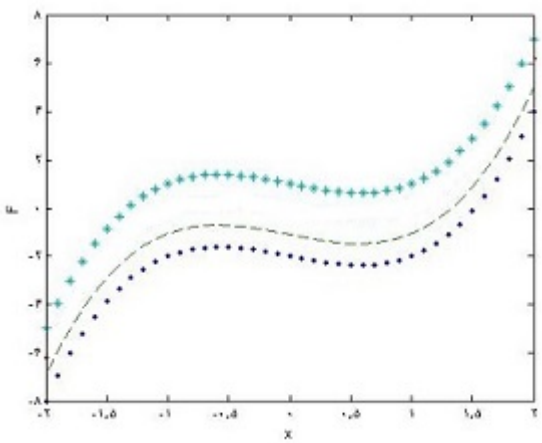

FIGURE 1. triangular fuzzy function $\widetilde{F} =< x^3 - x - 2, x^3 - x - \frac{1}{10}\cos x, x^3 - x + 1 >$.

4. Fuzzy Heat Equation

Let us consider heat equation with triangular fuzzy initial and boundary value problem

$$\begin{cases} u_t - au_{xx} = \widetilde{G}(x,t), \\ u(x,0) = \widetilde{\phi}(x), \\ u(0,t) = \widetilde{\psi}(t), \\ u(l,t) = \widetilde{F}(t), \end{cases} \tag{2}$$

where $a > 0$ is a constant and $\widetilde{G} =< G_1, G_m, G_2 >$, $\widetilde{\phi} =< \phi_1, \phi_m, \phi_2 >$, $\widetilde{\psi} =< \psi_1, \psi_m, \psi_2 >$ and $\widetilde{F} =< F_1, F_m, F_2 >$ are given triangular fuzzy functions.

Definition 4.1. Let $u(x,t) \in U$ be a real function and

$$\begin{cases} u_t - au_{xx} = g(x,t), \\ u(x,0) = \phi(x), \\ u(0,t) = \psi(t), \\ u(l,t) = f(t). \end{cases} \tag{3}$$

Also, let $\mu_1 = \mu_{\widetilde{G}}(g(*,*))$, $\mu_2 = \mu_{\widetilde{\phi}}(\phi(*))$, $\mu_3 = \mu_{\widetilde{\psi}}(\psi(*))$, $\mu_4 = \mu_{\widetilde{F}}(f(*))$ and $\mu = min\{\mu_1, \mu_2, \mu_3, \mu_4\}$. Then we say that the function $u(x,t)$ is an element of the solution set with the membership degree μ. We call the set of all functions $u(x,t)$ as a fuzzy solution of (2).

Theorem 4.2. *Let $u(x,t)$ be smooth and assume that $u_t - au_{xx} \leq 0$ be in the rectangular region $\{0 < x < l, 0 < t \leq T\}$ in space time domain. Then u gives its maximum either initially $(t = 0)$, or on the lateral sides $(x = 0$, or $x = l)$.*

Proof. [1] □

According to the Maximum principle, consider two below crisp problems

$$\begin{cases} u_t - au_{xx} = G_1(x,t), \\ u(x,0) = \phi_1(x), \\ u(0,t) = \psi_1(t), \\ u(l,t) = F_1(t), \end{cases} \tag{4}$$

and

$$\begin{cases} u_t - au_{xx} = G_m(x,t), \\ u(x,0) = \phi_m(x), \\ u(0,t) = \psi_m(t), \\ u(l,t) = F_m(t), \end{cases} \tag{5}$$

where $G_1(x,t) \leq G_m(x,t)$, $\phi_1(x) \leq \phi_m(x)$, $\psi_1(t) \leq \psi_m(t)$ and $F_1(t) \leq F_m(t)$. Subtracting (5) from (4) and denote $v(x,t) = u_1(x,t) - u_m(x,t)$, we have

$$\begin{cases} v_t - av_{xx} = G(x,t), \\ v(x,0) = \phi(x), \\ v(0,t) = \psi(t), \\ v(l,t) = F(t), \end{cases} \tag{6}$$

where $G(x,t) = G_1(x,t) - G_m(x,t) \leq 0$, $\phi(x) = \phi_1(x) - \phi_m(x) \leq 0$, $\psi(t) = \psi_1(t) - \psi_m(t) \leq 0$ and $F(t) = F_1(t) - F_m(t) \leq 0$. According to the maximum principle, we have $max 0 \leq x \leq l, 0 \leq t \leq T\ v(x,t) = max\{max_{0\leq x\leq l}\phi(x), max_{0\leq t\leq T}\psi(t), max_{0\leq t\leq T}F(t)\} \leq 0$, therefore $v(x,t) \leq 0$ and consequently $u_1(x,t) \leq u_m(x,t)$. Now assume

$$\begin{cases} u_t - au_{xx} = G_2(x,t), \\ u(x,0) = \phi_2(x), \\ u(0,t) = \psi_2(t), \\ u(l,t) = F_2(t), \end{cases} \tag{7}$$

where $G_m(x,t) \leq G_2(x,t)$, $\phi_m(x) \leq \phi_2(x)$, $\psi_m(t) \leq \psi_2(t)$ and $F_m(t) \leq F_2(t)$. Using maximum principle again for (6) and (7), with above process, we have $u_m(x,t) \leq u_2(x,t)$. Consequently, we have $u_1(x,t) \leq u_m(x,t) \leq u_2(x,t)$.

5. Example

In order to illustrate the performance of the presented method in solving fuzzy heat equation (2) and the effitiency of the method, the following examples are considered.

Example 5.1. Consider the nonhomogeneous heat equation

$$\begin{cases} u_t - u_{xx} = \widetilde{G}(x,t), \\ u(x,0) = \widetilde{\phi}(x), \\ u(0,t) = \widetilde{\psi}(t), \\ u(1,t) = \widetilde{F}(t), \end{cases} \tag{8}$$

where

$$a = 1,\ l = 1,\ T = 1,$$

nonhomogenous part :

$$\widetilde{G}(x,t) =< 10\cos(2t+x) + \sin(2t+x) - \frac{1}{10}(e^t(2+x-x^2) - 2)$$
$$,\ 10\cos(2t+x) + \sin(2t+x) - \frac{1}{20}(e^t(2+x-x^2) - 2)$$
$$,\ 10\cos(2t+x) + \sin(2t+x) + \frac{1}{10}(e^t(2+x-x^2) - 2) >,$$

fuzzy initial value :

$$\widetilde{\phi}(x) =< 5\sin x - \cos x,\ 5\sin x - \frac{1}{2}\cos x,\ 5\sin x + \cos x >,$$

fuzzy boundary values :

$$\widetilde{\psi}(t) =< 5\sin(2t) - e^{-t},\ 5\sin(2t) - \frac{1}{2}e^{2t},\ 5\sin(2t) - e^{-t} >,$$

and

$$\widetilde{F}(t) =< 5\sin(2t+1) - e^{-t}\cos 1,\ 5\sin(2t+1) - \frac{1}{2}e^{-t}\cos 1,\ 5\sin(2t+1) + e^{-t}\cos 1 > .$$

Now we must solve four below crisp problems. Using fourier transform method, we have

$$(9)\begin{cases} u_t - u_{xx} = G_1(x,t) = 10\cos(2t+x) + \sin(2t+x) - \frac{1}{10}(e^t(2+x-x^2)-2), \\ u(x,0) = \phi_1(x) = 5\sin x - \cos x, \\ u(0,t) = \psi_1(t) = 5\sin(2t) - e^{-t}, \\ u(1,t) = F_1(t) = 5\sin(2t+1) - e^{-t}\cos 1, \end{cases}$$

$$(10)\begin{cases} u_t - u_{xx} = G_m(x,t) = 10\cos(2t+x) + \sin(2t+x) - \frac{1}{20}(e^t(2+x-x^2)-2), \\ u(x,0) = \phi_m(x) = 5\sin x - \frac{1}{2}\cos x, \\ u(0,t) = \psi_m(t) = 5\sin(2t) - \frac{1}{2}e^{-t}, \\ u(1,t) = F_m(t) = 5\sin(2t+1) - \frac{1}{2}e^{-t}\cos 1, \end{cases}$$

and

$$(11)\begin{cases} u_t - u_{xx} = G_2(x,t) = 10\cos(2t+x) + \sin(2t+x) + \frac{1}{10}(e^t(2+x-x^2)-2), \\ u(x,0) = \phi_2(x) = 5\sin x + \cos x, \\ u(0,t) = \psi_2(t) = 5\sin(2t) + e^{-t}, \\ u(1,t) = F_2(t) = 5\sin(2t+1) + e^{-t}\cos 1, \end{cases}$$

where

$$u_1(x,t) = 5\sin(2t+x) - \frac{1}{10}(e^t-1)(x-x^2),\ u_m(x,t) = 5\sin(2t+x) - \frac{1}{20}(e^t-1)(x-x^2),$$

$$u_2(x,t) = 5\sin(2t+x) + \frac{1}{10}(e^t-1)(x-x^2),$$

are solutions of (9)-(11), respectively. obviously

$$u(x,t) =< -5\sin(2t+x) - \frac{1}{10}(e^t-1)(x-x^2)$$
$$,\ 5\sin(2t+x) - \frac{1}{20}(e^t-1)(x-x^2)$$
$$,\ 5\sin(2t+x) + \frac{1}{10}(e^t-1)(x-x^2) >,$$

is the fuzzy solution of the mentioned heat problem.

6. Conclusion

We can use the Maximum principle to solve any crisp problems with fuzzy kind of bundary and initial values.

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Unitary semi-ring on finite nexuses

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Abstract. This paper introduces and investigates a novel algebraic framework for structures composed of finite addresses, which we term *nexuses*. A nexus N is defined as a set of finite sequences (or addresses), and we define its order, $n(N)$, as the maximum integer value appearing among its constituent addresses. Within this context, we introduce two custom binary operations, $\oplus_n$ and $\odot_n$, parameterized by a natural number n. The central construction of this work is the generation of a semi-ring $\langle N\rangle_n$ from a given finite-order nexus N, where $n = n(N)$. This generated structure is shown to itself be a nexus, thereby preserving the core combinatorial properties of the original set while enriching it with a coherent algebraic architecture. Notably, the semi-ring $\langle N\rangle_n$ is unitary, with the multiplicative identity given by the unit address (1).

Keywords: Address, nexus, polynomial, semi-ring, unitary.

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1. Introduction

In 1980, Haristchain [3] in order to be able to conveniently handle the vast amount of varied data that defines a spatial structure, a sophisticated form of database was evolved which called a *plenix*. In 1984, Nooshin [5] defined the notion of a *nexus* as a mathematical object that represents the constitution of a plenix and a nexus was defined axiomatically by using the concept of the *address set*. In 2009, Bolourian's [1] introduced the notion of *nexus algebras* as an abstract algebraic structure and investigated its properties. Many authors work on nexuses and subnexuses. In 2018, Norouzi works on subnexuses on N-structure [6]. In 2019, Kamrani gave a structure of moduloid on a *nexus* and verified the concepts of *submoduloids*, *finitely generated submoduloids* and *prime submoduloids* on a *nexus* [2]. Recently, V. Nazemi Niya, H. Babaei and A. Rezaei [4] define a semi-ring on a nexus N and characterize all prime ideals of N and gave some properties of semi-ring homomorphism of nexuses. Further, they verify the concepts of quotient and localization on a semi-ring associated with a nexus N. But this semi-ring is not unitary unless the nexus N be a cyclic nexus and in this case if $N = \langle a\rangle$ be a cyclic nexus, $1 = a$.

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In this paper, we define a unitary semi-ring on some nexuses. At first, we assume that $n \in \mathbb{N}$ and a, b be two finite addresses. We define binary operations $a \oplus_n b$ and $a \odot_n b$. Let N be a nexuse with finite addresses such that $n(N) := \max\{a \in \mathbb{N} :$ there exist $(a_0, a_1, \cdots, a_k) \in N$ and $1 \leqslant i \leqslant k$, such that $a_i = a\}$ be finite number. Then we put $n := n(N)$ and assume $\langle N \rangle_n$ be a semi-ring generated on N with binary operations $a \oplus_n b$ and $a \odot_n b$ and also it is a nexuse. This semi-ring is unitary with $1 = (1)$. Then we verify some properties of this semi-ring.

2. Polynomials over a nexus N

Definition 2.1. Let $\mathbb{N}^* = \{0\} \cup \mathbb{N}$ and $0 \neq n \in \mathbb{N}$. We define the map $[\ ]_n$ on $\mathbb{N}^* = \{0\} \cup \mathbb{N}$ such that for every $a \in \mathbb{N}$, if $0 \leqslant a \lneqq n$, $[a]_n = a$ and if $a \geqslant n$, $[a]_n = n$.

Lemma 2.2. *Let $a, b, n \in \mathbb{N}$ with $a \leqslant b$, then $[a]_n \leqslant [b]_n$.*

Definition 2.3. For every $a, b \in \mathbb{N}$, we define $a+b = a \vee b = \max\{a, b\}$ and $a \cdot b = [ab]_n$. We denote the set $\mathbb{N}$ with actions $+$ and $\cdot$ by $\mathbb{N}_n$.

Theorem 2.4. *Let $n \in \mathbb{N}$ be arbitrary. Then $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.*

Hearifter, $\mathbb{N}_n$ is a semi-ring, it means, $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.

The following example shows that for every $n \in \mathbb{N}$, $(\mathbb{N}_n, +, \cdot, 0)$ is not a unitary semi-ring.

Example 2.5. Let $n < m \in \mathbb{N}$. So $1.m = [1.m]_n = [m]_n = n \neq m$. Hence $(\mathbb{N}_n, +, \cdot, 0)$ is not a unitary semi-ring.

Theorem 2.6. *Let R be a semi-ring. Then $R[x]$, the set of all polynomials on R is a semi-ring. If R is unitary semi-ring, then $R[x]$ is unitary semi-ring.*

Proposition 2.7. *The set $\mathbb{N}_n[x]$ of all polynimials on $\mathbb{N}_n$ is a semi-ring.*

Let $f(x) = \sum_{i=1}^{n} a_i x^i, g(x) = \sum_{i=1}^{m} b_i x^i \in \mathbb{N}_n[x]$. We have $f(x)g(x) = \sum_{i=0}^{n+m} c_i x^i$ such that $c_i = \bigvee_{j=0}^{i} [a_{i-j} b_j]_n$.

Example 2.8. Let $f(x) = 1 + 3x^2 + x^3, g(x) = 4x + 6x^2 \in \mathbb{N}_5[x]$.

$f(x) + g(x) = (0+1) + (0+4)x + (3+6)x^2 + (1+0)x^3 = (0 \vee 1) + (0 \vee 4)x + (3 \vee 6)x^2 + (1 \vee 0)x^3 = 1 + 4x + 6x^2 + x^3$.

$f(x)g(x) = (1.0) + (1.4 + 0.0)x + (1.6 + 0.4 + 3.0)x^2 + (1.0 + 0.6 + 3.4 + 1.0)x^3 = [0]_5 + ([4]_5 \vee [0]_5)x + ([6]_5 \vee [0]_5 \vee [0]_5)x^2 + ([0]_5 \vee [0]_5 \vee [12]_5 \vee [0]_5)x^3 = 1 + 4x + 5x^2 + 5x^3$.

Definition 2.9. Let R be a semi-ring and $R[x]$ be the semi-ring of all polynomials on R. We put $R\lfloor x \rfloor = \{f(x) = \sum_{i=1}^{m} a_i x^i \in R[x] : a_i \neq 0, 1 \leqslant i \leqslant m\}$.

Proposition 2.10. *Let R be a semi-ring such that for every $0 \neq a, b \in R$, $a+b \neq 0$ and $ab \neq 0$. and $R[x]$ be the semi-ring of all polynomials on R. Then $R\lfloor x \rfloor$ is a sub-semi-ring of $R[x]$.*

Proposition 2.11. *$\mathbb{N}_n\lfloor x \rfloor$ is a sub-semi-ring of $\mathbb{N}_n[x]$.*

Example 2.12. Let $f(x) = 1 + 6x + 3x^2 + x^3, g(x) = 8 + 4x + 6x^2 \in \mathbb{N}_7[x]$.

$f(x) + g(x) = (1+8) + (6+4)x + (3+6)x^2 + (1+0)x^3 = (1 \vee 8) + (6 \vee 4)x + (3 \vee 6)x^2 + (1 \vee 0)x^3 = 8 + 6x + 6x^2 + x^3$.

$f(x)g(x) = (1.8) + (1.4 + 6.8)x + (1.6 + 6.4 + 8.3)x^2 + (1.0 + 6.6 + 3.4 + 1.8)x^3 = [8]_7 + ([4]_7 \vee [48]_7)x + ([6]_7 \vee [24]_7 \vee [24]_7)x^2 + ([0]_7 \vee [36]_7 \vee [12]_7 \vee [8]_7)x^3 = 7 + (4 \vee 7)x + (6 \vee 7 \vee 7)x^2 + (0 \vee 7 \vee 7 \vee 7)x^3 = 7 + 7x + 7x^2 + 7x^3$.

Definition 2.13. Let N be a nexus and put

$$n(N) := \max\big\{a \in \mathbb{N} : \text{ there exist } (a_0, a_1, \cdots, a_k) \in N \text{ and } 1 \leqslant i \leqslant k, \text{ such that } a_i = a\big\}.$$

Define $\varphi_n : N \longrightarrow \mathbb{N}_n[x]$ by $\varphi_n\big((a_0, a_1, \cdots, a_k)\big) = \sum_{i=1}^{k} a_i x^i$ for all $(a_0, a_1, \cdots, a_k) \in N$.

Let N be a nexus and $a = (a_0, a_1, \cdots, a_k), b = (b_0, b_1, \cdots, b_t) \in N$ be two addresses of N. We define $a+b = (a_0 \vee b_0, a_1 \vee b_1, ..., a_l \vee b_l)$, where $l = k \vee t$ and $a \odot_n b = (c_0, c_1, \cdots, c_{k+t})$ such that $c_i = \bigvee_{j=0}^{i} [a_{i-j} b_j]_n$, where $n := n(N)$.

Example 2.14. Let $a = (1, 6, 3, 1)$, $b = (8, 4, 6)$ and $N = \langle a, b\rangle$, then $n(N) = 8$. Hence $a + b = \big((1 \vee 8), (6 \vee 4), (3 \vee 6), (1 \vee 0)\big) = (8, 6, 6, 1)$ and

$a \odot_8 b = \big((1.8), (1.4 + 6.8), (1.6 + 6.4 + 8.3), (1.0 + 6.6 + 3.4 + 1.8)\big) = \big([8]_8, \big([4]_8 \vee [48]_8\big), \big([6]_8 \vee [24]_8 \vee [24]_8\big), \big([0]_8 \vee [36]_8 \vee [12]_8 \vee [8]_8\big)\big) = \big((8, (4 \vee 8), (6 \vee 8 \vee 8), (0 \vee 8 \vee 8 \vee 8)\big) = (8, 8, 8, 8)$.

Theorem 2.15. *Let N be a nexus and $\langle N\rangle_n$ be a nexus closed under $\odot_n$ and closed under nexus. Then $\big(\langle N\rangle_n, +, \odot_n, (), (1)\big)$ is a unitary semi-ring.*

Example 2.16. Let $m \in \mathbb{N}$ and $N = \{(), (1), (2), \cdots (m)\}$. Then $n(N) = m$ and $\langle N\rangle_m = \{(), (1), (2), \cdots, (m)\} = N$.

Example 2.17. Let $N = \big\{(), (1), (2), (1, 1), (1, 2)\big\}$. Then $n(N) = 2$ and
$\langle N\rangle_2 = \big\{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (1, 1, 1), (1, 2, 1), (1, 2, 2), (2, 2, 1), (2, 2, 2), (1, 1, 1, 1), (1, 2, 2, 1), (1, 2, 2, 2), (2, 2, 2, 1), (2, 2, 2, 2), (1, 1, 1, 1, 1), \cdots\big\}$.
Be careful that $(2, 1, 1), (1, 1, 2) \notin \langle N\rangle_2$.

3. Some properties of semi-ring $\langle N\rangle_n$

Lemma 3.1. *Let a, b be two addresses with $l(a) = k$ and $l(b) = m$. Then $l(a \odot b) = k + m - 1$.*

Example 3.2. Let $a = (2, 3, 1)$ and $b = (3, 5, 2, 1)$. By Lemma 3.1, $l(a \odot_5 b) = 3 + 4 - 1 = 6$. In fact, $a \odot_5 b = \big(2 + 3x + x^2\big)\big(3 + 5x + 2x^2 + x^3\big) = [6]_5 + [10 \vee 9]_5 x + [4 \vee 15 \vee 3]_5 x^2 + [2 \vee 6 \vee 5]_5 x^3 + [3 \vee 2 \vee 5]_5 x^4 + [1]_5 x^5 = (5, 5, 5, 5, 5, 1)$.

Proposition 3.3. *Let N be a nexus with $n(N) = n$ and $s \in \mathbb{N}$ such that $s \leqslant n$. Then $(1, s) \in \langle N\rangle_n$ if and only if $(1, s) \in N$.*

Proposition 3.4. *Let N be a nexus with $n(N) = n$, $s_1, s_2 \in \mathbb{N}$ such that $s_1, s_2 \leqslant n$, $s_2 \leqslant s_1^2$ and $(1, s_1) \in N$. Then $(1, s_1, s_2) \in \langle N\rangle_n$ if and only if $(1, s_1, s_2) \in N$ or $1 \leqslant s_2 \leqslant s_1^2$.*

Example 3.5. Let N be cyclic nexus $N = \big\langle(1, 10)\big\rangle$. By Proposition 3.4, we have:

(I) $(1, 1, 1) \in \langle N\rangle_{10}$, but for every $2 \leqslant j \leqslant 10$, $(1, 1, j) \notin \langle N\rangle_{10}$.
(II) For every $1 \leqslant i \leqslant 4$, $(1, 2, i) \in \langle N\rangle_{10}$, but for every $5 \leqslant i \leqslant 10$, $(1, 2, i) \notin \langle N\rangle_{10}$.
(III) For every $1 \leqslant i \leqslant 9$, $(1, 3, i) \in \langle N\rangle_{10}$, but $(1, 3, 10) \notin \langle N\rangle_{10}$.
(IV) For every $4 \leqslant k \leqslant 10$ and for every $1 \leqslant i \leqslant 10$, $(1, k, i) \in \langle N\rangle_{10}$.

Proposition 3.6. *Let N be a nexus with $n(N) = n$, $s_1, s_2, s_3 \in \mathbb{N}$ such that $s_1, s_2, s_3 \leqslant n$. Then $(1, s_1, s_2, s_3) \in \langle N\rangle_n$ is product of two addresses of N if and only if there exist $s', s'' \in \mathbb{N}$ with $s', s'' \leqslant n$ such that one of the following coditions hold:*

(I) *$s' \leqslant s_1$ and $s'' \leqslant s'^2$ such that $s_2 = [s's_1]_n$ and $s_3 = [s''s_1]_n$.*
(II) *$s' \leqslant s_1$ and $s'' \leqslant s's_1 \leqslant s_1^2$ such that $s_2 = [s's_1]_n$ and $s_3 = [s's'']_n$.*
(III) *$s' \leqslant s_1$ and $s's_1 \leqslant s'' \leqslant s'^2$ such that $s_2 = s''$ and $s_3 = [s's'']_n$.*

Corollary 3.7. *Let $s \in \mathbb{N}$, N be a nexus and $n = s^3$. Then every $(1, s_1, s_2, s_3) \in \langle N \rangle_n$ is of the form:*

(I) *If $s_1 = s, 1 \leqslant k \leqslant s, s_2 = ks$, then $1 \leqslant s_3 \leqslant k^2 s$.*
(II) *If $s_1 = s, 1 \leqslant k \leqslant s-1, ks \lneqq s_2 \lneqq (k+1)s$, then $1 \leqslant s_3 \leqslant ks_2$.*

Example 3.8. Let N be a nexus, $(1,3) \in N$ and $n = 27$. All addresses of the form $(1,3,s_2,s_3) \in \langle N \rangle_{27}$ are:
$(1,3,3,i), 1 \leqslant i \leqslant 3$, $(1,3,6,i), 1 \leqslant i \leqslant 12$, $(1,3,9,i), 1 \leqslant i \leqslant 27$, $(1,3,4,i), 1 \leqslant i \leqslant 4$, $(1,3,5,i), 1 \leqslant i \leqslant 5$, $(1,3,7,i), 1 \leqslant i \leqslant 14$, $(1,3,8,i), 1 \leqslant i \leqslant 16$.

Proposition 3.9. *Let N be a nexus with $n(N) = n^4$, $s_1, s_2, s_3, s_4 \in \mathbb{N}$ such that $s_1, s_2, s_3, s_4 \leqslant n$. Then $(1, s_1, s_2, s_3, s_4) \in \langle N \rangle_n$ is product of two addresses of N if and only if*

(I) $s_1 = s, s_2 = ks, s_3 = kss', s_4 = j$, $1 \leqslant k \leqslant s$, $1 \leqslant s' \leqslant k$, $1 \leqslant j \leqslant k^2 ss'$.
(II) $s_1 = s, s_2 = ss', s_3 = i, s_4 = j$, $1 \leqslant k \leqslant s' \leqslant s$, $kss' \leqslant i \leqslant (k+1)ss'$, $1 \leqslant j \leqslant is'$.
(III) $s_1 = s, s_2 = ss', s_3 kss', s_4 = j$, $1 \leqslant j \leqslant k^2 ss'$.
(IV) $s_1 = s, s_2 = ss' \vee i, s_3 = sj \vee s'i, s_4 = t$, $1 \leqslant i \leqslant s^2$, $1 \leqslant j \leqslant s'^2$, $1 \leqslant t \leqslant ij$.

Example 3.10. We have:
$\big\langle (1,3) \big\rangle_3 = \big\{ (), (1), (1,j), (1,1,1), (1,2,j), (1,3,j), (1,1,1,1), (1,2,2,i), (1,2,3,j), (1,3,3,j), (1,1,1,1,1), (1,2,2,2,i), (1,2,3,2,1), (1,3,3,3,j), (1,1,1,1,1,1), (1,2,2,2,2,i), (1,2,3,3,2,1), (1,2,3,3,3,j), (1,3,3,3,3,j), \cdots \big\}$.
Where $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 3$.

4. Conclusion

We established that the semi-ring $(\mathbb{N}_n, \oplus_n, \odot_n, 0)$, defined on the set of natural numbers up to n with the specified operations, is *not unitary*. This absence of a multiplicative identity highlights a fundamental limitation in its algebraic structure, restricting the range of standard ring-theoretic concepts that can be applied. The central and more significant result of this study is the successful construction of a *unitary semi-ring* from a suitably closed nexus. Specifically, we have proven that if N is a nexus and its generated structure $\langle N \rangle_n$ is closed under both the multiplicative operation $\odot_n$ and the combinatorial principle of nexus formation, then the resulting system $\big(\langle N \rangle_n, \oplus_n, \odot_n, (), (1)\big)$ constitutes a unitary semi-ring. The identity element for the multiplicative operation $\odot_n$ is conclusively shown to be the address (1).

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Modeling Uncertainty in Logistics: Application of Interval-Valued Trapezoidal Fuzzy Numbers in Transportation Optimization

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Abstract: This research models the transportation problem in an imprecise environment using interval-valued trapezoidal fuzzy numbers (IVTrFNs). The primary innovation lies in optimization based on a weighted linear ranking function and the development of some of its theoretical concepts. This function converts IVTrFNs parameters into crisp values by incorporating the decision-maker's risk preference through the parameter z, thereby reducing the need for complex computations. The efficacy of the proposed method is validated through a numerical example involving two sources and three destinations. The results indicate that a shift in the decision-maker's attitude from conservative (z=1) to risk-seeking (z=0) can lead to an over threefold increase in the total cost. This finding underscores the importance of balancing risk and cost in imprecise environments and provides a practical framework for managers to design optimal transportation strategies.

Keywords: Interval-valued trapezoidal fuzzy number, Ranking method, Transportation problem.

1. Introduction

In recent years, numerous fuzzy models have been developed for the transportation problem. Liu and Kao [7] were the first to formulate the fuzzy transportation problem. Chandran and Kandaswamy [2] proposed a method for solving the problem with fuzzy cost, supply, and demand. Saini et al. [9] introduced a novel ranking method for solving the unbalanced fuzzy transportation problem using generalized triangular and trapezoidal fuzzy numbers. Baykasoglu and Subulan [1] employed the concept of constrained fuzzy arithmetic operations to develop a direct method based on the fuzzy ranking technique for solving the fully fuzzy transportation problem. In the domain of interval-valued fuzzy number modeling, Gupta and Kumar [6] and Ebrahimnejad [4] investigated multi-objective and linear problems, respectively. More recently, Ebrahimnejad et al. [5] utilized a novel optimization structure to solve the problem with interval-valued triangular fuzzy parameters. Despite these advancements, limited attention has been

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given to solving transportation problems using interval-valued trapezoidal fuzzy numbers. Focusing on this gap, this study employs a weighted ranking method to solve the transportation problem with interval-valued trapezoidal fuzzy parameters.

2. Basic Concepts

Definition 2,1. Let $\tilde{G}^l$ and $\tilde{G}^r$ be two trapezoidal fuzzy numbers at levels $\underline{\lambda}_{\tilde{G}}$ and $\overline{\lambda}_{\tilde{G}}$ on $\mathbb{R}$, respectively. An IVTrFN is represented as $\tilde{G} = \langle \tilde{G}^l, \tilde{G}^r \rangle = \langle \left(\underline{g}_1, \underline{g}_2, \underline{g}_3, \underline{g}_4; \underline{\lambda}_{\tilde{G}}\right), (\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4; \overline{\lambda}_{\tilde{G}}) \rangle$ at level $\langle \underline{\lambda}_{\tilde{G}}, \overline{\lambda}_{\tilde{G}} \rangle$, with the following lower and upper membership functions [5, 8]:

$$\mu_{\tilde{G}^l}(x) = \begin{cases} \dfrac{\left(x - \underline{g}_1\right)\underline{\lambda}_{\tilde{G}}}{\underline{g}_2 - \underline{g}_1}, & \underline{g}_1 \le x < \underline{g}_2 \\ \underline{\lambda}_{\tilde{G}}, & \underline{g}_2 \le x < \underline{g}_3 \\ \dfrac{\left(\underline{g}_4 - x\right)\underline{\lambda}_{\tilde{G}}}{\underline{g}_4 - \underline{g}_3}, & \underline{g}_3 \le x < \underline{g}_4 \\ 0, & OW \end{cases}, \quad \mu_{\tilde{G}^r}(x) = \begin{cases} \dfrac{(x - \overline{g}_1)\overline{\lambda}_{\tilde{G}}}{\overline{g}_2 - \overline{g}_1}, & \overline{g}_1 \le x < \overline{g}_2 \\ \overline{\lambda}_{\tilde{G}}, & \overline{g}_2 \le x < \overline{g}_3 \\ \dfrac{(\overline{g}_4 - x)\overline{\lambda}_{\tilde{G}}}{\overline{g}_4 - \overline{g}_3}, & \overline{g}_3 \le x < \overline{g}_4 \\ 0, & OW \end{cases} \quad (1)$$

Definition 2,2. Let $\tilde{G} = \langle \tilde{G}^l, \tilde{G}^r \rangle = \langle \left(\underline{g}_1, \underline{g}_2, \underline{g}_3, \underline{g}_4; \underline{\lambda}\right), (\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4; \overline{\lambda}) \rangle$ and $\tilde{H} = \langle \tilde{H}^l, \tilde{H}^r \rangle = \langle \left(\underline{h}_1, \underline{h}_2, \underline{h}_3, \underline{h}_4; \underline{\lambda}\right), (\overline{h}_1, \overline{h}_2, \overline{h}_3, \overline{h}_4; \overline{\lambda}) \rangle$ be two IVTrFNs at levels $\underline{\lambda}$ and $\overline{\lambda}$ on $\mathbb{R}$, and let δ be a non-negative scalar. The following arithmetic operations are defined [5, 8]:

1) $\tilde{G} + \tilde{H} = \langle \left(\underline{g}_1 + \underline{h}_1, \underline{g}_2 + \underline{h}_2, \underline{g}_3 + \underline{h}_3, \underline{g}_4 + \underline{h}_4; \underline{\lambda}\right), (\overline{g}_1 + \overline{h}_1, \overline{g}_2 + \overline{h}_2, \overline{g}_3 + \overline{h}_3, \overline{g}_4 + \overline{h}_4; \overline{\lambda}) \rangle$ (2)

2) $\delta\tilde{G} = \langle \left(\delta\underline{g}_1, \delta\underline{g}_2, \delta\underline{g}_3, \delta\underline{g}_4; \underline{\lambda}\right), (\delta\overline{g}_1, \delta\overline{g}_2, \delta\overline{g}_3, \delta\overline{g}_4; \overline{\lambda}) \rangle$ (3)

3. Mathematical Model of the Interval-valued Transportation Problem

The mathematical formulation of the interval-valued fuzzy transportation problem is given by Model (4). The primary objective is to determine the optimal shipment plan that minimizes total cost, where all parameters (costs, supply, and demand) are defined as IVTrFNs.

$$\begin{aligned} & min\ O = \textstyle\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij} y_{ij} \\ & s.t.\ \ \textstyle\sum_{j=1}^{n} y_{ij} = \tilde{a}_i,\ i = 1,2,\dots,m \\ & \textstyle\sum_{i=1}^{m} y_{ij} = \tilde{b}_j,\ j = 1,2,\dots,n \\ & y_{ij} \ge 0 \end{aligned} \quad (4)$$

Here, $\tilde{c}_{ij}$ is the IVTrF cost for transporting a unit from source i to destination j. The supply at each source and demand at each destination are known IVTrFNs $\tilde{a}_i$ and $\tilde{b}_j$.

4. Proposed Solution Methodology

Let $\tilde{G} = \langle \tilde{G}^l, \tilde{G}^r \rangle$ be an IVTrFN. Based on [3], the initial and end points of $\tilde{G}$ are calculated as:

$$S_z(\tilde{G}^l) = \frac{1}{2}\left\{z\underline{\lambda}\left(\underline{g}_1 + \underline{g}_2\right) + (1 - z)\underline{\lambda}\left(\underline{g}_3 + \underline{g}_4\right)\right\}, \quad (5)$$

$$S_z(\tilde{G}^r) = \frac{1}{2}\left\{z\overline{\lambda}(\overline{g}_1 + \overline{g}_2) + (1 - z)\overline{\lambda}(\overline{g}_3 + \overline{g}_4)\right\} \quad (6)$$

where $0 \le z \le 1$ is a parameter chosen by the decision-maker. The overall ranking is

$$S_z(\tilde{G}) = \frac{1}{4}\left\{z\left(\underline{\lambda}\left(\underline{g}_1 + \underline{g}_2\right) + \overline{\lambda}(\overline{g}_1 + \overline{g}_2)\right) + (1 - z)\left(\underline{\lambda}\left(\underline{g}_3 + \underline{g}_4\right) + \overline{\lambda}(\overline{g}_3 + \overline{g}_4)\right)\right\} \quad (7)$$

Theorem 4.1. The ranking function $S_z(\tilde{G})$ is linear: $S_z(\delta\tilde{G} + \tilde{H}) = \delta S_z(\tilde{G}) + S_z(\tilde{H})$.

Theorem 4.2. $S_z(\tilde{G})$ is monotonically increasing with respect to the all values of $\tilde{G} = \langle \tilde{G}^l, \tilde{G}^r \rangle$.

Theorem 4.3. $S_z(\tilde{G})$ is monotonically decreasing with respect to the parameter z.
This decreasing behavior occurs because a higher z assigns more weight to the lower bounds of the fuzzy interval, which typically represent less costly or more certain outcomes.
Solution Procedure.

i) All IVTrFN parameters (cost, supply, and demand) are converted into crisp values using the weighted linear ranking function $S_z(\tilde{G})$, based on the decision-maker's risk preference parameter z.

ii) The resulting crisp linear programming model is solved using MATLAB 2018 to obtain the optimal solution (shipment quantities and minimum total cost).

5. Numerical results

Consider a company with two sources s_1, s_2 and three destinations D_1, D_2 and D_3. The IVTrFN data, structured according to Model (4), are as follows [5]:

$$\tilde{C}_{11} = \langle[10,12,13,14;0.5],[9,12,13,15;1]\rangle, \tilde{C}_{21} = \langle[2,3,7,8;0.5],[1,3,7,9;1]\rangle$$
$$\tilde{C}_{12} = \langle[8,10,11,12;0.5],[7,10,11,14;1]\rangle, \tilde{C}_{22} = \langle[4,8,10,11;0.5],[3,8,10,12;1]\rangle$$
$$\tilde{C}_{13} = \langle[10,12,13,14;0.5],[9,12,13,16;1]\rangle, \tilde{C}_{23} = \langle[5,7,8,9;0.5],[4,7,8,10;1]\rangle$$
$$\tilde{a}_1 = \langle[2000,3000,4000,4500;0.5],[1500,3000,4000,5000;1]\rangle$$
$$\tilde{a}_2 = \langle[1200,1500,1800,1900;0.5],[1000,1500,1800,2000;1]\rangle$$
$$\tilde{b}_1 = \langle[1500,2000,2500,2700;0.5],[950,2000,2500,3000;1]\rangle$$
$$\tilde{b}_2 = \langle[900,1500,2000,2300;0.5],[850,1500,2000,2500;1]\rangle$$
$$\tilde{b}_3 = \langle[800,1000,1300,1400;0.5],[700,1000,1300,1500;1]\rangle$$

First, for different values of $z = 0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1$, the corresponding ranking value for each number is calculated. A summary of the results is presented in Table 1. The parameter z acts as the decision-maker's weight for combining the lower and upper parts of the fuzzy intervals. A value of $z = 1$ places maximum emphasis on the lower interval values, reflecting a conservative, risk-averse approach, while $z = 0$ highlights the upper interval values, reflecting a higher risk tolerance.

Table 1. Ranking of costs, supplies, and demands.

	$z=0$	$z=0.1$	$z=0.2$	$z=0.3$	$z=0.4$	$z=0.5$	$z=0.6$	$z=0.7$	$z=0.8$	$z=0.9$	$z=1$
$\tilde{C}_{11}$	10.375	10.138	9.9	9.6625	9.425	9.1875	8.95	8.7125	8.475	8.2375	8
$\tilde{C}_{12}$	9.125	8.8625	8.6	8.3375	8.075	7.8125	7.55	7.2875	7.025	6.7625	6.5
$\tilde{C}_{13}$	10.625	10.363	10.1	9.8375	9.575	9.3125	9.05	8.7875	8.525	8.2625	8
$\tilde{C}_{21}$	5.875	5.45	5.025	4.6	4.175	3.75	3.325	2.9	2.475	2.05	1.625
$\tilde{C}_{22}$	8.125	7.7375	7.35	6.9625	6.575	6.1875	5.8	5.4125	5.025	4.6375	4.25
$\tilde{C}_{23}$	6.625	6.3875	6.15	5.9125	5.675	5.4375	5.2	4.9625	4.725	4.4875	4.25
$\tilde{a}_1$	3312.5	3156.3	3000	2843.8	2687.5	2531.3	2375	2218.8	2062.5	1906.3	1750
$\tilde{a}_2$	1412.5	1367.5	1322.5	1277.5	1232.5	1187.5	1142.5	1097.5	1052.5	1007.5	962.5
$\tilde{b}_1$	2025	1940	1855	1770	1685	1600	1515	1430	1345	1260	1175
$\tilde{b}_2$	1662.5	1585	1507.5	1430	1352.5	1275	1197.5	1120	1042.5	965	887.5
$\tilde{b}_3$	1037.5	998.75	960	921.25	882.5	843.75	805	766.25	727.5	688.75	650

By modeling and solving each of these scenarios, the optimal solution and objective value are obtained, as shown in Table 2. As z increases from 0 to 1, the total transportation cost decreases from 36,806 units to 11,458 units. This trend indicates that decisions based on high risk-taking (weighting the upper values of costs, supply, and demand) lead to a significant increase in costs. The present analysis provides a basis for modeling transportation problems in real-world environments with imprecise data and facilitates the adoption of flexible strategies based on risk preferences.

Table 2. Optimal solution and objective value under different z scenarios.

	$z=0$	$z=0.1$	$z=0.2$	$z=0.3$	$z=0.4$	$z=0.5$	$z=0.6$	$z=0.7$	$z=0.8$	$z=0.9$	$z=1$
y_{11}	1650	1571.3	532.5	492.5	452.5	412.5	372.5	332.5	292.5	252.5	212.5
y_{12}	0	0	960	921.25	882.5	843.75	805	766.25	727.5	688.75	650
y_{13}	0	0	0	0	0	0	0	0	0	0	0
y_{21}	1662.5	1585	1507.5	1430	1352.5	1275	1197.5	1120	1042.5	965	887.5
y_{22}	375	368.75	1322.5	1277.5	1232.5	1187.5	1142.5	1097.5	1052.5	1007.5	962.5
y_{23}	1037.5	998.75	0	0	0	0	0	0	0	0	0
O^*	36806	33800	30823	27912	25141	22511	20020	17669	15459	13388	11458

6. Conclusion

This study presented an effective framework for modeling and solving the transportation problem in imprecise environments by employing IVTrFNs and a weighted linear ranking function. The numerical results clearly demonstrated that the decision-maker's risk preference, parameter z, has a direct and significant impact on the total cost and the optimal transportation plan.

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A Theoretical and Computational Study of a Defuzzification Function for the Generalized Fuzzy mTSP

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Abstract: This study presents a parametric approach for solving the Single-Depot Multiple Traveling Salesman Problem (mTSP) in a generalized trapezoidal fuzzy environment. The classical mTSP involves a set of cities and a number of salesmen, all of whom start from and return to a single depot (or home base), with each city required to be visited exactly once by one salesman. Our proposed algorithm, specifically designed for the single-depot mTSP, utilizes a two-stage architecture: in the first stage, a defuzzification process is applied to the fuzzy parameters, and in the second stage, the problem is formulated as a mathematical model and solved using MATLAB 2018. The primary innovation lies in the design of a numerical example alongside the development of related theorems concerning the defuzzification function. Furthermore, numerical results from its application to a seven-city problem with fuzzy costs are presented.

Keywords: Fuzzy number, Mixed Integer Linear Programming, Defuzzification function, mTSP.

1. Introduction

Solution methodologies for the mTSP are broadly classified into exact and approximate algorithms. Exact methods, such as mathematical programming and branch-and-bound, guarantee optimality but become computationally prohibitive for large-scale instances [6]. Conversely, approximate methods; encompassing heuristics and metaheuristics find high quality solutions within reasonable timeframes without this guarantee. A comprehensive survey of mTSP solutions was conducted by Bektas [2]. Among metaheuristic approaches, Yousefikhoshbakht et al. [8] developed a modified Ant Colony Optimization (ACO) algorithm. Camci [3] introduced a generalized formulation for the TSP. Other applied strategies include

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the mixed-integer linear programming model by Baltz et al. [1] and the two-phase heuristic proposed by Xu et al. [7].
Problem complexity increases when parameters such as travel costs are uncertain. In such environments, fuzzy systems provide an effective framework for uncertainty modeling. Although fuzzy logic has a history in optimization, the use of generalized trapezoidal fuzzy numbers, which accommodate varying membership levels, remains less explored. Ivohin and Yoshtin [5] presented a two-stage method for the fuzzy TSP, applying clustering followed by a genetic algorithm for route optimization. However, ranking functions with robust theoretical underpinnings have been relatively neglected in this context.

In this research, the travel cost between cities is modeled using generalized trapezoidal fuzzy numbers. These fuzzy parameters are then transformed into an equivalent deterministic model by employing a theoretically developed defuzzification method. The paper is structured as follows: Section 2 provides the problem definition, necessary notations, and the mathematical formulation of the mTSP in a fuzzy environment. Section 3 is dedicated to the development of the theoretical concepts. The numerical results from the experiment are presented in Section 4, and finally, the conclusion is provided in Section 5.

2. Problem Definition and Notation

The single-depot mTSP is a variant where all salesmen start and end at a common depot. The problem is defined on a complete directed graph G = (V, A), where V is the set of n nodes (cities) and A is the set of arcs connecting them. In our proposed mTSP, exact travel costs are uncertain and may vary due to factors like vehicle type or fuel consumption. Consequently, the problem is represented using a generalized trapezoidal fuzzy environment. Each arc (i, j) is assigned a non-negative, possibly asymmetric cost, denoted by the generalized trapezoidal fuzzy number $\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle$. Asymmetry means the cost from city i to j can differ from j to i. City 1 is designated as the depot. We formulate the problem using a Mixed-Integer Linear Programming (MILP) model.

Model (1):

$$\min \tilde{z} = \sum_{(i,j)\epsilon A} \tilde{c}_{ij} x_{ij}$$

$$\sum_{j=2}^{n} x_{1j} = m, \sum_{j=2}^{n} x_{j1} = m$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 2,3,\dots,n, \sum_{j=1}^{n} x_{ij} = 1, \quad i = 2,3,\dots,n,$$

$$u_i + (w-2)x_{1i} - x_{i1} \le w-1, \quad u_i + x_{1i} + (2-k)x_{i1} \le w-1; \; i = 2,3,\dots,n,$$

$$x_{ij}\epsilon\{0,1\}, \quad (i,j)\epsilon A.$$

Variables and Parameters:

1. m: Total number of salesmen.
2. x_{ij}: A binary decision variable that equals 1 if arc (i, j) is traversed in the solution, and 0 otherwise.
3. $\tilde{c}_{ij}$: The generalized trapezoidal fuzzy travel cost from city i to city j.
4. u_i: A continuous auxiliary variable used to eliminate subtours and control tour length, representing the sequence of visiting cities.
5. k, w: The minimum and maximum number of cities a salesman may visit, ensuring workload balance.

Objective Function: Minimize the sum of the total fuzzy travel costs for all salesmen.
Constraints:

1. Depot Flow: Exactly m salesmen depart from and return to the depot (city 1).
2. City Flow: For each non-depot city, the number of incoming arcs equals the number of outgoing arcs, and both are exactly one. This ensures each city is visited exactly once.
3. Subtour Elimination and Tour Size: Constraints utilizing u_i prevent subtours and enforce that the number of cities on any route is between k and w.
4. Binary Nature: The variables x_{ij} are binary.

3. Development of Theoretical Concepts

A ranking function is considered valid if it satisfies several logical theorems. To rank any generalized trapezoidal fuzzy number $\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle$, the following parametric relation can be used [4]:

$$\mathcal{M}_{\alpha}(\tilde{c}_{ij}) = (a_{ij} + d_{ij})(\mu_{ij} - \alpha) + (b_{ij} + c_{ij} - a_{ij} - d_{ij})\left(\frac{{\mu_{ij}}^2 - \alpha^2}{2\,\mu_{ij}}\right), 0 \le \alpha \le \mu_{ij}, (1)$$

Theorem 3,1. $\mathcal{M}_{\alpha}(\tilde{c}_{ij})$ is continuous and differentiable with respect to α on $(0, \mu_{ij})$.

Theorem 3,2. $\mathcal{M}_{\alpha}(\tilde{c}_{ij})$ is monotonically decreasing with respect to α on $(0, \mu_{ij})$.

Theorem 3,3. For any $\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle$, we have: $\mathcal{M}_{\mu_{ij}}(\tilde{c}_{ij}) \le \mathcal{M}_{\alpha}(\tilde{c}_{ij}) \le \mathcal{M}_{0}(\tilde{c}_{ij})$.

4. Numerical experiment

The travel costs (for both directions) between seven cities are provided below. Assuming $k = 4$ and $\mathrm{w} = 6$, two salesmen starting from the depot aim to find tours with the lowest total cost.

$$\tilde{c}_{11} = \tilde{c}_{22} = \tilde{c}_{33} = \tilde{c}_{44} = \tilde{c}_{55} = \tilde{c}_{66} = \tilde{c}_{77} = \langle (100,100,100,100); 1.0 \rangle$$
$$\tilde{c}_{12} = \langle (3.5,5.2,7.8,9.1); 0.7 \rangle, \tilde{c}_{13} = \langle (2.8,4.1,6.3,8.5); 0.6 \rangle,$$
$$\tilde{c}_{14} = \langle (5.7,7.3,9.6,11.2); 0.8 \rangle, \quad \tilde{c}_{15} = \langle (4.2,6.1,8.4,10.7); 0.5 \rangle,$$
$$\tilde{c}_{16} = \langle (7.9,9.5,12.3,14.8); 0.9 \rangle, \tilde{c}_{17} = \langle (6.4,8.2,10.9,13.5); 0.7 \rangle$$
$$\tilde{c}_{21} = \langle (2.7,4.3,6.8,8.9); 0.8 \rangle, \tilde{c}_{23} = \langle (5.9,7.6,10.1,12.3); 0.6 \rangle$$
$$\tilde{c}_{24} = \langle (4.8,6.7,9.2,11.8); 0.5 \rangle, \tilde{c}_{25} = \langle (8.6,10.9,13.7,16.2); 0.8 \rangle$$
$$\tilde{c}_{26} = \langle (7.2,9.4,12.1,14.9); 0.7 \rangle, \tilde{c}_{27} = \langle (10.8,13.2,16.3,19.1); 0.9 \rangle$$
$$\tilde{c}_{31} = \langle (4.1,5.9,8.3,10.6); 0.6 \rangle, \tilde{c}_{32} = \langle (6.8,8.7,11.4,13.9); 0.7 \rangle$$
$$\tilde{c}_{34} = \langle (8.9,11.2,14.1,16.8); 0.8 \rangle, \tilde{c}_{35} = \langle (7.5,9.8,12.6,15.3); 0.5 \rangle$$
$$\tilde{c}_{36} = \langle (11.7,14.3,17.5,20.8); 0.9 \rangle, \tilde{c}_{35} = \langle (10.2,12.9,16.1,19.3); 0.7 \rangle$$
$$\tilde{c}_{41} = \langle (5.6,7.4,10.2,12.7); 0.7 \rangle, \tilde{c}_{42} = \langle (8.3,10.6,13.7,16.4); 0.8 \rangle$$
$$\tilde{c}_{43} = \langle (10.1,12.8,16.2,19.5); 0.9 \rangle, \tilde{c}_{45} = \langle (12.9,15.7,19.2,22.8); 0.6 \rangle$$
$$\tilde{c}_{46} = \langle (11.3,14.1,17.6,21.2); 0.5 \rangle, \tilde{c}_{47} = \langle (15.8,18.9,22.7,26.4); 0.8 \rangle$$
$$\tilde{c}_{51} = \langle (7.2,9.5,12.8,15.6); 0.8 \rangle, \tilde{c}_{52} = \langle (10.4,13.2,16.7,19.9); 0.9 \rangle$$
$$\tilde{c}_{53} = \langle (12.7,15.6,19.3,22.8); 0.5 \rangle, \tilde{c}_{54} = \langle (15.3,18.4,22.3,26.1); 0.7 \rangle$$
$$\tilde{c}_{56} = \langle (17.9,21.2,25.1,29.3); 0.8 \rangle, \tilde{c}_{57} = \langle (16.4,19.7,23.6,27.5); 0.6 \rangle$$
$$\tilde{c}_{61} = \langle (9.1,11.8,15.3,18.6); 0.9 \rangle, \tilde{c}_{62} = \langle (12.6,15.7,19.6,23.2); 0.5 \rangle$$
$$\tilde{c}_{63} = \langle (15.2,18.6,22.7,26.5); 0.7 \rangle, \tilde{c}_{64} = \langle (18.1,21.5,25.8,29.7); 0.8 \rangle$$
$$\tilde{c}_{65} = \langle (20.7,24.3,28.6,32.9); 0.6 \rangle, \tilde{c}_{67} = \langle (22.9,26.7,30.9,35.2); 0.9 \rangle$$
$$\tilde{c}_{71} = \langle (11.3,14.2,18.1,21.7); 0.5 \rangle, \tilde{c}_{72} = \langle (14.9,18.3,22.8,26.9); 0.7 \rangle$$
$$\tilde{c}_{73} = \langle (17.8,21.4,25.9,30.2); 0.8 \rangle, \tilde{c}_{74} = \langle (20.9,24.8,29.5,34.1); 0.9 \rangle$$
$$\tilde{c}_{75} = \langle (23.8,27.9,32.7,37.4); 0.6 \rangle, \tilde{c}_{76} = \langle (26.7,30.9,35.8,40.6); 0.5 \rangle$$

After defuzzification using Equation (1) and obtaining the crisp cost matrix, the resulting values are incorporated into the MILP model. The model is then solved using MATLAB for two salesmen, yielding the following results:

For $\alpha = 0.0, 0.1, 0.2$, the routes are 1-4-6-2-3-1 and 1-5-7-1, with total costs of $\tilde{z}_{0.0} = 116.6200, \tilde{z}_{0.1} = 95.6673$, and $\tilde{z}_{0.2} = 74.7893$, respectively.
For $\alpha = 0.3, 0.4, 0.5$, the routes are 1-5-3-1 and 1-7-6-2-4-1, with total costs of $\tilde{z}_{0.3} = 52.7209, \tilde{z}_{0.4} = 30.7211$, and $\tilde{z}_{0.5} = 8.8070$, respectively.
As anticipated, the optimal total cost decreases as the value of α increases. At low values of α such as 0.0, 0.1, and 0.2, the system operates more conservatively and selects routes that consider the more pessimistic segments of the fuzzy numbers, specifically the values a and d in the quadruple representation, thereby carrying less risk. In contrast, as α increases, the model becomes more optimistic and assigns greater weight to the core segments with higher membership degrees, the values b and c. This ultimately leads to the selection of different routes with a lower overall cost. This behavior empirically confirms the monotonically decreasing nature of the defuzzification function $\mathcal{M}_\alpha(\tilde{c}_{ij})$, which was addressed in the theoretical section's theorems. Consequently, this method allows managers to explore and evaluate different solutions according to their specific risk tolerance levels by adjusting this parameter.

5. Conclusion

This study developed certain theoretical concepts for a parametric defuzzification function and utilized it to solve the single-depot mTSP with generalized trapezoidal fuzzy travel costs. A numerical experiment on a seven-city problem demonstrated the efficacy of the approach, showing that changes in the risk parameter α lead to variations in the optimal routes and total cost.

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Interior anti-ideals in BE-algebras

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Abstract. In this paper, we introduce and study the concept of an anti-ideal in a BE–algebra and explore the fundamental properties of these anti-ideals, establishing their foundational characteristics. Furthermore, we define the notion of interior anti-ideals in a BE–algebra and prove that a non-empty subset A of a BE–algebra X is an interior anti-ideal of X if and only if A is a right anti-ideal of X.

Keywords: BE-algebra, anti-ideal, bi-anti-ideal, interior anti-ideal.

AMS Mathematics Subject Classification [2020]: 06F35, 03G25, 20N20.

1. Introduction and Preliminaries

In 2007, H. S. Kim and Y. H. Kim [4] introduced the notion of a BE–algebra as a generalization of a dual BCK–algebra. In 2010, B. L. Meng [5] defined CI-algebras. Later on, in 2021, R. K. Bandaru et al. [2] introduced the concept of GE–algebras as a new generalization of BE-algebras and Hilbert algebrs. F. Schwarz [6] introduced anti-ideals of semigroups in 1953. Recently, M. Al-Tahan and S. Hoskova-Mayerova [1] defined interior anti-ideals and m-interior anti-ideals of a semigroup and studied their properties. In this note we try applied them to $CI/BE/GE$–algebras and give connection between interior anti-ideals and right anti-ideals.

Definition 1.1. A *CI–algebra* ([5]) $(X, *, 1)$ is a non-empty set X with a constant 1 and a binary operation " $*$ " satisfying the following axioms:

$$(\forall x \in X)\big(x * x = 1\big), \tag{1}$$

$$(\forall x \in X)\big(1 * x = x\big), \tag{2}$$

$$(\forall x, y, z \in X)\big(x * (y * z) = y * (x * z)\big). \tag{3}$$

A *BE–algebra* is a CI–algebra $(X, *, 1)$ satisfying the following axiom:

$$(\forall x \in X)\big(x * 1 = 1\big), \tag{4}$$

*Speaker.

Definition 1.2. A subset $I \neq \emptyset$ is called an ideal ([3]) of BE–algebra $(X, *, 1)$ if

$$(\forall x \in X, \forall a \in I)\big(x * a \in I\big), \ \big(i.e.\ X * I \subseteq I\big), \tag{5}$$

$$(\forall x \in X, \forall a, b \in I)\big((a * (b * x) \in I\big). \tag{6}$$

Further, if a subset $I \neq \emptyset$ of an BE–algebra X satisfies the following equality

$$X * I = I,$$

then we say that I is an *essence* of X. Note that $\{1\}$ and X itself are essences of X.

Definition 1.3. A subset $F \neq \emptyset$ is called a filter ([4]) of BE–algebra $(X, *, 1)$ if

$$1 \in F, \tag{7}$$

$$\text{If } x \in F \text{ and } x * y \in F, \text{ then } y \in F. \tag{8}$$

Definition 1.4. ([2]) An algebra $(X, *, 1)$ of type $(2, 0)$ is called a GE–algebra if it satisfies (1) and (2) and in the following axioms:

$$(\forall x, y, z \in X)\big(x * (y * z) = x * \big(y * (x * z)\big)\big). \tag{9}$$

2. Anti-ideals in BE-algebras

Recall that a groupoid is an algebraic structure $(G, *)$ consisting of a non-empty set G and a binary operation "$*$" in G i.e., $* : G \times G \to G$.

Definition 2.1. Let $(G, *)$ be a groupoid and $A \subseteq G$ be a non-empty set. Then

1. A is a left anti-ideal of G if $(G * A) \cap A = \emptyset$,
2. A is a right anti-ideal of G if $(A * G) \cap A = \emptyset$,
3. A is an anti-ideal of G if it is both left and right anti-ideal of G.

where $G * A = \{g * a : g \in G, a \in A\}$ and $A * G = \{a * g : a \in A, g \in G\}$.

Theorem 2.2. *A $CI/BE/GE$–algebra has no left anti-ideals.*

Proof. Let $(X; *, 1)$ be a $CI/BE/GE$–algebra. Then, for all $a \in A \subseteq X$, by (2), we have $a = 1 * a \in X * A$. Hence, $A \subseteq A * X$ which implies that $(X * A) \cap A = A$. Therefore A is not a left anti-ideal. □

Proposition 2.3. *Let $(X; *, 1)$ be a $CI/BE/GE$–algebra and $\emptyset \neq A \subseteq X$ be a right anti-ideal of X. Then*

(i) $1 \notin A$.
(ii) *If $a \in \{a\} * X$, then the singleton set $\{a\}$ is not a right anti-ideal,*
(ii) *If A is a right anti-ideal of X, then $X \setminus A$ is not a right anti-ideal.*

Example 2.4. Let $X = \{1, a, b, c, d\}$. Define a binary operation "$*_1$" on X in Cayley Table 1.

Then $(X; *_1, 1)$ is a BE–algebra. $\{b, d\}$ and $\{c, d\}$ are right anti-ideal of X.

Note. By (1) and Proposition 2.3, we see that a right anti-ideal of a $CI/BE/GE$–algebra X is not a subalgebra of X.

Proposition 2.5. *Let $(X; *, 1)$ be a $CI/BE/GE$–algebra and $\emptyset \neq A \subseteq X$ be a right anti-ideal of X. Then*

(i) *Every non-empty subset of A is a right anti-ideal.*
(ii) *Every non-empty intersection of A with any subset of X is a right anti-ideal.*

TABLE 1. BE–algebra $(X;\, *_1,\, 1)$

$*_1$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Notice that if I is an essence of a BE–algebra X, then it is not a left anti-ideal of X. Since $X * I = I$, we get $\big(X * I\big) \cap I = I \cap I = I \neq \emptyset$. Hence every essence is not an anti-ideal of X. Further, since every filter is an essence of X ([3, Th. 3.3]), we conclude that every filter is not an anti-ideal.

DEFINITION 2.6. Let $(G,\, *)$ be a groupoid and $A \subseteq G$ be a non-empty set. Then A is a bi-anti-ideal of G if $\big(A * (G * A)\big) \cap A = \emptyset$.

EXAMPLE 2.7. In Example 2.4, $A_1 = \big\{a\big\}$ and $A_2 = \big\{b, d\big\}$ are bi-anti-ideal of X.

Note. A bi-anti-ideal of a $CI/BE/GE$–algebra X is not a subalgebra of X. Further, the extension property is not valid for a bi-anti-ideal of X. Consider the Example 2.7. If we take $B = \big\{1, b, d\big\}$. Hence $A = \big\{b, d\big\} \subseteq B = \big\{1, b, d\big\}$, but B is not a bi-anti-ideal of X. Since $\big(B * (X * B)\big) = \big\{1, b, d\big\} * \big\{1, b, c, d\big\} = \big\{1, b, c, d\big\}$, and so $\big\{1, b, c, d\big\} \cap \big\{1, b, d\big\} = \big\{1, b, d\big\} \neq \emptyset$.

PROPOSITION 2.8. *Let $(X;\, *,\, 1)$ be a $CI/BE/GE$–algebra and $\emptyset \neq A \subseteq X$ be a bi-anti-ideal of X. Then*

(*i*) *Every non-empty subset of A is a bi-anti-ideal of X.*
(*ii*) *Every non-empty intersection of A with any subset of X is a bi-anti-ideal of X.*

THEOREM 2.9. *Let $(X;\, *,\, 1)$ be a BE–algebra then every right anti-ideal of X is a bi-anti-ideal.*

The following example shows that a bi-anti-ideal of X may not be a right anti-ideal.

EXAMPLE 2.10. Let $X = \big\{1, a, b, c, d\big\}$. Define a binary operation "$*_2$" on X in Cayley Table 2.

TABLE 2. BE–algebra $(X;\, *_2,\, 1)$

$*_2$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	1	1	1
c	1	1	b	1

Then $(X;\, *_2,\, 1)$ is a BE–algebra and $A = \big\{a\big\}$ is a bi-anti-ideal of X, but not a right anti-ideal of X, since $A *_2 X = \{a\} *_2 X = \{1, a\}$, and so $\big\{1, a\big\} \cap A = \{a\} \neq \emptyset$.

DEFINITION 2.11. Let $A \subseteq X$ be a non-empty set. Then A is an interior anti-ideal of X if $\big(X * (A * X)\big) \cap A = \emptyset$.

Example 2.12. In Example 2.4, $A_1 = \{a\}$ and $A_2 = \{b, d\}$ are interior anti-ideals of X.

Theorem 2.13. *Let $(X; *, 1)$ be a $BE/CI/GE$–algebra then every interior anti-ideal of X is a bi-anti-ideal of X.*

Proof. Let $A \subseteq X$ be an interior anti-ideal of X. By contradiction, suppose that A is not a bi-ant-ideal of X. Then $\big(A * (X * A)\big) \cap A \neq \emptyset$. Therefore, there exists $a \in \big(A*(X*A)\big)\cap A$. Hence, there exist $a', a'' \in A$ and $x \in X$, such that $a = a'*(x*a'') \in A*X$, on the other hand we have $a = 1 * a \in X * (A * X)$, and hence $\big(X * (A * X)\big) \cap A \neq \emptyset$, which is a contradiction. □

The following example shows that a bi-anti-ideal of X may not be an interior anti-ideal.

Example 2.14. In Example 2.10, $A = \{a\}$ is a bi-anti-ideal, but it is not an interior anti-ideal of X, since $X * (A * X) = X * (\{a\} * X) = X * (\{1, a\}) = \{1, a\}$, and so $\{1, a\} \cap \{a\} = \{a\} \neq \emptyset$.

Theorem 2.15. *Let $(X; *, 1)$ be a BE/CI–algebra and $A \subseteq X$ be a non-empty set. Then A is an interior anti-ideal of X if and only if A is a right anti-ideal of X.*

Proposition 2.16. *Let $(X; *, 1_X)$ and $(Y; \circ, 1_Y)$ be two $CI/BE/GE$–algebras, the map $f : X \longrightarrow Y$ be a homomorphism, $A \subseteq X$ and $B \subseteq Y$ be interior anti-ideals of X and Y, respectively. The following statements hold:*

(i) If f is an isomorphism, then $f(A)$ is an-interior anti-ideal of Y,
(ii) $f^{-1}(B)$ is an interior anti-ideal of X.

3. Conclusion

In this paper, we have introduced and investigated the concept of an anti-ideal in a $CI/BE/GE$–algebra. Furthermore, we defined the specific notion of an interior anti-ideal within a BE–algebra. The central result of our work is the proof of a crucial equivalence: a non-empty subset A of a BE–algebra X is an interior anti-ideal if and only if it is a right anti-ideal. For future work we will generalize and extend the notions of anti-ideals and interior anti-ideals to the fuzzy setting. One can define fuzzy anti-ideals and fuzzy interior anti-ideals and investigate their properties, level cuts, and the relationships between them. Also, One can develop the theory of prime anti-ideals, maximal anti-ideals, and irreducible anti-ideals in BE-algebras.

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Pattern Recognition in Student Career Selection

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Abstract: Pattern recognition plays a crucial role in modern student career selection by helping identify students strengths, interests, abilities, and potential career paths based on observable data. Instead of relying solely on manual judgment, counsellors and intelligent systems can use pattern recognition techniques to analyze large amounts of student information and uncover meaningful insights that support better career decisions. The thought of Fermatean fuzzy sets (FFSs) undeniably renders a valuable framework for addressing uncertainty and vagueness in decision-making scenarios. In pattern recognition, the ability to measure the distance between objects is crucial for tasks such as clustering, classification, and similarity assessment. Building upon the notion of distance between two entities and leveraging the information conveyed by the membership, non-membership, and hesitancy levels of Fermatean fuzzy sets, this paper introduces Euclidean distance measures between Fermatean fuzzy sets for pattern recognition purposes.

Keywords: Pattern Recognition, Student Career Selection.

1. Introduction

In 1965, Zadeh [6] invented the initial idea of the fuzzy set utilizing a membership degree, denoted as α, to depict the degree of belonging to set under examination. In this framework, each element of conceptual universe is redefined within the unit range [0, 1]. Owing to the unavailability of non-membership functions as well as contempt for capacity or possibility of indeterminacy edge, the notion of fuzzy sets theory seem to be unsatisfactory. To solve these inadequacies, the perception of intuitionistic fuzzy sets (IFSs) was developed by Atanassov in 1986. The structures of IFSs combines the belonging function (α), non-belongingness function (β) and the indeterminacy margin (π) yielding $\alpha + \beta \leq 1$ as well as $\alpha + \beta + \pi = 1$. IFSs provide a structure that is adaptable for elaborating ambiguity and uncertainty.

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Senapati et al. [4] introduced Fermatean fuzzy sets (FFSs). FFSs maintain belonging and non-belonging degrees in a manner ensuring that sum of cubes of grades falls within the range of 0 to 1. For example, in FFSs, $0.8^3 + 0.7^3 < 1$ in the given situation. The membership space of FFSs surpasses that of both IFSs and PFSs. In FFSs, the membership level (α) and non-membership grade (β) adhere to the conditions $0 \leq \alpha^3 + \beta^3 \leq 1$, where both α and β range between 0 and 1.

Euclidean distance measures between FFSs offer a novel approach to quantifying the similarity or dissimilarity between Fermatean fuzzy sets, thus enhancing pattern recognition methods. Garg [3] introduced a novel method for aggregating generalized PFSs using Einstein operations and demonstrated its application in decision-making processes. Senapati et al. [4] applied FFSs in multiple attribute decision making. Szmidt et al. [5] used similarity measure for IFSs in medical diagnostic. Pattern recognition is the one of the decision making problems. In this article, we discussed on career detection by pattern recognition based on Euclidean distance measurement.

2. Basic Definitions

In this section, we will explore concepts related to fuzzy sets, PFSs, and IFSs.

Definition 2.1 *[6] Fuzzy set F in a universal set X defined as*

$$F = \{\langle \xi, \alpha_F(\xi) \rangle : \xi \in X\},$$

$\alpha_F : X \to [0,1]$ represents a mapping known as the fuzzy membership level, which assigns each element in the set X a degree of membership ranging from 0 to 1.

The complement of α is defined by $\bar{\alpha}(\xi) = 1 - \alpha(\xi)$ for all $\xi \in X$ and denoted by $\bar{\alpha}$.

Definition 2.2 *An Intuitionistic fuzzy set (IFS) I in the set X is characterized as*

$$I = \{\langle \xi, \alpha_I(\xi), \beta_I(\xi) \rangle : \xi \in X\},$$

where the $\alpha_I(\xi)$ and $\beta_I(\xi)$ represents the worth of belonging and non-belonging of $\xi \in X$ respectively and satisfy the condition

$$0 \leq \alpha_I(\xi) + \beta_A I(\xi) \leq 1,$$

for all $\xi \in X$.

$h_I(\xi) = 1 - \alpha_I(\xi) - \beta_I(\xi)$ denotes degree of indeterminacy.

Definition 2.3 *[4] A Fermatean fuzzy set $\mathcal{P}$ in a finite universe of discourse X is furnished as*

$$\mathcal{P} = \{\langle \xi, \alpha_{\mathcal{P}}(\xi), \beta_{\mathcal{P}}(\xi) \rangle | \xi \in X\},$$

where $\alpha_{\mathcal{P}}(\xi) : X \to [0,1]$ denotes the worth of membership and $\beta_{\mathcal{P}}(\xi) : X \to [0,1]$ represents the worth to which the element $\xi \in X$ is not a member of the set $\mathcal{P}$, with the predicament that

$$0 \leq (\alpha_{\mathcal{P}}(\xi))^3 + (\beta_{\mathcal{P}}(\xi))^3 \leq 1,$$

for all $\xi \in X$.

The worth of indeterminacy $h_{\mathcal{P}}(\xi) = \sqrt[3]{1 - (\alpha_{\mathcal{P}}(\xi))^3 - (\beta_{\mathcal{P}}(\xi))^3}$.

3. Distance measurement of Fermatean Fuzzy Sets

This section provides a formal way to quantify the similarity or dissimilarity between two Fermatean fuzzy numbers using the Euclidean distance measure, taking into account the uncertainties represented by their membership, non-membership, and indeterminacy levels.

Definition 3.1 *(Property of distance) A set X with its elements referred to as points is defined as a metric space if, for any two points $\mathcal{P}_1, \mathcal{P}_2 \in X$, there exists a corresponding real number $d(\mathcal{P}_1, \mathcal{P}_2)$, known as the distance, satisfies condition belows:*

(a) $d(\mathcal{P}_1, \mathcal{P}_2) \geq 0$,

(b) $d(\mathcal{P}_1, \mathcal{P}_2) = 0 \Leftrightarrow \mathcal{P}_1 = \mathcal{P}_2$

(c) $d(\mathcal{P}_1, \mathcal{P}_2) = d(\mathcal{P}_2, \mathcal{P}_1)$

(d) $d(\mathcal{P}_1, \mathcal{P}_3) \leq d(\mathcal{P}_1, \mathcal{P}_2) + d(\mathcal{P}_2, \mathcal{P}_3)$, for any $\mathcal{P}_3 \in X$.

Functions possessing all these characteristics are denoted as distance functions or distance measures.

The Euclidean distance $d_E(\mathcal{P}_1, \mathcal{P}_2)$ between two FFSs $\mathcal{P}_1$ and $\mathcal{P}_2$ is calculated based on their respective membership (α), non-belonging (β), and hesitancy (π) levels for each ξ_i in X.

Definition 3.2 *The Euclidean distance $d_E(\mathcal{P}_1, \mathcal{P}_2)$ between two FFSs $\mathcal{P}_1$ and $\mathcal{P}_2$ is characterized by*

$$d_E(\mathcal{P}_1, \mathcal{P}_2) = \left[(\alpha^3_{\mathcal{P}_1}(\xi_i) - \alpha^3_{\mathcal{P}_2}(\xi_i))^2 + (\beta^3_{\mathcal{P}_1}(\xi_i) - \beta^3_{\mathcal{P}_2}(\xi_i))^2 + (\pi^3_{\mathcal{P}_1}(\xi_i) - \pi^3_{\mathcal{P}_2}(\xi_i))^2\right]^{\frac{1}{2}}, \quad (1)$$

$X = \{\xi_1, \xi_2, \dots, \xi_n\}$, $i = 1,2, \cdots, n$.

Definition 3.3 *The normalized Euclidean distance $d_{NE}(\mathcal{P}_1, \mathcal{P}_2)$ between FFSs $\mathcal{P}_1$ and $\mathcal{P}_2$ is characterized as*

$$d_{NE}(\mathcal{P}_1, \mathcal{P}_2)$$

$$= \left(\frac{1}{2n}\sum_{i=1}^{n}\left[(\alpha_{\mathcal{P}_1}^3(\xi_i)-\alpha_{\mathcal{P}_2}^3(\xi_i))^2+(\beta_{\mathcal{P}_1}^3(\xi_i)-\beta_{\mathcal{P}_2}^3(\xi_i))^2+(\pi_{\mathcal{P}_1}^3(\xi_i)-\pi_{\mathcal{P}_2}^3(\xi_i))^2\right]\right)^{\frac{1}{2}} \quad (2)$$

$$X=\{\xi_1,\xi_2,\dots,\xi_n\} \quad for \quad i=1,2,\dots,n.$$

Using the normalized Euclidean distance measurement method for FFSs in career determination problems is an interesting application.

Example 1 *Let set of careers denoted by $\eta=\{\eta_1,\eta_2,\eta_3,\eta_4\}$, where η_1,η_2,η_3, and η_4 represent Medical, Engineering, Pharmacy, and Nursing careers, respectively. This set encompasses various career paths that students may consider for their future endeavors. The set of subjects related to careers denoted by $\zeta=\{\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5\}$, where $\zeta_1,\zeta_2,\zeta_3,\zeta_4$, and ζ_5 represent English language, Physics, Chemistry, Biology, and Mathematics, respectively. These subjects are likely relevant to the careers listed in set η and are indicative of the academic requirements or interests associated with each career path.*

By utilizing FFSs and normalized Euclidean distance measurement, one can quantify the similarity or compatibility between a student's academic profile (represented as a Fermatean fuzzy set over subjects) and each career option. This approach permits a more nuanced assessment of the suitability of different careers based on the student's academic strengths and preferences.

The following table shows careers and related subjects requirements:

$$\eta_1=\{(\zeta_1,0.8,0.3),(\zeta_2,0.5,0.4),(\zeta_3,0.7,0.5),(\zeta_4,0.9,0.1),(\zeta_5,0.4,0.5)\}$$

$$\eta_2=\{(\zeta_1,0.7,0.3),(\zeta_2,0.8,0.5),(\zeta_3,0.7,0.6),(\zeta_4,0.6,0.4),(\zeta_5,0.8,0.3)\}$$

$$\eta_3=\{(\zeta_1,0.6,0.4),(\zeta_2,0.7,0.4),(\zeta_3,0.6,0.5),(\zeta_4,0.5,0.6),(\zeta_5,0.6,0.4)\}$$

$$\eta_4=\{(\zeta_1,0.7,0.5),(\zeta_2,0.6,0.3),(\zeta_3,0.5,0.4),(\zeta_4,0.8,0.3),(\zeta_5,0.5,0.4)\}$$

Each performance is delineated by Fermatean fuzzy numbers that consists of membership and non-belonging levels. After various examination and assessments a particular student's (say P) performance in different subjects are represented by Fermatean fuzzy numbers as follows:

$$P=\{(\zeta_1,0.6,0.4),(\zeta_2,0.5,0.3),(\zeta_3,0.7,0.2),(\zeta_4,,0.7,0.4),(\zeta_5,0.5,0.4)\}$$

Our objective is to ascertain which careers (η_i, where $i=1,2,3,4$) the student P belongs to. Since the task of determining careers aligns with a pattern recognition problem, we can employ the recognition rule outlined below:

$$k=\min_{1\le i\le 4}\{d_{NE}(\eta_i,P)\}, \qquad (3)$$

then we assign the student P to the particular career.

By the Eq.(2), we can obtain the normalized Euclidean distance as follows:

$d_{NE}(\eta_1, P) = 0.4220$, $d_{NE}(\eta_2, P) = 0.5642$, $d_{NE}(\eta_3, P) = 0.2886$, and $d_{NE}(\eta_4, P) = 0.2715$.

Hence, we can assign the student P to the career η_4.

4. Conclusion

Pattern recognition transforms student career counselling from a subjective, guess-based process into a structured, data-driven method. By analyzing academic performance, personality, behavior, and interests, counsellors can uncover meaningful patterns that match students with the most suitable career paths. This leads to more informed decisions, better performance, and greater satisfaction in future careers.

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Riemann solitons properties on Morris-Thorne wormhole spacetime

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Abstract: The aim of this paper is to study the geometry of wormholes (WH) based on the curvatures accepted by this spacetime. In this paper, we investigate the curvature properties of Morris-Thorne wormholes using the concepts of almost Riemann solitons and almost Riemann gradient solitons, which lead to the solution of the Einstein field equations by considering the Tachibana tensor obtained from the energy-momentum tensor of the wormhole. This tensor satisfies some quasi-symmetric conditions and is consistent with Weyl and Riemann. A Morris-Thorne wormhole is a spherically symmetric solution of the Einstein field equations with the cosmological constant. It is known that such a spacetime has several types of symmetries, such as generalized Ricci pseudosymmetry, generalized Ricci image pseudosymmetry, pseudosymmetry due to Weyl symmetric curvature, semisymmetry due to harmonic curvature, etc. Also, it is an Einstein manifold and a special quasi-Einstein manifold.

Keywords: Morris-Thorne wormhole spacetime, Riemann soliton, Gradient Riemann soliton.

1.Introduction

A wormhole is a geometric passage-like structure that connects different universes or distinct regions of a universe. They are hypothetical structures in space-time that Wormholes are hypothetical structures in space-time that are classified into two categories: interuniverse and intrauniverse wormhole [2]. The difference between these two categories of wormholes is at the level of general geometry and the level of general topology. Ellis and Bronnikov introduced an example of wormhole and the corresponding solution nowadays known as Ellis–Bronnikov wormhole [12]. Within framework of general relativity (GR), static and spherically symmetric Lorentzian wormhole solutions are a subject of extensive research. Lemos et al. reviewed the properties of wormholes and analyzed the Morris–Thorne metric with the inclusion of a cosmological constant. Rahaman et al. treated the cosmological constant as a function of the radial coordinate, derived wormhole solutions [5]. Garcia et al. studied wormholes using scalar curvature and Lagrangian density of matter to obtain an exact solution. Duplessis et al. applied scale-free gravity, discovering solutions for wormholes and black holes [9]. Zubair et al. utilized gravity to study symmetric, spherical, and static wormhole solutions. These wormholes were represented using different metric tensors [6]. To ascertain the geometry of a semi-Riemannian manifold curvatures are the effective means and symmetry of a semi-Riemannian manifold is explained locally by the curvature restriction $\nabla R = 0$[13]. This curvature restriction is replaced by some weaker conditions to generalize the concept of symmetry [8]. Einstein and Rosen through coordinate transformations, derived a mathematical alternative to the singularities of spacetime and proposed the existence of bridge through spacetime [11]. they used the theory of general relativity to elaborate this idea These bridges connect two different points in spacetime [14]. This shortcut is known as Einstein-Rosen bridge [1]. this theory led to the idea of wormholes in 1962. Morris and Thorne presented the simplest metric in 1988[3]. This paper is to present the curvature properties of Morris-Thorne wormhole metric, and that several

important geometric structures are fulfilled by this metric, such as, Ricci generalized pseudosymmetry, Ricci generalized projective pseudosymmetry, pseudosymmetric Weyl conformal tensor etc. Additionally, it is noticed that the Tachibana tensor $Q(T,R)$ is linearly dependent with the tensors $R.R$ and $Q(g,R)$ as well as with the tensors $P.R$ and $Q(g,R)$,which provides some geometric structures of pseudosymmetric type. In this paper, we extend the Levi-Civita connection in the definition of spacetime and study the geometric structures by this metric. The organization of this paper is as follows. Section 2 describes the prerequisites and basic concepts related to solitons and WH spacetime. An initial survey of the geometric properties known by WH spacetime and the Riemann properties of solitons that the spacetime admits is focused on in Section 3. In Section 4, the result and theorem are presented.

.2. Preliminaries

In this section, we outline essential definitions and concepts that are fundamental to the discourse of this study. Consider spacetime as a Lorentzian manifold, which serves as framework for the analysis of cosmological models.

Definition1: Let (M,g) a pseudo-Riemannian manifold and let R the Riemann curvature tensor corresponding to the metric g. The concept of Riemann flow on the manifold (M,g) is defined by the equation $\frac{\partial}{\partial_t}G(t) = -2R(g(t))$,

where $G = \frac{1}{2}g \odot g$ and $\odot$ signifies the Kulkarni-Nomizu product. For two $(0,2)$ −tensors, ω and θ, this product is defined as follows [10]

$$(\omega \odot \theta)(Z_1,Z_2,Z_3,Z_4) = \omega(Z_1,Z_4)\theta(Z_2,Z_3) + \omega(Z_2,Z_3)\theta(Z_1,Z_4)$$
$$-\omega(Z_1,Z_3)\theta(Z_2,Z_4) - \omega(Z_2,Z_4)\theta(Z_1,Z_3),$$

for any vector fields Z_1,Z_2,Z_3,Z_4.

Definition2: Let (M,g) denote a pseudo-Riemannian manifold and let R represent the Riemann curvature tensor corresponding to the metric g If the following condition holds

$$2R + \mu g \odot g + g \odot L_\xi g = 0,$$

for a smooth vector field ξ, then the manifold M is classified as a Riemann soliton (RS) and is denoted as (M^n,g,μ,ξ) and μ is treated as a constant [7]. The classification of the Riemann soliton is based on the value of μ: it is categorized as expanding when $\mu > 0$, steady when $\mu = 0$, and shrinking when $\mu < 0$, if $\xi = gerad\varphi$ for some smooth function φ, the Riemann soliton assumes a particular form

$$2R + \mu g \odot g + 2g \odot \nabla^2\varphi = 0,$$

and is termed a gradient Riemann soliton. When the parameter μ is a smooth function, the terms Riemann soliton and gradient Riemann soliton are referred to as almost Riemann soliton and almost gradient Riemann soliton, respectively [4,10]. We shall examine the characteristics of the vectors within this manifold. Let (M^4,g) be a Lorentzian manifold. in spherical coordinates (t,l,v,φ), the metric tensor of WH spacetime is expressed as

$$ds^2 = -c^2dt^2 + dl^2 + (b^2+l^2)dv^2 + (b^2+l^2)sin^2vd\varphi^2,$$

where t is the global time, l is the radial coordinate, b is the constant and c is the speed of light.

3. Main Results and Their Proofs

In this section, we state and prove the main theorems, derive the main results, and investigate homogeneous Riemann solitons on the space-time WH. Let X be a vector with field

$$X = X^1(x_1,x_2,x_3,x_4)\partial_1 + X^2(x_1,x_2,x_3,x_4)\partial_2 + X^3(x_1,x_2,x_3,x_4)\partial_3 + X^4(x_1,x_2,x_3,x_4)\partial_4,$$

To examine the geometric properties of WH, first the Levi-Civita connection and then the Ricci tensor $Ric = \frac{-2b^2}{(b^2+l^2)^2}dl^2$ were calculated. Then Einstein's equations were obtained as follows

$$\lambda c^2 dt^2 - \frac{b^4\lambda + 2b^2\lambda l^2 + \lambda l^4 + 2b^2}{\left(b^2+l^2\right)^2}dl^2 - \lambda\left(b^2+l^2\right)sin^2 v d\varphi^2,$$

After calculating the Lie derivative, the Ricci soliton solutions are obtained as follow

$$\left\{0\,,\left(\frac{\partial_{X_3}}{\partial_l}(b^2+l^2)+\frac{\partial_{X_2}}{\partial_v}\right)c^2\,,\left(\frac{\partial_{X_2}}{\partial_t}-\frac{\partial_{X_1}}{\partial_l}c^2\right)(b^2+l^2),\dots\right\},$$

By obtaining the Ricci operator $Q = -\frac{2b^2\partial_l}{(b^2+l^2)^2}dl$ and calculating the covariant derivative of the two sets of solutions, the following was obtained

$$\{b=b,\quad l=0,\quad v=v\}\ ,\ \{b=0,\quad l=l,\quad v=v\}$$

Then, by calculating the locally symmetric of the space $\nabla R = 0$ and calculating the Weyl tensor, the conformally flatness was checked. And finally, the gradient Ricci soliton was obtained

$$GDF = \frac{\Lambda(-f_l + Hf_t)\partial_t}{3} - \frac{\Lambda f_t\partial_l}{3} - \frac{\Lambda f_v\partial_v}{3} - \frac{\Lambda f_\varphi\partial_\varphi}{3},$$

we solve the Riemann soliton equations for the case where $H(x_1,x_2,x_3)$ is dependent only on x_3.assuming that $f(x_1,x_2,x_3,x_4) = c_1$, where $f(x_1,x_2,x_3,x_4)$ is an arbitrary function, the Riemann soliton equations for the WH space-times is obtained as follows

(3.1) $\partial_1 X^2 = 0$

(3.2) $\partial_2 X^3 + \partial_1 X^2 = 0$

(3.3) $\partial_1 X^3 + \partial_4 X^2 = 0$

(3.4) $\partial_3 X^4 + \partial_4 X^1 + \partial_2 X^3 H = 0$

(3.5) $\partial_1 X^4 + \partial_3 X^2 + \partial_2 X^4 H = 0$

(3.6) $\Lambda x_2 + 3\partial_1 X^1 x_3 + \mu x_4 - X^2 = 0$

(3.7) $\Lambda x_2 + 3\partial_3 X^3 x_4 + \mu x_4 - X^2 = 0$

After solving these equations, the results are obtained as follows

$\mu = -\Lambda$

$$X^1(x_1,x_2,x_3,x_4) = \frac{c_1 H^2 x_3^2}{4} + \frac{(x_4^2 c_5 - c_7 x_1^2 + c_3 x_2)H - x_3^2 c_4}{2} + c_5 x_3 - c_2$$

$$X^2(x_1,x_2,x_3,x_4) = \frac{c_2x_4^2 - (c_1x_3 + c_6)x_2 + c_4x_2 - c_5}{2} + c_3x_4^2$$

$$X^3(x_1,x_2,x_3,x_4) = \frac{(c_2x_4 - 2Hc_4x_2 + 2c_1x_3 + c_7)x_2}{2}$$

$$X^4(x_1,x_2,x_3,x_4) = \frac{(c_1x_3x_4 - c_2x_1^2) + (c_1x_4 + c_6x_2 - 2c_5)Hx_2}{4} - c_7x_1 + 2c_3x_3$$

So $c_1, \dots, c_7$ that are arbitrary coefficients. Now let us give the the main results in this paper.

Theorem 3.1: The Morris-Thorne wormhole metric satisfies geometric structures such as the Riemann soliton and Ricci generalized pseudosymmetry $R.R = Q(Ric, R)$ and the Ricci generalized projectively pseudosymmetric $P.R = \frac{2}{3}Q(Ric, R)$.

4. Conclusion

Therefore, according to the above theorem, we can conclude that the Morris-Thorne wormhole metric satisfies geometric structures such as the Pseudosymmetric Weyl conformal curvature and curvature pseudo-symmetric manifold due to the Weyl conformal curvature and WH Ricci tensor is Riemann compatible, Weyl conformal compatible and conharmonic compatible.

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Riemann Solitons on the Peres Metric

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Abstract. In this paper, we investigate the existence of Riemann solitons in the Peres metric, a four-dimensional Lorentzian pp-wave spacetime. After recalling the definition of Riemann solitons and the curvature properties of this metric, we derive the determining equations for the potential soliton vector fields and obtain these vector fields. This analysis shows that the null wave nature of the Peres metric naturally supports stable soliton structures. This study contributes to the understanding of Riemann solitons in Lorentzian geometry and their role in mathematical physics.

Keywords: Riemann soliton, Ricci soliton, Peres metric, General Relativity

AMS Mathematics Subject Classification [2020]: 58E11, 53B30, 53C50 (at least 1 and at most 3)

1. Introduction

Geometric flows are a central subject in modern mathematics and physics. Since Hamilton's introduction of the Ricci flow [5], self-similar solutions called Ricci solitons have played a crucial role in understanding geometric evolution. Generalizing this, one may consider Riemann solitons, defined by soliton equations involving the full Riemann curvature tensor [7]. Exact solutions of Einstein's equations provide a natural testing ground for geometric flow concepts. The Peres metric [3], first introduced in 1959, describes a gravitational wave background in four dimensions. The metric belongs to the class of plane-fronted waves with parallel rays (pp-waves), but its explicit form differs from the canonical Brinkmann representation. In particular, in coordinates (x_1, x_2, x_3, x_4) the Peres metric is written as:

$$ds^2 = dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2 - 2f(x_1 + x_4, x_2, x_3)\,(dx_1 + dx_4)^2, \tag{1}$$

where f is a smooth function. When f is harmonic in the transverse coordinates (x, y), the metric satisfies Einstein's vacuum equations [6].

In this paper, we analyze the existence of Riemann solitons on the Peres metric. Section 2 reviews the definition of Riemann solitons and basic properties of the metric. Section

*Speaker.

3 derives the soliton conditions and discusses possible solutions. Section 4 provides the conclusion.

2. Riemann Solitons

Let (M, g) be a pseudo-Riemannian manifold and R be the Riemann curvature tensor corresponding to the metric g. The Riemann flow on manifold (M, g) is introduced as [1,4]:

$$\frac{\delta}{\delta t}G(t) = -2R(g(t)),$$

where $G = \frac{1}{2}g \odot g$ is the Kulkarni-Nomizu product, where for two $(0, 2)$-tensors ω and θ, it is determined by:

$$\begin{aligned}(\omega \odot \theta)(Y_1, Y_2, Y_3, Y_4) = \omega(Y_1, Y_4)\theta(Y_2, Y_3) + \omega(Y_2, Y_3)\theta(Y_1, Y_4)\\ - \omega(Y_1, Y_3)\theta(Y_2, Y_4) - \omega(Y_2, Y_4)\theta(Y_1, Y_3),\end{aligned}$$

for any vector fields Y_1, Y_2, Y_3, Y_4. If there exists a smooth vector field V where $\mathcal{L}_V g$ denote the Lie derivative of the metric g in direction V, such that the following equation is true

$$2R + \mu\, g \odot g + g \odot \mathcal{L}_V g = 0, \tag{2}$$

then n-dimensional complete pseudo-Riemannian manifold (M, g) is called Riemann soliton (or RS) [1], and denoted by (M^n, g, μ, V). Here μ is a constant. The Riemann soliton is called expanding, steady, or shrinking according as $\mu > 0$ or $\mu = 0$, or $\mu < 0$. If $V = \nabla\phi$ for some smooth function ϕ, then the RS becomes

$$2R + \mu\, g \odot g + 2g \odot \nabla^2\phi = 0, \tag{3}$$

and the soliton is called gradient Riemann soliton. If μ is a smooth function, then RS and gradient RS is called almost Riemann soliton (or ARS) and almost gradient Riemann soliton, respectively.

3. Main Results

In this section, we establish and prove the main theorem regarding the characterization of Riemann solitons on the homogeneous Peres metric. Let us consider a smooth vector field of the form

$$\begin{aligned}X =& X_1(x_1, x_2, x_3, x_4)\partial_1 + X_2(x_1, x_2, x_3, x_4)\partial_2 + X_3(x_1, x_2, x_3, x_4)\partial_3\\ &+ X_4(x_1, x_2, x_3, x_4)\partial_4,\end{aligned}$$

where, for simplicity, we denote $\partial_i = \frac{\partial}{\partial x_i}$. For brevity, we omit explicitly writing the variables (x_1, x_2, x_3, x_4) for in the following calculations.

By substituting this general vector field into the soliton condition (equation (2) in the Riemann Solitons section), and after a series of lengthy but straightforward tensorial computations (similar to those found in [2, 4]), we arrive at the following system of determining partial differential equations:

(4) $\partial_1 X_2 = 0$

(5) $-\partial_3 X_4 - \partial_4 X_3 = 0$

(6) $-\partial_2 X_4 - \partial_1 X_4 = 0$

(7) $-2\partial_2 X_2 - 2\partial_1 X_1 = 0$

(8) $-\partial_3 X_3 + \partial_1 X_2 + \partial_1 X_1 = 0$

(9) $-\partial_2 X_3 + \partial_3 X_2 - \partial_3 X_1 = 0$

(10) $-\partial_2 X_4 + \partial_4 X_2 - \partial_4 X_1 = 0$

(11) $-\partial_4 X_4 + \partial_2 X_2 - \partial_2 X_1 = 0$

(12) $-\partial_4 X_4 - \partial_1 X_2 - \partial_1 X_1 = 0$

(13) $-\partial_4 X_4 + \partial_1 X_2 + \partial_1 X_1 = 0$

(14) $-\partial_3 X_3 - \partial_2 X_2 + \partial_1 X_2 + \partial_2 X_1 - \partial_1 X_1 = 0$

(15) $-\partial_4 X_4 - \partial_2 X_2 + \partial_1 X_2 + \partial_2 X_1 - \partial_1 X_1 = 0$

(16) $2\partial_3 X_2 - \partial_2 X_3 = 0$

(17) $-\partial_2 X_3 - \partial_1 X_3 = 0$

(18) $-\partial_2 X_2 + \partial_2 X_1 = 0$

(19) $-\partial_4 X_4 - \partial_3 X_3 = 0$

(20) $-\partial_1 X_3 - \partial_3 X_2 + \partial_3 X_1 = 0$

(21) $-\partial_1 X_4 - \partial_4 X_2 - \partial_4 X_1 = 0$

(22) $\partial_3 X_3 + \partial_2 X_2 - \partial_2 X_1 = 0$

(23) $-\partial_3 X_3 - \partial_1 X_2 - \partial_1 X_1 = 0$

(24) $-\partial_1 X_4 - \partial_2 X_4 - \partial_4 X_1 = 0$

To obtain *vector fields* of this system PDE$_s$, we proceed as follows; from equation (1), we have:

$$X_2 = X_2(x_2, x_3, x_4)$$

Since X_2 depends on x_2 and according to equation (18) in general, this says:

$$X_1 - X_2 = F(x_1, x_3, x_4) \tag{25}$$

Equivalently:

$$X_1(x_1, x_2, x_3, x_4) = X_2(x_2, x_3, x_4) + F(x_1, x_3, x_4) \tag{26}$$

By substituting equation (26) into equation (7), we have: $\partial_2 X_2 + \partial_1(X_2 + F) = 0$ and by substituting equation (1), the last relationship is in the form of $\partial_2 X_2 + \partial_1 F = 0$, which implies:

$$F(x_1, x_3, x_4) = -x_1 G(x_3, x_4) + H(x_3, x_4) \tag{27}$$

$$\partial_1 F = -G(x_3, x_4) \tag{28}$$

From the substitution of $\partial_2 X_2 + \partial_1 F = 0$ into (28) we have:

$$X_2(x_2, x_3, x_4) = x_2 G(x_3, x_4) + K(x_3, x_4) \tag{29}$$

By substituting (26), (27) and (28) become

$$X_1 = X_2 + F = x_2 G + K - x_1 G + H \tag{30}$$

We can absorb $K + H$ into a single function L, so for simplicity:

$$X_1 = (x_2 - x_1) G(x_3, x_4) + L(x_3, x_4) \tag{31}$$

where G, K and L are arbitrary functions of x_3, x_4.

Now, using equation (17) and integrating, we have:

$$X_3 = M(x_1 - x_2, x_3, x_4) \tag{32}$$

and similarly:

$$X_4 = N(x_1 - x_2, x_3, x_4) \tag{33}$$

By placing the relations (29) to (33) in $\mathrm{PDE_s}$ and simplifying and integrating them, the vector fields become as follows:

$$\begin{aligned}
X_1(x_1, x_2, x_3, x_4) &= -\frac{c_2 x_3}{2} + c_7 \\
X_2(x_1, x_2, x_3, x_4) &= -c_4 x_4 + \frac{c_2 x_3}{2} + c_6 \\
X_3(x_1, x_2, x_3, x_4) &= c_1 x_4 + (-x_1 + x_2)c_2 + c_3 \\
X_4(x_1, x_2, x_3, x_4) &= (x_1 - x_2)c_4 - c_1 x_3 + c_5
\end{aligned}$$

where $c_1, c_2, \ldots, c_7$ are arbitrary real constants of integration.

4. Conclusion

In this paper, we examined the existence of Riemann solitons on the Peres metric and derived explicit conditions for their realization. Moreover, the explicit vector fields obtained illustrate how geometric evolution interacts with wave-like spacetimes. This contributes to the broader study of solitonic structures in Lorentzian geometry and mathematical physics.

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On characterizations of hypersurfaces in a Kenmotsu space form

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Abstract. This paper introduces hypersurfaces in Kenmotsu space forms and investigates them under specific geometric conditions. We establish several relations involving key operators, including the structure Jacobi operator, its D-recurrent variant, and the recurrent structure Jacobi operator. Additionally, we explore properties such as local symmetry, weakly constant holomorphic curvature, and other inequality constraints.

Keywords: Jacobi operator, Kenmotsu manifold, Kenmotsu spce form

AMS Mathematics Subject Classification [2020]: 53C25, 53C40

1. Introduction

To investigate the geometry of an unknown manifold, it is often both convenient and insightful to first embed it into a well-understood ambient manifold and then study its geometric properties in parallel with those of the host space. This approach has given rise to the rich and comprehensive theory of submanifolds, which has evolved into an independent and vibrant field of research. Submanifold theory finds significant applications in mechanics, physics, and beyond.

The notion of Kenmotsu manifolds was introduced by K. Kenmotsu [4], who proved that a locally Kenmotsu manifold is a warped product $I \times_f N$, where I is an interval, N is a Kähler manifold, and the warping function is $f(t) = se^t$ with $s \neq 0$ a constant. Kenmotsu manifolds have been extensively studied by numerous authors, including G. Pitis [5], De and Pathak [2], Jun, De and Pathak [3], Binh, Tamássy, De and Tarafdar [1], Venkatesha [6], and many others. In particular, De and Pathak [2] showed that a three-dimensional Kenmotsu manifold with η-parallel Ricci tensor has constant scalar curvature.

This paper investigates hypersurfaces in Kenmotsu space forms $M^{2n+1}(c)$ of constant ϕ-sectional curvature c, under the following geometric conditions: local symmetry, recurrence and D-recurrence of the structure Jacobi operator, weakly constant holomorphic curvature, and additional inequality constraints.

*Speaker.

2. Preliminaries

A differentiable manifold $\widetilde{M}^{2n+1}$ is said to admit an almost contact structure if there exist a (non-vanishing) vector field ξ, a 1-form η, and a $(1,1)$-tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2n$ at every point of $\widetilde{M}^{2n+1}$. A manifold $\widetilde{M}^{2n+1}$ equipped with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold and will be denoted by $(\widetilde{M}^{2n+1}, \phi, \xi, \eta)$.

Let $\widetilde{M}^{2n+1}$ be a manifold endowed with an almost contact structure (ϕ, ξ, η). A Riemannian metric $\widetilde{g}$ on $\widetilde{M}^{2n+1}$ is said to be compatible with this structure if

$$\widetilde{g}(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields $X, Y \in \mathfrak{X}(\widetilde{M}^{2n+1})$. In this case, the quadruple $(\phi, \xi, \eta, \widetilde{g})$ is called an almost contact metric structure on $\widetilde{M}^{2n+1}$, and the manifold is denoted by $(\widetilde{M}^{2n+1}, \phi, \xi, \eta, \widetilde{g})$. It is well known that every almost contact manifold admits at least one compatible metric. Moreover,

$$\eta(X) = \widetilde{g}(X, \xi),$$

for all $X \in \mathfrak{X}(\widetilde{M}^{2n+1})$, so that η is the $\widetilde{g}$-dual of the characteristic vector field ξ.

A manifold $\widetilde{M}^{2n+1}$ is said to be a contact manifold if it carries a global one-form η such that

$$\eta \wedge (d\eta)^n \neq 0,$$

everywhere on M. The one-form η is called the contact form.

A submanifold M of a Riemannian contact manifold $(\widetilde{M}^{2n+1}, \phi, \xi, \eta, \widetilde{g})$ tangent to ξ is called

- invariant if $\phi(T_pM) \subseteq T_pM$ for every $p \in M$,
- anti-invariant if $\phi(T_pM) \subseteq T_p^{\perp}M$ for every $p \in M$.

A submanifold M tangent to ξ of a contact manifold $\widetilde{M}^{2n+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M such that:

(1) $TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
(2) D is invariant by ϕ, i.e., $\phi(D_p) \subset D_p$, for each $p \in M$;
(3) $D^{\perp}$ is anti-invariant by ϕ, i.e., $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, for each $p \in M$.

Let $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ be a $(2n+1)$-dimensional contact manifold such that

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi, \qquad (\widetilde{\nabla}_X \phi)Y = \widetilde{g}(\phi X, Y)\xi - \eta(Y)\phi X,$$

where $\widetilde{\nabla}$ is the Levi-Chivita connection of $\widetilde{M}$, then $\widetilde{M}$ is called a Kenmotsu manifold.

A plane section $\pi \subset T_x\widetilde{M}$ is called a ϕ-section if $\phi(\pi_x) \subseteq \pi_x$ for each $x \in \widetilde{M}$. The manifold $\widetilde{M}$ is said to have constant ϕ-sectional curvature if the sectional curvature of every ϕ-section is constant. A Kenmotsu space form is a Kenmotsu manifold of constant ϕ-sectional curvature. In this case the Riemannian curvature tensor field $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c+3}{4}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \\ -\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\widetilde{g}(Y,Z)\eta(X) - \widetilde{g}(X,Z)\eta(Y)]\xi\end{aligned}$$

$$-\widetilde{g}(\phi Y, Z)\phi X + \widetilde{g}(\phi X, Z)\phi Y + 2\widetilde{g}(\phi X, Y)\phi Z\}$$

for each $X, Y, Z \in \chi(\widetilde{M})$.

3. Hypersurfaces of a Kenmotsu Space Forms

Let (M^{2n}, g) be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$, tangent to ξ. Let N be a local unit normal vector field on M. Then the anti-invariant distribution $D^{\perp}$ is one-dimensional and spanned by $U = -\phi N$.

Let $\widetilde{\nabla}$ and ∇ denote the Levi-Civita connections of the Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ and the hypersurface M^{2n}, respectively. Then, by the Gauss formula and the Kenmotsu structure equations,

$$\nabla_X \xi + g(AX, \xi)N = X - \eta(X)\xi,$$

where A is the shape operator of M.

Lemma 3.1. *Let M^{2n} be a hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ tangent to ξ, and let A denote the shape operator of M. Then $A\xi = 0$.*

Lemma 3.2. *Let M^{2n} be a hypersurface of a Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ tangent to the vector field of ξ and A be the shape operator of M. Then $AU = \alpha U$.*

Lemma 3.3. *Let M^{2n} be a hypersurface of a Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ tangent to the vector field of ξ and A be the shape operator of M. Suppose there exists a vector field X such that both X and FX are principal vectors. Then $c = -1$.*

4. Main results

A hypersurface M of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ is said to be locally symmetric if it satisfies

$$\nabla R = 0.$$

Theorem 4.1. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ do not admit any real hypersurface M^{2n} that is locally symmetric.*

A hypersurface M^{2n} of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ is said to have a recurrent structure Jacobi operator if there exists a 1-form ω on M such that

$$(\nabla_X R_\xi)Y = \omega(X)R_\xi(Y),$$

for each $X, Y \in \chi(M)$ where ω is a one-form on M.

Theorem 4.2. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ do not admit any real hypersurface M^{2n} with a recurrent structure Jacobi operator.*

Corollary 4.3. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ do not admit any real hypersurface M^{2n} with a parallel structure Jacobi operator.*

Theorem 4.4. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ do not admit any real hypersurface M^{2n} with a D-recurrent structure Jacobi operator.*

Corollary 4.5. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ do not admit any real hypersurface M^{2n} with a D-parallel structure Jacobi operator.*

Let M^{2n} be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$, tangent to ξ. For each $p \in M$, let

$$H_p(M) = \{v \in T_pM \mid \phi v \in T_pM\}$$

be the ϕ-invariant subspace of T_pM (i.e., the holomorphic distribution D_p). Let $H(M)$ denote the smooth distribution $\bigcup_{p\in M} H_p(M)$. For any $X \in H(M)$, define the function

$$H(X) = g\big(R(X, \phi X)\phi X, X\big).$$

The hypersurface M is said to have weakly constant holomorphic curvature if $H(X)$ is a constant on $H(M)$.

THEOREM 4.6. *Let M^{2n} be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ tangent to ξ and with weakly constant holomorphic curvature. Then $c = -1$.*

THEOREM 4.7. *Kenmotsu space forms $\widetilde{M}^{2n+1}(c)$ with $c \neq -1$ do not admit any real hypersurface M^{2n} with weakly constant holomorphic curvature.*

THEOREM 4.8. *Let M^{2n} be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(-1)$ tangent to ξ and with weakly constant holomorphic curvature. Then the shape operator A of M has exactly n distinct eigenvalues.*

THEOREM 4.9. *Let M^{2n} be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(c)$ tangent to ξ. If the shape operator A commutes with ϕ, i.e.,*

$$\phi AX = A\phi X \quad \text{for all } X \in \mathfrak{X}(M),$$

then

$$c = -1.$$

THEOREM 4.10. *Let M^{2n} be a real hypersurface of a Kenmotsu space form $\widetilde{M}^{2n+1}(-1)$ tangent to ξ and satisfying the commutative condition*

$$\phi AX = A\phi X \quad \text{for all } X \in \mathfrak{X}(M).$$

Then the shape operator A of M has exactly two distinct eigenvalues.

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On characterizations of hypersurfaces in a Sasakian space form

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Abstract. Let M be a real hypersurface in a Sasakian space form $\overline{M}(c)$. In this paper, we prove that if $R_U L_U = L_U R_U$ holds on M, then M is a Hopf hypersurface, where R_U and L_U denote the Jacobi operator with respect to the structure vector field and the operator induced by the Lie derivative along the unit normal vector field U, respectively. Furthermore, we provide a complete characterization of such Hopf hypersurfaces in $\overline{M}(c)$.

Keywords: Hopf hypersurfaces, Commuting operators, Sasakian manifold

AMS Mathematics Subject Classification [2020]: 53C25, 53C40

1. Introduction

Let $\overline{M}(c)$ be a Sasakian space form endowed with the metric g. Let M be a real hypersurface of $\overline{M}(c)$ such that the structure vector field ξ is tangent to M. Denote by ϕ the contact structure on $\overline{M}(c)$ and by N a local unit normal vector field to M. Then $U = -\phi N$ is a tangent vector field to M, referred to as the induced normal vector field on M. The hypersurface M inherits a metric structure (F, g, ξ, η, U, u) induced from the contact metric structure and the endomorphism ϕ of $\overline{M}(c)$. A hypersurface M is called a Hopf hypersurface if the plane spanned by ξ and U is invariant under the shape operator A of M [1]. Hypersurfaces in Sasakian space forms have been extensively studied in [2,3].

The induced operator L_U on the hypersurface M is defined via the Lie derivative of the metric along U as follows:

$$(\mathcal{L}_U g)(X, Y) = g(L_U X, Y)$$

for all vector fields X, Y on M, where $\mathcal{L}_U$ denotes the Lie derivative with respect to the induced normal vector field U.

*Speaker.

2. Preliminaries

A differentiable manifold $\widetilde{M}^{2m+1}$ of odd dimension is said to admit an almost contact structure if there exist tensor fields ϕ, ξ, and η of types $(1,1)$, $(0,1)$, and $(1,0)$, respectively, satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity endomorphism on the tangent spaces at every point of $\widetilde{M}^{2m+1}$. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2m$ at every point of $\widetilde{M}^{2m+1}$. A manifold $\widetilde{M}^{2m+1}$ equipped with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold and is denoted by $(\widetilde{M}^{2m+1}, \phi, \xi, \eta)$.

Let $\widetilde{M}^{2m+1}$ be a manifold equipped with an almost contact structure (ϕ, ξ, η). A Riemannian metric g on $\widetilde{M}^{2m+1}$ is said to be compatible with this structure if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y on $\widetilde{M}^{2m+1}$. In this case, the quadruple (ϕ, ξ, η, g) is called an almost contact metric structure on $\widetilde{M}^{2m+1}$. Substituting $Y = \xi$ into the compatibility condition yields

$$\eta(X) = g(X, \xi)$$

for every vector field X tangent to $\widetilde{M}^{2m+1}$, which shows that η is the metric dual of the characteristic vector field ξ.

A contact manifold $\widetilde{M}^{2m+1}$ is a $(2m+1)$-dimensional manifold equipped with a global contact 1-form η such that

$$\eta \wedge (d\eta)^m \neq 0$$

everywhere on $\widetilde{M}$.

A submanifold $M \subset \widetilde{M}^{2m+1}$ tangent to the Reeb vector field ξ is called

- invariant if $\phi(T_pM) \subset T_pM$ for all $p \in M$,
- anti-invariant if $\phi(T_pM) \subset T_p^{\perp}M$ for all $p \in M$.

A submanifold M of a Riemannian contact manifold $\widetilde{M}^{2m+1}$, tangent to the Reeb vector field ξ, is called a contact CR-submanifold if there exist a pair of orthogonal smooth distributions D and $D^{\perp}$ on M such that:

(1) $TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
(2) D is invariant by ϕ, i.e., $\phi(D_p) \subset D_p, \forall p \in M$;
(3) $D^{\perp}$ is anti-invariant by ϕ, i.e., $\phi(D_p^{\perp}) \subset T_p^{\perp}M, \forall p \in M$.

Let $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ be a $(2n+1)$-dimensional contact manifold such that

$$\overline{\nabla}_X \xi = -\phi X \qquad , \qquad (\overline{\nabla}_X \phi)Y = \widetilde{g}(X, Y)\xi - \eta(Y)X,$$

then $\widetilde{M}$ is called a Sasakian manifold. By tacking into account that, the plane section π of $T\widetilde{M}$ is called a $\phi-$section if $\phi\pi_x \subseteq \pi_x$ for any $x \in \widetilde{M}$, a Sasakian space form is the Sasakian manifold of the constant $\phi-$ sectional curvature. The Riemannian curvature tensor field $\widetilde{R}$ of the Sasakian space form is given by [4, 5]

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c+3}{4}&\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \\ &-\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\widetilde{g}(Y,Z)\eta(X) - \widetilde{g}(X,Z)\eta(Y)]\xi \\ &\quad -\widetilde{g}(\phi Y, Z)\phi X + \widetilde{g}(\phi X, Z)\phi Y + 2\widetilde{g}(\phi X, Y)\phi Z\},\end{aligned}$$

for any $X, Y, Z \in \chi(\widetilde{M})$.

3. Hypersurfaces in the Sasakian space form

Let (M, g) be a real hypersurface in a Sasakian space form $\overline{M}(c)$, tangent to the Reeb vector field ξ, with unit normal vector field N. Then the tangent bundle decomposes as

$$TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi,$$

where D is a ϕ-invariant subspace and $D^{\perp}$ is an one dimensional subspace, that is spanned by $U = -\phi(N)$, and is the orthogonal component of D.

Moreover, it is clear that $\phi TM \subseteq TM \oplus SpanN$. Hence, we have for any tangent vector field X the following decomposition in the tangent and the normal components:

$$\phi X = FX + u(X)N. \tag{1}$$

It is easily shown that F is a skew-symmetric linear endomorphism that acts on T_xM. Since the structure vector field ξ is tangent to M, (1) implies

$$F\xi = 0, \quad FU = 0, \quad g(U, X) = u(X), \quad u(\xi) = g(U, \xi) = 0, \quad u(U) = 1. \tag{2}$$

Next, by applying ϕ to (1) and using (2), we also have

$$F^2X = -X + \eta(X)\xi + u(X)U, \quad u(FX) = 0.$$

We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\overline{M}$ and M, respectively. Then the Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

for any vector fields X, Y tangent to M. Here and in the sequel h denotes the second fundamental form and A is the shape operator corresponding to the normal vector field N. Therefore

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N.$$

Definition 3.1. [1] Let A denote the shape operator of the hypersurface M in the Sasakian space form $\overline{M}(c)$. If the 2-dimensional plane span$\{\xi, U\}$ is A-invariant, then M is called a Hopf hypersurface of $\overline{M}(c)$.

By taking the covariant derivative of both sides of equation (1) and equating the tangential and normal components, we obtain:

$$\begin{aligned}
&(\nabla_Y F)X = -g(Y, X)\xi + \eta(X)Y - g(AY, X)U + u(X)AY,\\
&(\nabla_Y u)X = g(FAY, X),\\
&\nabla_X U = FAX.
\end{aligned}$$

On the other hand, since the structure vector ξ is tangent to M, we get

$$\begin{aligned}
\nabla_X \xi &= FX,\\
A\xi &= U.
\end{aligned}$$

Moreover, for any vector fields X, Y, Z tangent to M by applying the equation (1), the Gauss and the Codazzi equations hold

$$\begin{aligned}
R(X, Y)Z = &\frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\}\\
&-\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi\\
&-g(FY, Z)FX + g(FX, Z)FY + 2g(FX, Y)FZ\}
\end{aligned}$$

$$+g(AY,Z)AX - g(AX,Z)AY,$$
$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c-1}{4}\{u(X)FY - u(Y)FX - 2g(FX,Y)U\}.$$

4. Main results

We first prove that the Lie derivative operator L_U satisfies

$$L_U = FA - AF \quad \text{on } M,$$

and that the induced normal vector field $U = -\phi N$ is Killing if and only if $L_U = 0$. In particular, we establish that:

THEOREM 4.1. *Let M be a real hypersurface in the Sasakian space form $\overline{M}(c)$ with $\xi \in \Gamma(TM)$ and $L_U = 0$ on M. Then M is of one of the following two types:*

- *M is a locally isometric to*
$$S^n(r_1) \times S^n(r_2), \qquad (r_1^2 + r_2^2 = 1).$$
- *M is locally the product $M' \times \zeta$, where M' is a totally geodesic manifold and ζ is a geodesic curve of M.*

For the curvature tensor field R on a real hypersurface M, the Jacobi operator R_X with respect to a unit vector field X is defined by

$$R_X(Y) = R(Y,X)X \quad \text{for all } Y \in \Gamma(TM).$$

Also, we obtain:

THEOREM 4.2. *Let M be a real hypersurface in the Sasakian space form $\overline{M}(c)$ with $\xi \in \Gamma(TM)$ and satisfying $R_U L_U = 0$ on M. Then M is one of the hypersurfaces listed in Theorem 4.1.*

Finally, we study real hypersurfaces in the Sasakian space form $\overline{M}(c)$ for which the Jacobi operator R_U and the Lie derivative operator L_U commute, i.e.,

$$[R_U, L_U] = 0,$$

and obtain the following:

THEOREM 4.3. *Let M be a real hypersurface in the Sasakian space form $\overline{M}(c)$ with $\xi \in \Gamma(TM)$ and satisfying*

$$R_U L_U = L_U R_U \quad \text{on } M.$$

Then M is one of the hypersurfaces listed in Theorem 4.1.

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On characterizations of Contact CR Submanifolds of maximal Contact CR dimension of Sasakian Space Form

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Abstract. In this paper, we investigate contact CR submanifolds of contact CR dimension in Sasakian space form and introduce the general structure of these submanifolds and then studying structures of this submanifols with the condition $h(FX,Y)+h(X,FY)=g(FX,Y)\zeta$, for the normal vector field ζ, which is nonzero, and we classify these submanifolds.

Keywords: Sasakian space form, Contact CR Submanifolds of maximal Contact CR dimension, Submanifold

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1. Introduction

Let M be a connected $(n+1)$-dimensional submanifold of codimension $q+1$ in a Sasakian space form $(\overline{M},\phi,\xi,\eta,g)$, where $n>5$. It is known that if the maximal ϕ-invariant subspace of each tangent space to M is $(n-1)$-dimensional, then M admits a naturally induced metric structure [1–3]. In the hypersurface case $(q=0)$, this maximal ϕ-invariant subspace is necessarily $(n-1)$-dimensional and coincides with the maximal holomorphic subspace when the ambient space $\overline{M}$ is a Sasakian space form. For arbitrary codimension $q+1$, however, less detailed results are available, though further developments may be anticipated.

Kim and etc. studied in [4] the maximal dimensional contact CR-submanifold in unit sphere which satisfy the condition

$$h(FX,Y)+h(X,FY)=0$$

and determined such submanifolds under the additional condition, where F denotes a skew-symmetric endomorphism induced from ϕ acting on the tangent bundle TM and h the second fundamental form on M. Also Okunura and etc. studied in [4] the maximal dimensional contact CR-submanifold in complex space form with same condition.

*Speaker.

Afterward studied the maximal dimensional contact CR-submanifold in unit sphere and complex space form which satisfy the condition

$$h(FX,Y)+h(X,FY)=g(FX,Y)\zeta$$

for a normal vector field ζ to M. Recently in [5] Kim and etc. studied the maximal dimensional contact CR-submanifold in Sasakian space form which satisfy the condition

$$h(FX,Y)+h(X,FY)=0.$$

In this paper we study $(n+1)-$dimensional contact CR-submanifold of $(n-1)$ contact CR-dimension in a Sasakian space form and determine such submanifolds in a complete simply connected Sasakian space form of constant ϕ-holomorphic sectional curvature c under assumption that

$$h(FX,Y)+h(X,FY)=g(FX,Y)\zeta.$$

2. Preliminaries

A differentiable manifold $\widetilde{M}^{2n+1}$ is said to admit an almost contact structure if there exist a (non-vanishing) vector field ξ, a one-form η, and a $(1,1)$-tensor field ϕ satisfying

$$\eta(\xi)=1, \quad \phi^2=-I+\eta\otimes\xi,$$

where I denotes the identity endomorphism on the tangent spaces. These conditions imply that $\phi\xi=0$, $\eta\circ\phi=0$, and that the endomorphism ϕ has rank $2n$ at every point of $\widetilde{M}^{2n+1}$. A manifold $\widetilde{M}^{2n+1}$ equipped with an almost contact structure (ϕ,ξ,η) is called an almost contact manifold and will be denoted by $(\widetilde{M}^{2n+1},\phi,\xi,\eta)$.

Let $\widetilde{M}^{2n+1}$ be a manifold equipped with an almost contact structure (ϕ,ξ,η). A Riemannian metric $\widetilde{g}$ on $\widetilde{M}^{2n+1}$ is said to be compatible with this structure if

$$\widetilde{g}(\phi X,\phi Y)=\widetilde{g}(X,Y)-\eta(X)\eta(Y)$$

for all vector fields X,Y on $\widetilde{M}^{2n+1}$. In this case, the quadruple $(\phi,\xi,\eta,\widetilde{g})$ is called an almost contact metric structure on $\widetilde{M}^{2n+1}$, and the manifold is denoted by $(\widetilde{M}^{2n+1},\phi,\xi,\eta,\widetilde{g})$.

A manifold $\widetilde{M}^{2n+1}$ is said to be a contact manifold if it admits a global one-form η, called the contact form, such that, $\eta\wedge(d\eta)^n\neq 0$ at every point of $\widetilde{M}^{2n+1}$.

Let $(\widetilde{M},\phi,\xi,\eta,\widetilde{g})$ be a $(2n+1)$-dimensional contact metric manifold such that

$$\widetilde{\nabla}_X\xi=\phi X, \qquad (\widetilde{\nabla}_X\phi)Y=-\widetilde{g}(X,Y)\xi+\eta(Y)X$$

for all vector fields X,Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection of $\widetilde{g}$. Then $\widetilde{M}$ is called a Sasakian manifold.

A plane section $\pi\subset T_x\widetilde{M}$ at a point $x\in\widetilde{M}$ is called a ϕ-section if $\phi(\pi)\subseteq\pi$. The Sasakian manifold $\widetilde{M}$ is said to have constant ϕ-sectional curvature if the sectional curvature of every ϕ-section is constant (say, equal to $c\in\mathbb{R}$).

3. CR maximal dimensional submanifold structure

Let $\overline{M}$ be a $(2m+1)-$dimensional almost contact metric manifold with structure (ϕ,ξ,η,g). Let M be an $(n+1)-$dimensional submanifold tangent to the structure vector field ξ of $\overline{M}$ and denote by D_x the ϕ-invariant subspace of the tangent space T_xM of M at $x\in M$. Then ξ cannot be contained in D_x at any point $x\in M$. Thus the assumption $\dim D_x^{\perp}=2$ being constant and equal to 2 at each point $x\in M$ yields that M can be dealt with a contact CR-submanifold, where $D_x^{\perp}$ denotes the complementary orthogonal

subspace to D_x in T_xM. In fact, if there exists a non-zero vector U which is orthogonal to ξ and contained in $D_x^{\perp}$, then

$$N := \phi U \tag{1}$$

must be normal to M.

Moreover, it is clear that $\phi TM \subseteq TM \oplus SpanN$. Hence we have, for any tangent vector field X and for a local orthogonal basis $N_{ii=1,\dots,p}(N_1 := N, p := 2m-n)$ of normal vectors to M, the following decomposition in tangential and normal components:

$$\phi X = FX + u(X)N, \qquad \phi N_\alpha = \sum_{j=2}^{p} PijN_j, \qquad i = 2,\dots,p. \tag{2}$$

It is easily shown that F is a skew-symmetric endomorphism acting on T_xM and $P_{ij} = -P_{ji}$. Since the structure vector field ξ is tangent to M, Sasakian conditions with (1) and (2) imply

$$F\xi = 0, FU = 0, u(X) = g(U,X), u(\xi) = \eta(U) = 0. \tag{3}$$

Next, applying ϕ to (2) and using Sasakian conditions with (1), (2) and (3), we also have

$$F^2X = -X + u(X)U + \eta(X)\xi, \qquad u(FX) = 0.$$

On the other hand, it is clear from Sasakian conditions and (3) that

$$\phi N = -U, \tag{4}$$

which combined with (2) yields the existence of a local orthonormal basis $\{N, N_a, N_{a^*}\}_{a=1,\dots,q}$ of normal vectors to M such that

$$N_{a^*} := \phi N_a, \qquad a = 1,\dots,q := \frac{(p-1)}{2}. \tag{5}$$

We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}(c)$ and M, respectively, and by $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{6}$$

$$\overline{\nabla}_X N = -AX + \nabla_X^{\perp} N = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*}\}, \tag{7}$$

for any tangent vector fields X, Y to M. Here and in the sequel h denotes the second fundamental form and A, A_a, A_{a^*} the shape operators corresponding to the normals N, N_a, N_{a^*}, respectively. They are related by

$$h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*}\}. \tag{8}$$

Also we have

$$(\nabla_Y F)X = -g(Y,X)\xi + \eta(X)Y - g(AY,X)U + u(X)AY,$$
$$\nabla_X U = FAX, \qquad (\nabla_Y u)X = g(FAY,X).$$

On the other hand, since ξ is tangent to M, from Sasakian conditions combined with (6) and (8) yields

$$\nabla_X \xi = FX, (\nabla_X \eta)Y = g(FX,Y),$$
$$\eta(AX) = g(A\xi,X) = u(X), i.e., A\xi = U,$$

$$A_a\xi = 0, A_{a^*}\xi = 0, a = 2, \ldots, q.$$

If the ambient manifold $\overline{M}$ is a Sasakian space form $\overline{M}(c)$, i.e., a Sasakian manifold of constant ϕ-holomorphic sectional curvature c, then its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \end{aligned}$$

for any vector fields X, Y, Z tangent to $\overline{M}$.

4. Main results

In this section, let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $n-1$, immersed in a Sasakian manifold $\overline{M}$. We adopt the notation established in the previous section.

We assume that the equality

$$h(FX,Y) - h(X,FY) = g(FX,Y)\zeta \tag{9}$$

holds on M for a normal vector field ζ to M. We also use the orthonormal basis (5) of normal vectors to M and set

$$\zeta = \rho N + \sum_{a=1}^{q}(\rho_a N_a + \rho_{a^*} N_{a^*}).$$

Lemma 4.1. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $n-1$, immersed in a Sasakian space form $\overline{M}(c)$. Assume that the condition verified in 9 holds. Then the shape operator A has constant eigenvalues.*

Lemma 4.2. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $n-1$, immersed in a Sasakian space form $\overline{M}(c)$. Assume that the condition verified in 9 holds. Then the function ζ is constant and non-zero.*

Theorem 4.3. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $n-1$, immersed in a Sasakian space form $\overline{M}(c)$. Assume that the condition verified in 9 holds. Then M is locally isometric to one of the following:*

- *the Riemannian product $S^1 \times M'$, where M' is a submanifold of $\overline{M}$;*
- *the Riemannian product $C \times M'$, where C is a geodesic curve and M' is a totally geodesic hypersurface of $\overline{M}$;*
- *the Riemannian product $M_1 \times M_2$, where M_1 and M_2 are submanifolds of $\overline{M}$.*

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G- Manifolds with Negative Curvature

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Abstract. We classify cohomogeneity two UND-Riemannian G-manifolds under the condition $M^G \neq \emptyset$.

Keywords: Lie group, Isometry, Manifold, cohomogeneity.

AMS Mathematics Subject Classification [2020]: topology differential geometry

1. INTRODUCTION

A classic theorem about Riemannian manifolds of non-positive curvature [12] states that a homogeneous Riemannian manifold M of non positive curvature is simply connected or it is diffeomorphic to a cylinder over a torus (i.e, it is diffeomorphic to $R^k \times T^s$, $k + s = dimM$). It is interesting to reduce the homogeneity condition to weaker conditions and see what happens to the topology of M. When M is homogeneous then there is a connected and closed subgroup G of the isometries of M such that the orbit space of the action of G on M, $\frac{M}{G}$ is a one point set. A weaker condition is that $dim\frac{M}{G} = 1$ or 2 (i.e, M be a cohomogeneity one or cohomogeneity two G-manifold). There are some interesting theorems about topological properties of cohomogeneity one G-manifolds of non-positive curvature under conditions on G and M (see [1], [9], [10], [11]). There is a topological characterization of cohomogeneity one UND-Riemannian manifolds (Riemannian manifolds with the property that the universal covering manifold decomposes as a direct product of negatively curved manifolds) in [10]. Following the papers [7] and [8], where the first author proved various results about topological properties of cohomogeneity two negatively curved G-manifold M under some special conditions on M or G, we are going to consider some cohomogeneity two UND- manifolds in the present paper. We topologically characterize a UND-manifold M which is acted on isometrically by a connected and closed subgroup G of isometries, under the condition that the fixed point set of the action is not empty. Our main result is Theorem 3.5.

2. Preliminaries

*Speaker.

We will use the following definitions, facts and symbols in the proof of the main theorem.

Definition 2.1. A differentiable function $f : M \to R$ on a complete Riemannian manifold M is called convex (strictly convex) if for each geodesic $\gamma : R \to M$ the composed function $f \circ \gamma : R \to R$ is convex (strictly convex), that is $(fo\gamma)'' \geq 0$ ($(f \circ \gamma)'' > 0$).

Fact 2.2 (see [2], [5]).
(1) Let M be a simply connected Riemannian manifold of nonpositive curvature and $\delta \in Iso(M)$. The squared displacement function $d_\delta^2 : M \to M$ defined by $d_\delta^2(x) = (d(x, \delta x))^2$ is a convex function.

(2) If in (1) M has negative curvature then d_δ^2 is strictly convex except at the minimum point set.

(3) Let M be a simply connected Riemannian manifold of negative curvature and δ be an isometry on M without fixed point. If there is a geodesic γ such that $\delta(\gamma) = \gamma$ then the image of γ is the minimum point set of the function $d_\delta^2 : M \to M$.

(4) If $f : M \to R$ is a convex function defined on a complete Riemannian manifold M then the minimum point set C of f is a totally convex subset of M (i.e, it contains every geodesic segment with endpoints inside C).

(5) Let M be a complete Riemannian manifold of nonpositive curvature. A submanifold S of M is closed and totally convex if and only if S is totally geodesic and the exponential map $exp : \perp S \to M$ is a diffeomorphism. Where, $\perp S$ denotes the normal bundle of S.

Fact 2.3 (see [3], [6]). Let M be a Riemannian manifold and G be a connected subgroup of $Iso(M)$, and let $\widetilde{M}$ be the universal Riemannian covering manifold of M with the covering map $\kappa : \widetilde{M} \to M$ and deck transformation group Δ. Then, there is a connected covering $\widetilde{G}$ of G with the covering map $\pi : \widetilde{G} \to G$, such that $\widetilde{G}$ acts isometrically on $\widetilde{M}$ and
(1) Each $\delta \in \Delta$ maps $\widetilde{G}$-orbits on to $\widetilde{G}$-orbits.
(2) If $x \in M$ and $\widetilde{x} \in \pi^{-1}(x)$ then $\kappa(\widetilde{G}(\widetilde{x})) = G(x)$.
(3) $\widetilde{M}^{\widetilde{G}} = \kappa^{-1}(M^G)$.
(4) The deck transformation group centralizes $\widetilde{G}$ (i.e., for each $\delta \in \Delta$ and $\widetilde{g} \in \widetilde{G}$, $\delta\widetilde{g} = \widetilde{g}\delta$).
(6) If $\widetilde{M}^{\widetilde{G}}$ is a one point set then $M = \widetilde{M}$.

Fact 2.4. If G is a closed and connected subgroup of the isometries of a Riemannian manifold M, then the set of the fixed points of the action of G on M, $M^G = \{x \in M : G(x) = x\}$, is a totally geodesic submanifold of M.

Lemma 2.5 ([7]). *If M is a connected and complete cohomogeneity k Riemannian G-manifold then $k > dimM^G$.*

Fact 2.6. ([4]) If M is a Riemannian manifold of negative curvature, $\widetilde{M}$ is its universal covering manifold and Δ is the deck transformation group such that for a geodesic γ in $\widetilde{M}$, $\Delta(\gamma) = \gamma$, then Δ is isomorphic to $(Z, +)$.

Fact 2.7. A vector bundle over circle S^1 is diffeomorphic to a cylinder over S^1 or it is diffeomorphic to a cylinder over the mobious band B.

3. Results

Definition 3.1. We say that a Riemannian manifold M is universally and negatively decomposable (UND), when its universal covering manifold $\widetilde{M}$ decomposes as $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2 \times ... \times \widetilde{M}_k$ such that for each i, $\widetilde{M}_i$ has negative curvature, and each $\delta \in \Delta$ decomposes as $\delta = \delta_1 \times \delta_2 \times ... \times \delta_k$, $\delta_i \in ISO(\widetilde{M}_i)$.

Example 3.2. If M is a direct product of negatively curved manifolds then M is a UND-manifold.

Example 3.3. If the factors of de Rham decomposition of $\widetilde{M}$ have negative curvatures and Δ is a subset of the connected component of $ISO(\widetilde{M})$ then M is a UND-manifold (see [7] vol. 1, page 240).

Lemma 3.4. *Let M be a UND-Riemannian manifold and let $\widetilde{M}$ be its universal covering, by the covering map $k : \widetilde{M} \to M$. If there is a geodesic γ in $\widetilde{M}$ and an element δ in the center of Δ such that $\delta(\gamma) = \gamma$. Then M is a vector bundle over a circle S^1.*

Proof: Suppose that $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2 \times ... \times \widetilde{M}_l$, $\Delta = \Delta_1 \times \Delta_2 \times .... \times \Delta_l$, $\delta = \delta_1 \times \delta_2 \times .... \times \delta_l$, $\gamma = \gamma_1 \times \gamma_2 .... \times \gamma_l$ such that for each i, $\delta_i \in \Delta_i$ and γ_i is a geodesic in $\widetilde{M}_i$. Since δ belongs to the center of Δ, for each i, δ_i commutes with the elements of Δ_i. Consider the convex functions $\widetilde{f}_i : \widetilde{M}_i \to R$ defined by $\widetilde{f}_i(a) = d^2(a, \delta_i a)$ and put

$$\widetilde{f} : \widetilde{M} \to R, \quad f(x) = \sum_i f_i(x_i), \quad x = (x_1, ..., x_l).$$

$\widetilde{f}$ is a convex function on $\widetilde{M}$. Now put

$$f : M \to R, \quad f(x) = \widetilde{f}(\kappa^{-1}(x))$$

f is well defined because if $y, z \in \kappa^{-1}(x)$ then there is a σ in Δ such that $\sigma(y) = z$, so

$$\widetilde{f}(z) = \widetilde{f}(\sigma y) = \sum_i f_i(\sigma_i y_i) = \sum d^2(\sigma_i y_i, \delta_i \sigma_i y_i) =$$

$$\sum d^2(\sigma_i y_i, \sigma_i \delta_i y_i) = \sum d^2(y_i, \delta_i y_i) = \widetilde{f}(y)$$

Put $\lambda = \kappa \circ \gamma$. Since by assumptions, $\delta(\gamma) = \gamma$, then for all i, $\delta_i \gamma_i = \gamma_i$. By Fact 2.3, the image of γ_i is the minimum point set of the function $\widetilde{f}_i : \widetilde{M}_i \to R$. Then, the image of γ is the minimum point set of $\widetilde{f}$, and the image of λ must be the minimum point set of f. Now, by Fact 2.2(4), the image of λ is totally convex, so it is simply closed geodesic in M and diffeomorphic to S^1. Then, by Fact 2.2(5), M is a vector bundle over S^1.

Theorem 3.5. *Let M^{n+2} be a nonsimply connected UND-Riemannian manifold which is*

of cohomogeneity two under the action of G a closed and connected subgroup of isometries such that $M^G \neq \emptyset$. Then
(a) M^G is diffeomorphic to S^1.
(b) M is diffeomorphic to $S^1 \times R^{n+1}$ or $B \times R^n$, where B is the mobious band.

Proof: Keeping the symbols of Fact 2.3, If $\widetilde{M^{\widetilde{G}}}$ is a one point set then by Fact 2.3(6), M must be simply connected, which is in contrast with the assumptions. If $dim\widetilde{M^{\widetilde{G}}} \geq 2$ then by Lemma 2.5, the cohomogeneity of the action of G on M must be bigger than 3. Therefore, $dim\widetilde{M^{\widetilde{G}}} = 1$, and $\widetilde{M^{\widetilde{G}}}$ is equal to the image of a geodesic γ. By Fact 2.3(4), the elements of Δ and $\widetilde{G}$ are commutative. So, for each $g \in \widetilde{G}$, $x \in \gamma$ and $\delta \in \Delta$, we have

$$g\delta(x) = \delta g(x) = \delta(x) \Rightarrow \delta(x) \in \gamma.$$

Thus, Δ is a discrete subgroup of the isometries such that $\Delta(\gamma) = \gamma$ and by Fact 2.6, it must be isomorphic to Z. Now, by Lemma 3.4, M is a vector bundle over a circle S^1 and by Fact 2.7, it is diffeomorphic to $S^1 \times R^{n+1}$ or $B \times R^n$. Also, $\widetilde{M^{\widetilde{G}}}$ is diffeomorphic to R (the image of γ), Δ is isomorphic to $\pi_1(M) = Z$ and $\Delta(\gamma) = \gamma$. Thus M^G is diffeomorphic to $\frac{R}{Z} = S^1$.

Example 3.6. Consider a geodesic γ in H^{n+1}, $n > 2$. Let G_1 be the group of rotations about γ, G_2 the group of all transvections along γ and Δ be a discrete subgroup of G_2 which must be isomorphic to Z. The manifold $M = \frac{H^n}{\Delta}$ is diffeomorphic to $S^1 \times R^n$, it is of cohomogeneity two nuder the action of $G = G_1$ and $M^G = S^1$.

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Organic 3D Forms with Local Learning in Parametric Design

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Abstract: This paper presents a simple yet innovative method for generating organic 3D forms in parametric design using a lightweight local learning algorithm. In this approach, each layer of curves adapts based on the previous layer, producing smooth, wave-like shapes with a natural and coherent appearance.Unlike traditional methods that rely solely on random noise, local learning allows the geometry to self-organize, providing unique outputs in every execution. The method is lightweight, computationally efficient, and does not require training data, while offering high flexibility. Its applications are extensive and include architectural and interior design, creation of organic structures, fractal-inspired design, digital sculpting, product and industrial design, automotive and aerospace components, fashion and jewelry design, virtual reality and game environments. Moreover, it is valuable in mathematics, particularly in computational geometry, fractal and nonlinear geometry, shape analysis, and mathematical modeling, serving as a tool for both research and education in exploring complex geometric patterns and structures.

Keywords: Parametric Design, Fractal Geometry, Local Learning, Adaptive Growth, Computational Design

1. Introduction

In recent decades, parametric design has emerged as an advanced approach in architecture and digital art, enabling the creation of complex and organic forms using computational algorithms [1]. This approach has been particularly applied in landmark projects such as the Eden Project in the United Kingdom and the Beijing National Stadium (Bird's Nest) to develop structures with complex geometries and optimized performance [2]. One of the main challenges in parametric design is generating forms that possess natural and harmonious characteristics. In this context, local learning algorithms have been proposed as a novel method, where new forms are generated based on information from previous layers, producing geometries that closely resemble natural structures [3]. This method is especially effective in organic and nature-inspired design and allows unique forms to be produced in each execution [3,4]. Beyond design and artistic applications, local learning algorithms can serve as tools in mathematics for geometric modeling and analysis of complex structures. In particular, they allow the simulation of natural patterns

and self-organizing structures [4]. The aim of this study is to introduce a lightweight, flexible, and efficient method for generating organic 3D forms using local learning in parametric design. This approach can be applied in architectural projects, digital art, and scientific research, enabling the creation of coherent, aesthetically pleasing, and unique geometries in every execution.

2. Methodology

In this study, organic 3D forms are generated using a parametric approach with Python in the Rhino/Grasshopper environment. This method enables the creation of smooth and natural geometry by combining parametric settings, local learning, and procedural noise. The main parameters include the number of layers (num_layers), the number of points per layer (points_per_layer), base radius (radius_base), height (z_height), petal count (petal_count), twist rate (twist_rate), local learning adjustment (adjust_rate), and noise amplitude (noise_amp). These parameters allow precise control over the shape, complexity, and natural variation of the forms. In each layer, points are distributed evenly around a circle, and their radial distances are modulated using a sinusoidal wave to create petal-like undulations. Local learning adjusts the radius of each point based on the corresponding point in the previous layer to ensure smooth transitions between layers, while small random noise simulates natural irregularities. Interpolated curves are created for each layer, and a loft operation produces a continuous 3D surface. Using lofting allows precise control over the surface curvature [5], and local adjustments enable the geometry to self-organize, producing unique and natural outputs for each execution [6]. The method is flexible, allowing designers to explore various organic shapes by adjusting parameters. Both curves and lofted surfaces are output, providing further manipulation and integration into digital design, architectural modeling, or computational geometry studies.

3. Experimental Results and Visualization

The experimental implementation of the parametric algorithm generated a variety of 3D forms with smooth and coherent surfaces, demonstrating the impact of local learning and procedural noise. Multiple simulations were conducted by varying key parameters, including the number of layers, points per layer, petal count, twist rate, and local learning adjustment. Each simulation produced unique organic shapes, illustrating the algorithm's ability to generate a wide range of forms while maintaining structural harmony. Visualization of the generated curves and lofted surfaces revealed that the local learning mechanism effectively minimized abrupt transitions between layers, producing smooth and continuous surfaces. The sinusoidal undulations combined with controlled noise introduced subtle natural variations, giving the forms a lifelike appearance. Higher petal counts and increased twist rates resulted in more intricate patterns, while lower values produced simpler, wave-like forms. These variations demonstrate the flexibility of the method in exploring different levels of geometric complexity and aesthetic characteristics. Recent studies have shown that parametric self-organizing methods can enhance the design workflow and enable rapid exploration of complex forms [7]. Additionally, the use of visualization in the design process improves understanding of geometry and allows for quick parameter adjustments [8]. The lofted surfaces were rendered in Rhino/Grasshopper, allowing real-time inspection and manipulation of the forms. The separation of curves and lofted surfaces as outputs enabled detailed analysis and further modifications, such as scaling, rotation, or additional parametric adjustments. Graphical representations confirmed that the method can produce complex 3D geometries suitable for architectural modeling, digital sculpting, or experimental design studies. Overall, the experimental results indicate that combining parametric control, local learning,

and procedural noise produces unique, self-organizing 3D forms. The visualizations validate the method's effectiveness in generating smooth, natural-looking surfaces and highlight its potential applications in architecture, digital art, computational design, and geometry-based research. Figure 1 shows the algorithm and key code in the Grasshopper environment, while Figures 2 and 3 display results obtained from different parameter values.

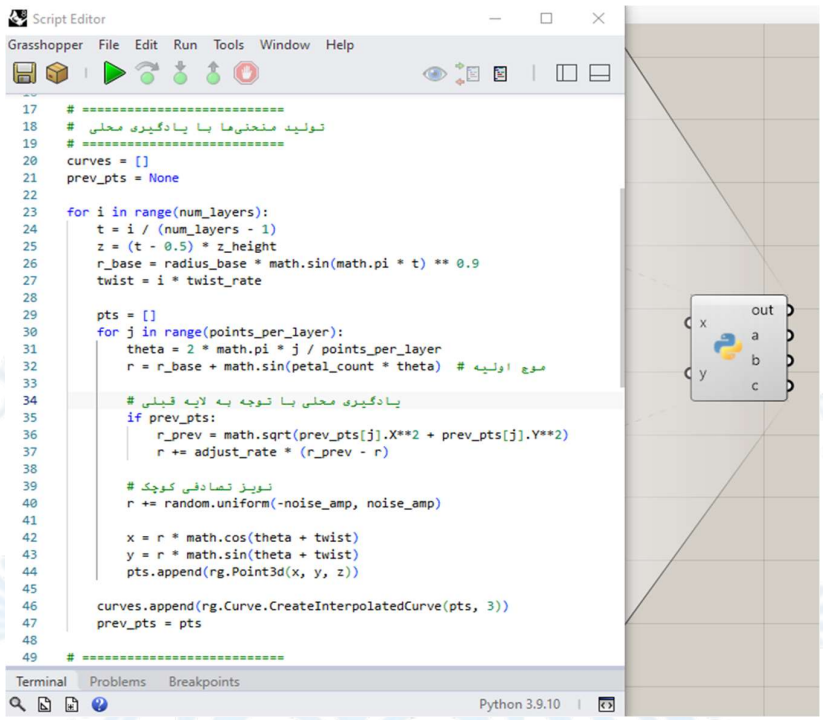

Figure 1. Parametric algorithm in Grasshopper, Source: Author.

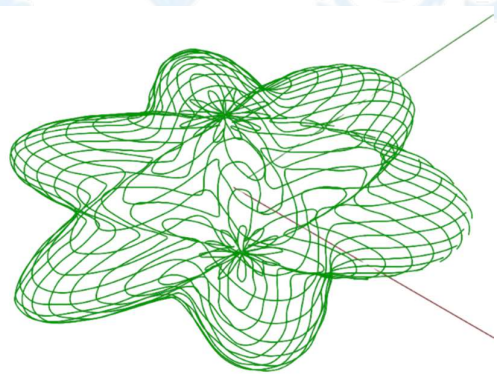

Figure 2. Generated form with parameters: num_layers = 10, points_per_layer = 100, radius_base = 4.5, z_height= 5.0, petal_count = 5, twist_rate = 4°, adjust_rate = 0.2, noise_amp = 0.05

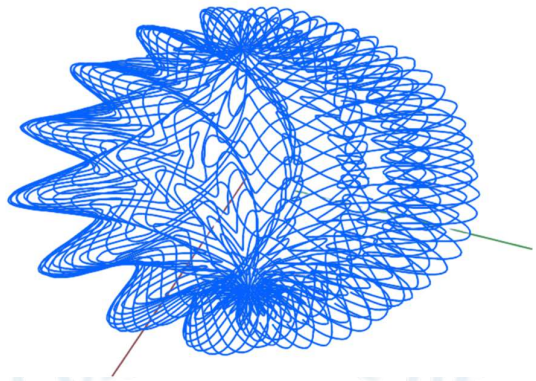

Figure 3. Generated form with parameters:num_layers=20, points_per_layer=120, radius_base=3.8,z_height=6.0, petal_count=5, twist_rate=3°, adjust_rate=0.4, noise_amp=0.02.

4. Results

This research introduced an efficient approach that combines the concepts of local learning and parametric design to generate organic 3D forms. The proposed algorithm gradually adjusts geometric parameters such as radius and angle in each layer, producing continuous, natural, and unique surfaces. The integration of procedural noise and local learning enables the structures to self-organize and resemble natural patterns. Experimental results demonstrated that the proposed method, while maintaining computational simplicity and requiring no training data, can generate a wide variety of forms with different levels of complexity and aesthetic quality. This framework is applicable not only in architectural and artistic design but also in mathematical modeling, computational geometry, and education. It bridges computational design with adaptive artificial intelligence, paving the way for future intelligent design systems.

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On 1-maximal hypersurfaces in Lorentzian hyperbolic 5-space

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Abstract. In this paper, we talk about the 1-maximal hypersurfaces in the standard Lorentzian hyperbolic space $\mathbb{H}^5_1$ with sectional curvature -1. Let M^4 be a Riemannian hypersurface of $\mathbb{H}^5_1$ isometrically immersed by the position map $\mathbf{x} : M^4 \to \mathbb{H}^5_1$. The hypersurface M^4 is called *1-maximal* if its second mean curvature is identically zero. We assume that the position map $\mathbf{x}$ satisfies the C-biharmonicity condition $\mathrm{C}^2\mathbf{x} = 0$, where C is the Cheng-Yau operator. We show that if a C-biharmonic hypersurface M^4 has at most two distinct principal and constant mean curvature, then it has to be 1-maximal.

Keywords: C-biharmonic, spacelike, mean curvature, hyperbolic space.

AMS Mathematics Subject Classification [2020]: 53A30, 53B30, 53C43.

1. Introduction

The theory of harmonic maps is a central topic in geometric analysis with numerous applications in differential geometry. The study of higher order energy functionals defined on the space of all maps between two Riemannian manifolds is a new and interesting research direction. Despite the complexity of the associated Euler-Lagrange equations, which are high order elliptic PDEs, there have been found many interesting examples of maps which are critical points for such energies. Especially in the case of the bienergy functional, in the last two decades, there have been obtained many relevant examples and classification results for biharmonic submanifolds. In a different setting, Bang-Yen Chen defined the notion of biharmonic submanfolds in a Euclidean space with the equation $\Delta^2 x = 0$, where x is the position vector field. In particular, the famous conjecture due to Chen says that any biharmomonic submanifolds in a Euclidean space is minimal. These are the important motivations for studying biharmonic submanifolds in the literature. Chen himself has affirmed the minimality of biharmonic surfaces in Euclidean 3-space. In fact, he has taken the first step in verifying his famous conjecture ([2]). However, some

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mapping spaces do not contain harmonic maps. There are many cases that the conjecture is confirmed under some additional conditions. In the last two decades, these subjects have been studied in the context of semi-Riemannian manifolds (see [1, 5]). The Cheng-Yau map C is the linearized version of the 1-variation of the ordinary mean curvature function. One of interesting subjects in this context is the biharmonic spacelike hypersurface of Lorentz 5-space forms. We study on weakly C-biharmonic spacelike hypersurfaces in the 5-dimensional Lorentz pseudo-sphere and pseudo-hyperbolic space.

2. Preliminaries

First, we recall some prerequisites from [3–5]. We deal with the n-dimensional (pseudo) Euclidean space $\mathbb{E}^m_\nu$ of index $\nu = 0, 1, 2$, equipped with the product defined by $\langle \mathbf{v}, \mathbf{w}\rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{i=\nu+1}^{m} v_i w_i$, for every vectors $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ in $\mathbb{E}^m$. In fact, we use the 5-dimensional Lorentzian space forms with the following general notation

$$\overline{M}^5_1(c) = \begin{cases} \mathbb{S}^5_1(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^5 = \mathbb{E}^5_1 & (\text{if } c = 0) \\ \mathbb{H}^5_1(-r) & (\text{if } c = -1/r^2), \end{cases}$$

where, for $r > 0$, $\mathbb{S}^5_1(r) = \{\mathbf{v} \in \mathbb{E}^6_1 | \langle \mathbf{v}, \mathbf{v}\rangle = r^2\}$ denotes the 5-pseudosphere of radius r and curvature $1/r^2$, and $\mathbb{H}^5_1(-r) = \{\mathbf{v} \in \mathbb{E}^6_2 | \langle \mathbf{v}, \mathbf{v}\rangle = -r^2, v_1 > 0\}$ denotes the pseudo-hyperbolic 5-space of radius $-r$ and curvature $-1/r^2$. In the canonical cases $c = \pm 1$, we get the de Sitter 5-space $\mathbb{S}^5_1 := \mathbb{S}^5_1(1)$ and Lorentzian hyperbolic 5-space $\mathbb{H}^5_1 = \mathbb{H}^5_1(-1)$.

We study the spacelike hypersurfaces M^4 of the canonical Lorentzian hyperbolic 5-space $\mathbb{H}^5_1$ isometrically immersed by $\mathbf{x} : M^4 \to \mathbb{H}^5_1$. As usual, $\chi(M^4)$ stands for the set of all smooth tangent vector fields on M^4. We use ∇, ∇^0 and $\bar{\nabla}$ for the Levi-Civita connections on M^4, $\mathbb{E}^6_2$ and $\mathbb{H}^5_1$, respectively. The Weingarten formula on M^4 is $\bar{\nabla}_V W = \nabla_V W + \langle \mathbb{S}V, W\rangle \mathbf{n}$, for each $V, W \in \chi(M^4)$, where $\mathbf{n}$ is a unit normal timelike vector field on M^4 and $\mathbb{S}$ is its related shape operator. Furthermore, the Gauss formula is $\nabla^0_V W = \bar{\nabla}_V W + \langle V, W\rangle \mathbf{x}$.

The Cheng-Yau map C is the linearized operator arisen from normal variation of the second mean curvature vector field. One may find exact statements in [4]. The C-biharmonic hypersurfaces play an important role in the theory of 1-maximal hypersurfaces. We study the weakly C-biharmonic spacelike hypersurfaces in the 5-dimensional Lorentzian hyperbolic 5-space.

Let $\mathbf{x} : M^4 \to \mathbb{H}^5_1$ be a spacelike hypersurface in the 5-dimensional Lorentzian hyperbolic 5-space. With respect to a unit normal vector field $\mathbf{n}$ and a suitable tangent frame of unit principal directions of M^4 as $\mathcal{E} = \{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$, it has a diagonal shape operator S. Denoting the eigenfunctions of $\mathbb{S}$ by $\kappa_1, \ldots, \kappa_4$, we define the ℓth elementary function $s_\ell := \sum_{1 \le j_1 < \ldots < j_\ell \le 4} \kappa_{j_1} \ldots \kappa_{j_\ell}$ (for $\ell = 0, 1, \cdots, 4$). The *ℓth mean curvature* of M^4 is defined by $H_\ell := \frac{(-1)^\ell}{\binom{5}{\ell}} s_\ell$. A Riemannian hypersurface $\mathbf{x} : M^4 \to \mathbb{H}^5_1$ is said to be *ℓ-maximal* if $H_{\ell+1} \equiv 0$ on M^4. We will denote the ordinary mean curvature of M^5 by $H := H_1 = \frac{1}{4} tr(\mathrm{S})$ and its ordinary mean curvature vector filed by $\mathbf{H} := H\mathbf{n}$. The second mean mean curvature vector field of M^5 is defined by $\mathbf{H}_2 := H_2 \mathbf{n}$. The normalized scalar curvature (i.e. R) of M^4 satisfy the equality $R = 1 - \frac{1}{20} H_2$.

The hypersurface $\mathbf{x} : M^4 \to \mathbb{H}^5_1$ is said to be *isoparametric* if all of its principal curvatures are constant.

The well-known Newton transformations on the hypersurface, $\mathrm{N}_r : \chi(M) \to \chi(M)$, is defined by

$$\mathrm{N}_0 = I,\ \mathrm{N}_r = s_r I - \mathrm{S} \circ \mathrm{N}_{r-1},\ (r = 1, \ldots, 4),$$

where, I is the identity map.

The *Cheng-Yau operator* arisen from the first variation of the second mean curvature of M^4 denoted by $\mathrm{C} : \mathcal{C}^\infty(M^4) \to \mathcal{C}^\infty(M^4)$ is defined by the formula $\mathrm{C}(f) := tr(\mathrm{N}_1 \circ \nabla^2 f)$, where, ∇^2 is metrically equivalent to the Hessian tensor defined by $< \nabla^2 f(X), Y >=< \nabla_X \nabla f, Y >$ for every $X, Y \in \chi(M)$. For a Riemannian hypersurface $\mathbf{x} : M^4 \to \mathbb{H}^5_1$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}^6_t$ we use the decomposition $\mathbf{a} = \mathbf{a}^\top + \mathbf{a}^\perp$ where $\mathbf{a}^\top \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^\perp$ is its normal component, and we have the following formula from [4]:

$$\mathrm{C}\mathbf{x} = 12H_2\mathbf{n} + 12H_1\mathbf{x}. \tag{1}$$

$$\begin{aligned}\mathrm{C}^2\mathbf{x} &= 24\,(\mathrm{N}_2\nabla H_2 + \mathrm{N}_1\nabla H_1 - 9H_2\nabla H_2)\\ &+ 12[\mathrm{C}H_2 - 12H_2(2H_1H_2 - H_3) + 12H_1H_2]\mathbf{n}\\ &+ 12[\mathrm{C}H_1 - 12(H_2^2 - H_1^2)]\mathbf{x}.\end{aligned}$$

Definition 2.1. (a) The equation $\mathrm{C}^2\mathbf{x} = 0$ is called the C-*biharmonicity condition* for hypersurface $\mathbf{x} : M^4 \to \mathbb{H}^5_1$.
(b) M^4_1 is called *weakly* C-*biharmonic* if it satisfies the following conditions

$$\begin{aligned}&\text{(i) } \mathrm{N}_2\nabla H_2 + \mathrm{N}_1\nabla H_1 = 9H_2\nabla H_2\\ &\text{(ii) } \mathrm{C}H_2 = 12H_2(2H_1H_2 - H_3) - 12H_1H_2.\end{aligned} \tag{2}$$

The equations $d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}$ are known as the structure equations of space form $\mathbb{H}^5_1$. So, restricted to M^4, we have $\omega_5 = 0$ and then, $0 = d\omega_5 = \sum_{i=1}^{4} \omega_{5,i} \wedge \omega_i$.

A lemma due to Cartan gives the decomposition $\omega_{5,i} = \sum_{j=1}^{4} h_{ij}\omega_j$ for smooth functions h_{ij} satisfying the equality $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_5$ where B is the second fundamental form of M^4. The mean curvature H is given by $H = \frac{1}{4}\sum_{i=1}^{4} h_{ii}$. So, the structure equations of M are

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j,\ \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{4} R_{ijkl}\omega_k \wedge \omega_l,$$

for $i, j = 1, 2, 3, 4$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$ where R_{ijkl} denotes the components of the Riemannian curvature tensor of M^4. Now, let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{4} h_{ik}\omega_{jk} + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ijk}\omega_k,$$

and by the Codazzi equation we get

$$h_{ijk} = h_{ikj}. \tag{3}$$

One can choose $e_1, \ldots, e_4$ such that $h_{ij} = \kappa_i \delta_{ij}$. On the other hand, the Levi-Civita connection of M^4 satisfies $\nabla_{e_i} e_j = \sum_k \omega_{jk}(e_i) e_k$, and we have $e_i(k_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j)$ and

$$\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l)$$

whenever i, j, l are distinct.

3. Main results

In this section, we present some theorems on the weakly C-biharmonic spacelike hypersurfaces in the Lorentz hyperbolic 5-space $\mathbb{H}^5_1$. Assuming some additional conditions we show that such a hypersurface has to be 1-maximal.

Theorem 3.1. Let $\mathbf{x} : M^5 \to \mathbb{H}^5_1$ be a weakly C-biharmonic spacelike hypersurface such that whose mean curvature is constant and second mean curvature is non-constant. Also, assume that it has exactly three distinct principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$ according to an orthonormal basis $\{w_1, \ldots, w_4\}$ of principal directions satisfying $w_4(\lambda_2) = 0$. Then, the following equality occurs:

$$w_1(\lambda_2) w_1(\lambda_1 + 2\lambda_2) = \frac{1}{2} \lambda_2 (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{4}$$

Theorem 3.2. *Every weakly C-biharmonic spacelike hypersurface of $\mathbb{H}^5_1$ with a principal curvature of multiplicity 4 has to be 1-maximal.*

Theorem 3.3. *Assume that M^4 is a weakly C-biharmonic spacelike hypersurface of $\mathbb{H}^5_1$ with exactly two distinct principal curvatures, respectively of multiplicities 1 and 3. Then it has to be 1-maximal.*

Theorem 3.4. *Assume that M^4 is a weakly C-biharmonic spacelike hypersurface of $\mathbb{H}^5_1$ with exactly two distinct principal curvatures of multiplicities 2. The it is 1-maximal..*

Theorem 3.5. *Let M^4 be a weakly C-biharmonic spacelike hypersurface of $\mathbb{H}^5_1$ with exactly two distinct principal curvatures and constant ordinary mean curvature. Then, it has to be 1-maximal.*

Theorem 3.6. Let $\mathbf{x} : M^4 \to \mathbb{H}^5_1$ be a weakly C-biharmonic spacelike hypersurface with exactly three distinct principal curvatures constant ordinary mean curvature. Then it has to be 1-maximal.

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A Reinformance Learning Approach for Generating Fractal Curves and Surfaces

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Abstract: This research presents a reinforcement learning-based framework for generating three-dimensional fractal curves and surfaces using adaptive growth algorithms in the Grasshopper environment. The model simulates the natural evolution of branching structures, where each stage adjusts growth parameters through a simple reward mechanism that balances geometric uniformity and spatial expansion. The system is controlled by eight parameters: number of primary branches (a), fractal depth (b), base branch length(c), horizontal and vertical angle variation (d, e), scale reduction factor (f), vertical growth shift per iteration (g), and number of sub-branches per node (h). Through iterative learning, the algorithm produces diverse self-similar geometries and connected fractal surfaces that evolve dynamically in three-dimensional space. The framework demonstrates strong potential across multiple disciplines: in computational design and bio-inspired architecture for generating organic forms; in digital art for procedural aesthetics; in computational biology for modeling vascular and plant growth patterns; in physics and earth sciences for simulating natural morphogenesis; and in data science for analyzing multi-scale geometric structures. This adaptive model establishes a foundation for intelligent fractal generation and future integration with advanced machine learning systems.

Keywords: Fractal geometry, Reinforcement learning, Adaptive growth, Computational design, Grasshopper.

1. Introduction

Nature has always been a profound source of inspiration for understanding hidden order within complexity. From the branching of trees to the crystallization of minerals, apparent randomness often conceals an underlying mathematical harmony. Fractal geometry provides a framework for representing these self-similar patterns, bridging mathematics, art, and science. In recent years, the integration of fractal geometry with artificial intelligence algorithms has enabled the exploration of generative systems that reproduce adaptive and nature-inspired behaviors. Among these methods, reinforcement learning offers a promising approach for modeling self-organizing growth, as each iteration can dynamically adapt based on environmental feedback and geometric

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balance [1, 8]. Moreover, deep neural models have been applied to synthesize self-similar patterns with natural appearances [2], while parametric platforms such as Grasshopper allow designers to merge computational precision with organic expression [3, 5]. In addition, genetic and evolutionary algorithms have demonstrated how algorithmic creativity can emerge from a set of simple rules [4, 7]. Despite these advancements, most existing studies have primarily focused on aesthetic or optimization aspects, paying less attention to adaptive and learning-based frameworks that treat fractals as dynamic and evolving structures. The present study aims to fill this gap by proposing a reinforcement learning–based parametric framework for generating three-dimensional fractal curves and surfaces. The model, implemented in the Grasshopper environment, simulates adaptive growth in which angles, scaling ratios, and branch heights evolve through iterative, reward-driven updates. In this framework, the input parameters include the number of main branches (a) growth depth (b), base length of each branch (c), horizontal branching angle (d), vertical tilt angle (e), scale reduction ratio (f), vertical growth shift per iteration (g) and number of sub-branches (h). Dynamic adjustment of these parameters enables the algorithm to exhibit learning behavior and generate a variety of organic growth structures, dynamic surfaces, and self-similar curves. The primary goal of this approach is to establish a dynamic balance between order and randomness, offering potential applications in computational design, bio-inspired modeling, materials science, and data visualization.

2. Methodology

In this study, a parametric framework is developed to simulate natural fractal growth within the Grasshopper–Python environment. The algorithm is based on recursive branching, inspired by self-similar patterns observed in natural structures. The process begins from a central point, and in each iteration, new branches are generated with varying angles, lengths, and heights relative to their parent branch. This recursive process continues until the defined depth is reached, resulting in a set of smooth, continuous three-dimensional curves. Finally, the curves are lofted into an organic surface using the *Loft* function. The input parameters, labeled from a to h, include the number of main petals, growth depth, branching angle, scale reduction ratio, height along the z-axis, randomness factor for angular variation, learning rate, and reward coefficient. These parameters control the overall behavior, structural complexity, and final morphology of the fractal surface. Adjusting each parameter produces a distinct geometric response during the growth process, generating a wide spectrum of symmetrical, organic, and stochastic forms. As Hensel et al. [9] demonstrated, parametric design enables the simulation of natural behaviors through simple local rules and feedback relationships, allowing geometry to emerge from iterative and gradual growth. Following the insights of Stanley et al. [10], the proposed framework can be further extended by integrating reinforcement learning and evolutionary optimization methods to autonomously adapt growth parameters and evolve self-organizing geometries toward dynamic geometric equilibrium.

3. Experimental Results and Visualization

This experiment demonstrates the process of generating three-dimensional fractal structures based on recursive branching and an adaptive reinforcement learning mechanism within the Grasshopper–Python environment. The algorithm starts from a central point and, at each iteration, produces branches with variable angles, deviations, and lengths that evolve adaptively through an updating mechanism. Each branch, depending on the user-defined input parameters including the number of petals (a), growth depth (b), base length (c), horizontal and vertical angles (d, e), scale decay ratio (f), vertical growth factor (g), and number of sub-branches (h) —

divides into smaller sub-branches. Figures 1 and 2 illustrate the implementation of the algorithm in Grasshopper, showing the parametric settings and a key section of the Python script. Figures 3 and 4 present the results obtained from different input parameter values, demonstrating the influence of branching angles and scale ratios on the geometry and surface continuity. The implementation of the algorithm in the Grasshopper environment, including the parametric settings and a key portion of the Python script, is presented in Figure 1.

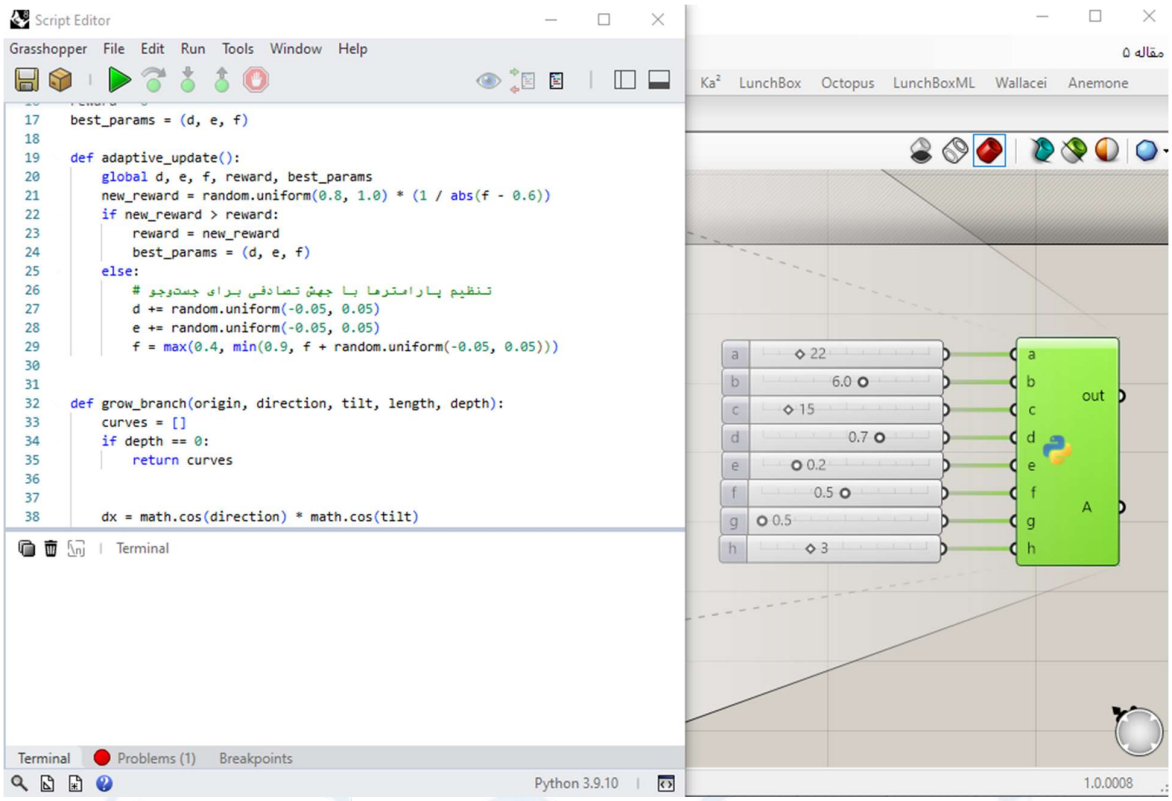

Figure 1. Implementation of the algorithm in Grasshopper with key parametric settings and Python cod. Source: Authors.

Figures 2 and 3 show the results of varying input parameters, illustrating the effects of branching angles and scale ratio on geometry and surface continuity.

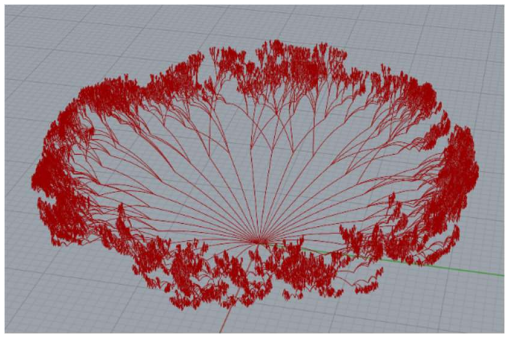

Figure 2. The output obtained from a = 29, b = 7.2, c = 12, d = 0.7, e = 0.2, f = 0.5, g=0.5 and h = 0.8.

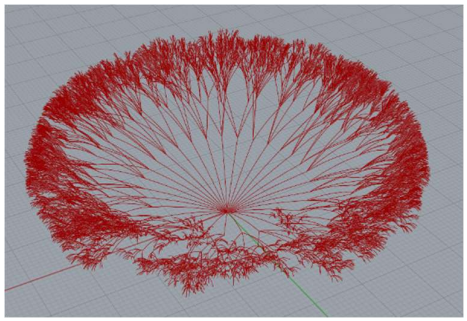

Figure 3. The result obtained from a = 31, b = 5.7, c = 17, d = 0.2, e = 0.7, f = 0.6, and h = 0.8.

This code consists of 79 lines and Approximately, its time and space complexity is $O(ah^b)$.

4. Results

The proposed reinforcement learning algorithm successfully generated a variety of three-dimensional fractal curves and surfaces. By dynamically adjusting geometric parameters such as angles, scales, and growth depth, the system produced self-similar and continuous structures with both symmetry and natural variation. The results confirmed the model's ability to simulate adative and organic growth behaviors within the Grasshopper–Python environment.

5. Conclusion

This study presented an intelligent, adaptive model for fractal generation based on reinforcement learning. The framework effectively balances order and randomness, enabling the creation of living and evolving fractal forms. The approach offers broad potential applications in computational design, bio-inspired architecture, digital art, and scientific modeling, providing a foundation for future integration with advanced AI systems.

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A Parabolic Analysis of Arches under Uniform Vertical load

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Abstract: In this paper, an analytical practical–mathematical approach is presented to determine the equation of parabolic arches that exhibit funicular behavior (i.e., with negligible internal bending moments) under uniform vertical loading. The main objective of the study is to develop a systematic and applicable method for constructing and analyzing parabolas passing through three defined points: the origin (0,0), the crown point (2.5, f), and the return to the horizontal axis at (5,0). Accordingly, three different values of the crown height (f) are considered, and the associated parameters including the parabolic coefficient, horizontal thrust due to uniform load, and the approximate required cross-sectional area for common construction materials are calculated and analyzed. This study aims to elucidate the geometric effects on internal force equilibrium and the structural efficiency of parabolic arches. The numerical and analytical results are presented step-by-step to provide a clear and comprehensive understanding of the structural behavior of arches subjected to uniform vertical loading.

Keywords: Parabolic Arch, Uniform Vertical Load, Analytical Practical–Mathematical Approach, Horizontal Thrust, Structural Efficiency.

1.Introduction

Arches have long been recognized in structural engineering and architecture as one of the most efficient and effective load-bearing systems. These structures can carry vertical loads primarily through axial compression, especially in parabolic arches, without generating significant internal bending moments. Among various forms, parabolic arches are considered ideal funicular shapes under uniform load due to their geometric compatibility with the load distribution. Recent studies have shown that parabolic arches can closely approximate funicular behavior under uniform loading [1]. For example, the optimization of funicular arches under equally spaced point loads demonstrated that increasing the number of point loads makes the arch shape converge toward

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the ideal parabolic form [2]. Moreover, recent research indicates that geometric parameters, such as the crown height and span of the arch, significantly affect internal forces, horizontal thrust at the supports, and the required cross-sectional dimensions [3]. The primary objective of this study is to present a practical analytical method for determining the equation of parabolic arches that exhibit funicular behavior under uniform vertical load. In this regard, three different crown heights are considered, and related parameters including the parabolic coefficient, horizontal thrust, and approximate cross-sectional area required for common construction materials are calculated and analyzed. This study aims to clarify the geometric effects on internal force equilibrium and structural efficiency of parabolic arches.

2. Mathematical Model and Formulation

Assume the span of the arch is L = 5 m, and it is modeled as a parabolic curve with roots at x = 0 and x = L. The suitable form that satisfies these boundary conditions is expressed as:

$$y(x) = ax(L - x) \qquad (1)$$

If the vertex of the parabola lies at the midpoint x = L/2 with a height f, the coefficient a can be determined by substituting $x = \frac{L}{2}$ and $y = f$, resulting in:

$$a = -\frac{4f}{L^2} \qquad (2)$$

Therefore, the equation of the parabolic arch becomes:

$$y(x)=(-\frac{4f}{L^2})\ x\ (L-x) \qquad (3)$$

For a uniformly distributed vertical load with intensity q per unit horizontal length, the horizontal thrust H transmitted to the supports is given by the classical expression [4]:

$$H = \frac{qL^2}{8f} \qquad (4)$$

This relationship indicates that increasing the rise f leads to a decrease in H, implying that taller arches require less horizontal thrust under a constant load intensity [5].

The above equation is derived from the theory of funicular shapes, which are used to describe the equilibrium geometry of structures under pure axial compression [6].
Several modern studies have re-examined these classical results to improve their applicability to complex load conditions and numerical analysis techniques [2].

3. Numerical Evaluation

To illustrate the analytical formulation presented in the previous section, a numerical study is conducted for a parabolic arch with a clear span of L = 5 m. Three different crown heights are considered to examine the geometric influence on the structural response:

$$f_1 = 2.4455, f_2 = 2.9339, f_3 = 2.1805$$

For a parabolic shape defined by $y(x) = ax(L - x)$, the coefficient a is obtained using the geometric relation:

$$a = -\frac{4f}{L^2} \qquad (5)$$

Hence, the equations of the three arches, in factored form, become:

$$y_1(x) = -0/39128x(5 - x)$$
$$y_2(x) = -0/469424x(5 - x)$$

$$y_3(x) = -0/34884x(5-x)$$

3.1 Horizontal Thrust under Uniform Load

For comparison, a uniform vertical load of intensity q = 10 $\frac{kN}{m}$ is applied.

The horizontal thrust H at the supports for a parabolic arch under uniform loading is expressed as:

$$H = \frac{qL^2}{8f} \qquad (6)$$

Substituting the given values yields :

$$f_1 = 2.4455 \rightarrow H_1 = 12.7786\ kN$$
$$f_2 = 2.9339 \rightarrow H_2 = 10.6514\ kN$$
$$f_3 = 2.9339 \rightarrow H_3 = 10.6514\ kN$$

As expected, increasing the rise f reduces the horizontal thrust for a constant loading intensity, demonstrating the geometric efficiency of higher arches.

3.2 . Estimation of Required Cross-Sectional Area

An initial estimation for the minimum required cross-sectional area to resist the horizontal axial thrust H, under the allowable compressive stress of the material (σ allow), is given by:

$$A = \frac{H \times 1000}{\sigma allow}$$

Representative allowable stresses for typical construction materials are assumed as:

Material	Σ allow (MPa)
Masonry	5
Concrete	20
Structural steel	160

Accordingly, for q = 10 kN/m and the calculated thrust values, the approximate cross-sectional areas are:

Arch (f)	H (kN)	A_steel (m²)	A_concrete (m²)	A_masonry (m²)
2.4455	12.7786	7.9866×10^{-5}	6.3893×10^{-4}	2.5557×10^{-3}
2.9339	10.6514	6.6571×10^{-5}	5.3257×10^{-4}	2.1303×10^{-3}
2.1805	14.3316	8.9572×10^{-5}	7.1658×10^{-4}	2.8663×10^{-3}

4. Discussion

Under uniform vertical loading, the parabolic arch represents the ideal or *funicular* shape of equilibrium. Three arches with different rise heights can exhibit this moment-free state, yet increasing the rise (*f*) reduces the horizontal thrust at the supports ($H \propto 1/f$) a key factor in optimizing foundation dimensions and construction costs. For materials with lower strength, such as masonry, larger cross-sections are required to resist axial forces. Nevertheless, this analysis is based on simplified assumptions, and more advanced studies should account for local bending effects, shear forces, and lateral loads such as wind or seismic actions. Figure 1 shows the three studied arches on the coordinate axes and illustrates the best arch from an engineering perspective.

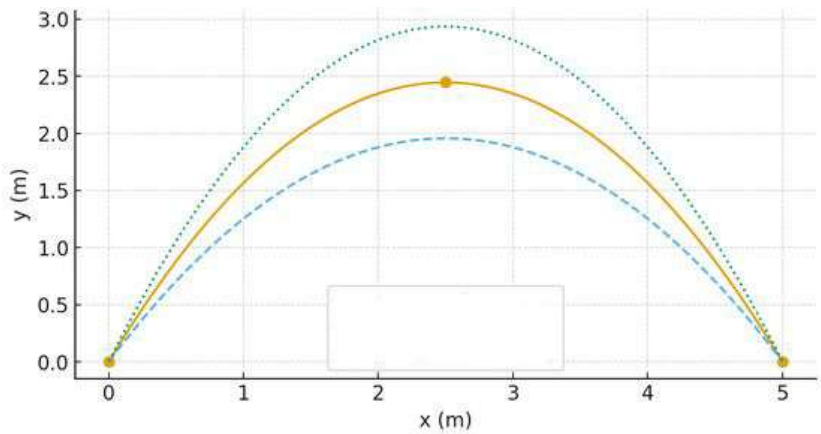

Figure1. Three studied arches on the axes, showing the best arch from an engineering perspective

5. Result

This research illustrated an example where three parabolic arches with the same span (5 m) but different rises can all exhibit funicular behavior under a uniform vertical load. From an engineering perspective, however, the arch with the larger rise (f_2 =2.9339) is more favorable, as it generates lower horizontal thrust for the same load intensity q, reducing the requirements for cross-section and foundation. The provided numerical values (horizontal force and estimated section areas for sample materials) offer a simple, practical tool for quick comparison, but the final design should be based on more detailed analyses and standard criteria.

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On the Expected Dimension of Brill–Noether Loci of Stable Rank-2 Vector Bundles with Fixed Determinant

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Abstract. This paper investigates the Brill–Noether loci $B^k_{2,L}$ of stable rank–2 vector bundles with fixed determinant over a smooth projective curve C. Focusing on the case $k = 1$, explicit families of stable bundles are constructed using extension theory, and it is shown that these families have the expected Brill–Noether dimension. This provides a concrete and constructive realization of the expected components of $B^1_{2,L}$, complementing classical Petri map results for line bundles and recent developments in non-abelian Brill–Noether theory.

Keywords: Brill–Noether loci, Stable vector bundles, Extensions, Moduli spaces

AMS Mathematics Subject Classification [2020]: 14H60, 14D20, 14H40

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$, and let L be a line bundle of degree $d > 0$. The moduli space $U_C(2, L)$ of stable rank–2 vector bundles with determinant L has dimension $3g - 3$. For each integer $k \geq 1$, the Brill–Noether locus is

$$B^k_{2,L} := \{E \in U_C(2, L) \mid h^0(C, E) \geq k\}.$$

The expected dimension of $B^k_{2,L}$ is given by

$$\rho(g, d, k) = 3g - 3 - k\big(k - \chi(E)\big),$$

where

$$\chi(E) = h^0(C, E) - h^1(C, E),$$

and for rank 2 bundles $\chi(E) = d + 2(1 - g)$ by Riemann–Roch.

For line bundles ($r = 1$), Petri's theorem ensures that for a general curve the Brill–Noether loci have the expected dimension and are smooth at general points. For rank 2 bundles with canonical determinant, Teixidor proved the injectivity of the Petri map for a general curve, implying analogous statements for B^k_{2,K_C} [5]. Mukai studied non-abelian Brill–Noether loci in connection with Fano threefolds, providing striking geometric realizations in special cases [3]. Earlier foundational work of Newstead [4] and Le Potier [2]

developed the moduli-theoretic and cohomological framework underlying these constructions. Hirschowitz [1] applied Horace-type methods to study the geometry of these loci further.

In this note we give a constructive approach for general determinant and $k = 1$, producing explicit irreducible families of stable bundles of the expected dimension inside $B^1_{2,L}$ by using nontrivial extensions of line bundles.

2. Main results

Fix a line bundle $L \in \mathrm{Pic}^d(C)$ with $d > 0$. Consider nontrivial extensions

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Extension classes are parametrized by

$$Ext^1(L, \mathcal{O}_C) \cong H^1(C, L^{-1}).$$

By Serre duality and Riemann–Roch

$$h^1(C, L^{-1}) = h^0(C, K_C \otimes L) = d + g - 1,$$

so the projective extension space $\mathbb{P}H^1(C, L^{-1})$ has dimension $d + g - 2$. Each nonzero extension class yields a bundle E with at least one global section (coming from $\mathcal{O}_C$), hence a point of $B^1_{2,L}$. The zero class corresponds to the split bundle $\mathcal{O}_C \oplus L$, which is excluded by projectivization.

Theorem. *Let C be a smooth curve of genus $g \geq 2$ and $L \in \mathrm{Pic}^d(C)$ with $d > 0$ and $h^0(C, L^{-1}) = 0$. Then the projective extension space $\mathbb{P}H^1(C, L^{-1})$ maps generically injectively into $B^1_{2,L}$. For a general extension, the resulting bundle E is stable. Therefore $B^1_{2,L}$ contains an irreducible component of dimension $d + g - 2 = \rho(g, d, 1)$.*

Sketch of proof. Nontrivial extensions correspond to nonzero elements of $H^1(C, L^{-1})$, and the associated bundles always have a nonzero section from $\mathcal{O}_C$. The dimension computation uses Riemann–Roch as above. For stability, consider a line subbundle $M \subset E$. If $M \subset \mathcal{O}_C$ then $\deg M = 0 < d/2$. If M maps nontrivially to L, then the extension class must satisfy certain linear equations in $H^1(C, L^{-1})$. For each fixed M, the locus of extensions admitting M is a proper linear subspace. Since there are finitely many possibilities for $\deg M$ that can destabilize, the complement of these subspaces is nonempty and open, so a general extension yields a stable bundle. This argument is standard in the study of extension constructions [2, 4].

3. Example

For $g = 3$, $d = 4$ we have $h^1(C, L^{-1}) = 6$, so $\mathbb{P}^5$ parametrizes nontrivial extensions. These map into $B^1_{2,L}$ to form a 5-dimensional irreducible family of stable bundles. This coincides with the expected number

$$\rho(3, 4, 1) = d + g - 2 = 4 + 3 - 2 = 5.$$

4. Conclusion

Explicit families of stable rank–2 vector bundles with fixed determinant and at least one section were constructed using extension theory. These families have the expected Brill–Noether dimension and give a concrete realization of components of $B^1_{2,L}$, complementing the Petri map results of Teixidor [5] and Mukai's geometric constructions [3].

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Non-Abelian Brill–Noether Loci of Stable Rank-2 Bundles with Fixed Determinant: The Case $k = 2$

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Abstract. This paper studies the Brill–Noether loci $B^2_{2,L}$ of stable rank–2 vector bundles with fixed determinant L on a smooth projective curve C. Building on the classical Petri map results for line bundles and Teixidor's injectivity theorem for rank–2 bundles with canonical determinant, explicit families of stable bundles with two independent global sections are constructed using extension theory. It is shown that these families have the expected Brill–Noether dimension and fit naturally into the general framework of non-abelian Brill–Noether theory developed by Mukai and others.

Keywords: Brill–Noether loci, Stable vector bundles, Extensions, Moduli spaces

AMS Mathematics Subject Classification [2020]: 14H60, 14D20, 14H40

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$ and let $L \in \mathrm{Pic}^d(C)$ be a line bundle of degree $d > 0$. The moduli space $U_C(2, L)$ of stable rank–2 vector bundles with determinant L has dimension $3g - 3$. For each integer $k \geq 1$, the Brill–Noether locus is defined by

$$B^k_{2,L} := \{E \in U_C(2, L) \mid h^0(C, E) \geq k\}.$$

The expected dimension of $B^k_{2,L}$ is

$$\rho(g, d, k) = 3g - 3 - k\big(k - \chi(E)\big),$$

where

$$\chi(E) = h^0(C, E) - h^1(C, E) = d + 2(1 - g).$$

For line bundles, Petri's theorem guarantees that the Brill–Noether loci on a general curve have the expected dimension and are smooth at general points. For rank–2 bundles with canonical determinant, Teixidor [5] proved injectivity of the Petri map for general curves, giving an analogue of Petri's theorem in higher rank. Mukai [3] studied non-abelian Brill–Noether theory in connection with Fano threefolds, uncovering rich geometry of these loci. Newstead [4] and Le Potier [2] developed foundational tools for extension-theoretic constructions of vector bundles, while Hirschowitz [1] introduced Horace-type methods that remain central to dimensional estimates.

In this paper, we treat the case $k = 2$ for general determinant. Instead of relying only on Petri map injectivity, we construct explicit families of stable bundles with two independent global sections, using suitable extension sequences and choices of 2-dimensional subspaces of sections.

2. Extension Construction for $k = 2$

Fix $L \in \mathrm{Pic}^d(C)$ with $d > 0$ and assume $h^0(C, L^{-1}) = 0$. Consider extensions of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \otimes M^{-1} \longrightarrow 0,$$

where M is a line bundle of degree m. Such extensions are classified by

$$\mathrm{Ext}^1(L \otimes M^{-1}, M) \cong H^1(C, M^2 \otimes L^{-1}).$$

A general extension class corresponds to a non-split bundle E of determinant L.

To produce two independent global sections of E, we choose a 2-dimensional subspace

$$V \subset H^0(C, E),$$

and consider the evaluation map

$$V \otimes \mathcal{O}_C \longrightarrow E.$$

Its image is a rank–1 subsheaf of E, which corresponds to a line bundle M and gives rise to the extension above. This construction parametrizes families of bundles with $h^0(E) \geq 2$.

By Serre duality and Riemann–Roch, the dimension of the projective extension space is

$$\dim \mathbb{P}H^1(C, M^2 \otimes L^{-1}) = h^1(C, M^2 \otimes L^{-1}) - 1 = h^0(C, K_C \otimes M^{-2} \otimes L) - 1.$$

For general choices of M, this dimension equals the expected Brill–Noether number $\rho(g, d, 2)$, providing a candidate for an irreducible component of $B^2_{2,L}$.

3. Main Result

Theorem. *Let C be a smooth projective curve of genus $g \geq 2$ and $L \in \mathrm{Pic}^d(C)$ with $d > 0$ and $h^0(C, L^{-1}) = 0$. Then for a general line bundle M and a general extension class in*

$$\mathbb{P}H^1(C, M^2 \otimes L^{-1}),$$

the associated bundle E is stable and has at least two independent global sections. The family of such bundles forms an irreducible subvariety of $B^2_{2,L}$ of dimension equal to the expected Brill–Noether number $\rho(g, d, 2)$.

Sketch of proof. The idea of the proof relies on extension spaces of line bundles. Fix a line bundle M of degree m. Non-trivial extensions

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \otimes M^{-1} \longrightarrow 0$$

are parametrized by the projective space

$$\mathbb{P}H^1(C, M^2 \otimes L^{-1}),$$

whose dimension is computed using Serre duality and the Riemann–Roch theorem. Varying M in $\mathrm{Pic}^m(C)$ gives a parameter space of the expected Brill–Noether dimension $\rho(g, d, 2)$.

For each extension, choosing a two-dimensional subspace $V \subset H^0(C, E)$ and applying the evaluation map determines M, ensuring that $h^0(E) \geq 2$. Stability holds for a general

extension class because destabilizing subbundles impose proper linear conditions on the extension space; hence a general extension corresponds to a stable bundle. The resulting family is irreducible and has the expected dimension, giving the desired component of $B^2_{2,L}$.

4. Example

Let $g = 4$, $d = 8$, and choose M of degree $m = 2$. Then by Riemann–Roch,

$$h^1(C, M^2 \otimes L^{-1}) = h^0(C, K_C \otimes M^{-2} \otimes L) = 10,$$

so $\mathbb{P}^9$ parametrizes extensions. These extensions give bundles with two independent sections and fill a 9-dimensional family inside $B^2_{2,L}$, matching $\rho(4, 8, 2)$.

5. Conclusion

Explicit families of stable rank–2 bundles with fixed determinant and two independent global sections were constructed using extension theory. This provides a direct geometric construction of components of $B^2_{2,L}$ of the expected dimension, extending the classical $k = 1$ construction and complementing Petri map results for rank–2 bundles [5]. Mukai's geometric perspective [3] and the foundational work of Newstead [4] and Le Potier [2] provide the broader framework in which these families naturally fit.

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3D Cup Modeling Using Bessel–Gamma Functions and Octopus Machine Learning

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Abstract: In this research, an innovative method is presented for three-dimensional modeling of artistic, cup-like shapes, where Bessel and Gamma functions are used to generate the base form. To determine the optimal function parameters (n, a, b) and to optimize the height range and surface smoothness, the Octopus Machine Learning metaheuristic algorithm is employed. The final output is displayed with appealing coloring, three-dimensional effects, and soft lighting, demonstrating that the combination of analytical mathematics and artificial intelligence enables the creation of complex and visually captivating artistic forms.

Keywords: Bessel functions, Gamma functions, 3D modeling, Octopus Machine Learning, AI art.

1.Introduction

Three-dimensional modeling of artistic and geometric shapes, especially complex and unconventional forms, has long attracted the attention of researchers in computer science, mathematics, and digital art. The use of special functions such as Bessel and Gamma functions enables the creation of base forms with precise geometric features. Studies have shown that Bessel functions can be used to generate diverse three-dimensional geometries and simulate curved, oscillatory surfaces [1]. Additionally, Gamma functions, with their nonlinear behavior, allow designers to adjust the slope and height of surfaces [1-2]. In the field of parameter optimization, metaheuristic algorithms play a critical role. The Octopus algorithm, inspired by the social and foraging behavior of octopuses, has been introduced as a novel metaheuristic algorithm capable of efficiently searching parameter spaces and finding optimal combinations. Recent research indicates that this algorithm performs well in complex optimization problems, including 3D shape generation and surface design [3]. The combination of mathematical modeling with artificial intelligence and metaheuristic algorithms enables the creation of artistic shapes with complex geometry and unique aesthetic characteristics. Previous studies have shown that the simultaneous use of mathematical functions and machine learning or metaheuristic algorithms can facilitate the generation of precise and customizable 3D forms [2-3]. The objective of this study is

to present an innovative method for three-dimensional modeling of artistic shapes using Bessel and Gamma functions, while optimizing the model parameters with the Octopus algorithm. The program input variables include radius (r), angle (θ), and the parameters n, a, and b, which represent the number of oscillations, scale, and surface slope, respectively. By adjusting these parameters, the final outputs can be displayed with appealing coloring, soft lighting, and three-dimensional effects.

2. Methodology

2.1 Overview

This study proposes an innovative approach for three-dimensional modeling of artistic, cup-like shapes by combining mathematical functions and metaheuristic optimization. The base surface is generated using Bessel and Gamma functions, while the Octopus Machine Learning algorithm is employed to optimize the model parameters, ensuring smoothness, appropriate height variation, and aesthetic appeal. Previous studies have shown that 3D analytic frameworks such as SPHARM (spherical harmonics) can be used to model complex structural shapes from continuous surface maps [4], which is highly analogous to using Bessel and Gamma functions for geometric modeling.

2.2. Mathematical Modeling

The artistic shape is defined in polar coordinates (r,θ) as follows:

$$Z(r,\theta) = a.J_n(r).sin(\theta) + b.\Gamma(r+1) \qquad (1)$$

where $J_n(r)$ is the Bessel function of the first kind of order n, and Γ is the Gamma function. Parameters a and b control the amplitude and slope of the surface, while n determines the number of oscillations in the radial direction. The polar grid (r,θ) is discretized at high resolution to ensure a smooth surface and accurate 3D rendering.

2.3 . Parameter Optimization with the Octopus Algorithm

To determine the optimal parameters n, a, b and maximize visual appeal and surface smoothness, a metaheuristic optimization inspired by octopus behavior is used. The Octopus Algorithm proceeds as follows:

1. **Initialization**: A random population of candidate solutions is generated within predefined bounds: n∈[0,6], a∈[0.3,2.5], b∈[0.005,0.115].
2. **Evaluation**: Each candidate is scored based on two criteria: surface amplitude (range of Z) and surface roughness (standard deviation of the gradient magnitude). The fitness function is defined as:

$$\text{Score}=\alpha\cdot\text{Amplitude}-\beta\cdot\text{Roughness} \qquad (2)$$

where α and β are weighting coefficients.

3. **Movement and Update**: Each individual moves in parameter space toward the current best solution with stochastic perturbations to avoid local optima.
4. **Iteration**: The population is updated over multiple generations until convergence criteria are met.

2.4 Implementation in Python

The method is implemented in Python using NumPy, SciPy, and Matplotlib for 3D visualization. A high-resolution polar grid is created with np.meshgrid, and the surface is rendered using a custom colormap and lighting. The optimized shape $(n_{best}, a_{best}, b_{best})$ produces a smooth, well-scaled 3D model for artistic and computational use.

3.Analysis of Findings

3.1. Optimization Outcomes

The Octopus Algorithm was executed for 20 generations with a population size of 25. The fitness evaluation function successfully guided the population toward an optimal solution that balanced surface amplitude and smoothness. The final optimized parameters were obtained as: n_{best}=3, a_{best}=1.45, b_{best}t=0.058. These values produced a 3D shape with smooth, well-defined oscillations and an aesthetically appealing overall form. The convergence of the fitness score over generations is shown in Figure 1, demonstrating rapid improvement within the 10 generations and stabilization toward the final solution.

```
Generation 01 | Score: 401759.9340 | n=4.78, a=2.38, b=0.1110
Generation 02 | Score: 416203.9289 | n=4.87, a=2.50, b=0.1150
Generation 03 | Score: 416203.9289 | n=4.67, a=2.50, b=0.1150
Generation 04 | Score: 416203.9289 | n=4.59, a=2.50, b=0.1150
Generation 05 | Score: 416203.9289 | n=4.54, a=2.50, b=0.1150
Generation 06 | Score: 416203.9289 | n=4.77, a=2.50, b=0.1150
Generation 07 | Score: 416203.9289 | n=4.92, a=2.50, b=0.1150
Generation 08 | Score: 416203.9289 | n=5.03, a=2.50, b=0.1150
Generation 09 | Score: 416203.9289 | n=4.61, a=2.50, b=0.1150
Generation 10 | Score: 416203.9289 | n=4.64, a=2.50, b=0.1150
```

Figure 1. The results 20 generation

3.2. 3D Surface Visualization

The optimized surface was rendered in 3D using a customized turquoise-green colormap and soft lighting. Figures 2 showes different views of the surface, highlighting the smooth radial oscillations and curvature generated by the Bessel and Gamma components. The combination of $J_n(r)$ and $\Gamma(r+1)$ allowed for both radial variations and gradual height transitions, producing a cup-like, visually appealing form.

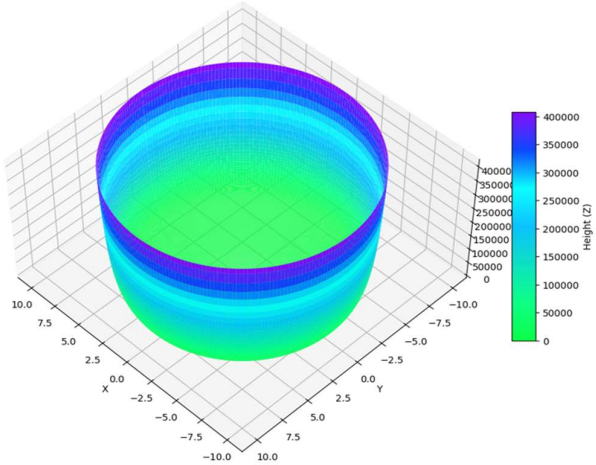

Figure 2. The result of optimized surface.

3.3. Surface Roughness and Amplitude Analysis

Previous studies on 3D shape modeling, such as SPHARM-based frameworks [4], also aim to capture complex surface geometries. Compared with these approaches, the current method provides more intuitive control over surface features through parameters n, a, b and leverages metaheuristic optimization to adaptively balance smoothness and visual amplitude. The results highlight the potential of combining classical mathematical functions with AI-based optimization for artistic and computational modeling applications.

3.4. Potential Applications

The optimized shapes have potential applications in computational art, digital sculpture, virtual reality modeling, and scientific visualization where precise control of complex 3D surfaces is required. The method allows for scalable adjustment of surface features by changing input parameters, offering flexibility for both aesthetic and functional design purposes.

4.Result

The proposed framework generates 3D artistic shapes by combining Bessel and Gamma functions with the Octopus Machine Learning algorithm. It achieves a balance between surface smoothness and amplitude, creating visually appealing and mathematically precise forms. Compared to traditional methods, it provides better control and adaptability for diverse design applications. The results highlight the potential of integrating mathematical modeling with AI optimization for computational art, digital sculpture, and scientific visualization. Future work will explore more complex shapes and improve efficiency through hybrid and parallel optimization.

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Graph and Combinatorics

Edge Coalitions in Graphs

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Abstract. Haynes et al. (2020) introduced and investigated the concept of coalition in graphs [2]. Their study examined this concept from a vertex-based perspective, whereas in this paper, we extend the investigation to an edge-based perspective of graphs.

Keywords: Edge coalition, edge coalition partition, edge dominating sets.

AMS Mathematics Subject Classification [2020]: 05C69

1. Introduction

Below, we introduce the essential definitions and terms. For further explanations and more detailed notation and terminology, please refer to [1].
Let $G = (V, E)$ be a graph of size m, an edge e is referred to as a full edge if its degree is equal to $m-1$, meaning that e is adjacent to all other edges in the graph. For example, in a star graph S_n, every edge is a full edge. A subset E_i of E is referred to as a singleton set if $|E_i| = 1$, indicating that it contains exactly one edge. A unicyclic graph is a connected graph that contains exactly one cycle. The double star $S(n, m)$, where $n \geq m \geq 0$, is defined as the union of two stars, $K_{1,n}$ and $K_{1,m}$ connected by an edge joining their central vertices. A set $S \subseteq V$ is a dominating set of a graph G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. An edge-dominating set in G is a subset D of the edge set $E(G)$ with the property that for each edge $e \in E(G) \setminus D$ there exists at least one $e^{'} \in D$ adjacent to e. The minimum size of any edge dominating set D of G is denoted by $\gamma'(G)$ of G, in other words $\gamma'(G) = |D|$.

In [2], Haynes et al. introduced the concept of coalition for vertices, and subsequently examined various properties of the associated parameter in [3–5]. A coalition in a graph $G = (V, E)$ consists of two disjoint sets of vertices V_1 and V_2, neither of which is a dominating set but whose union $V_1 \cup V_2$ is a dominating set. A coalition partition in a graph G of order $n = |V|$ is a vertex partition $\pi = \{V_1, \cdots, V_k\}$ such that every set V_i

*Speaker.

of π either is a dominating set consisting of a single vertex of degree $(n-1)$, or is not a dominating set but forms a coalition with another set $V_j \in \pi$.

In this paper, we introduce the concept of edge coalitions in graphs. We study $EC(G)$ for specific classes of graphs, derive an optimal upper bound for $EC(G)$, and characterize the family of graphs for which $EC(G)$ equals the size of G.

2. Preliminary results

Definition 2.1. Let $G = (V, E)$ be a graph. An **edge coalition** in G is defined as two disjoint sets of edges E_1 and E_2, where neither E_1 nor E_2 is an edge-dominating set, but their union $E_1 \cup E_2$ forms an edge dominating set. We refer to E_1 and E_2 as forming an edge coalition, and call them edge coalition partners.

Definition 2.2. An **edge coalition partition**, henceforth called an *ec*-partition, in a graph G is an edge partition $\pi = \{E_1, \cdots, E_k\}$ such that every set E_i of π is either a singleton edge-dominating set, or is not an edge-dominating set but forms an edge coalition with another set E_j in π. The edge coalition number $EC(G)$ equals the maximum order k of an *ec*-partition of G, if one such *ec*-partition exists, an *ec*-partition of G having order $EC(G)$ is called an $EC(G)$-partition.

Example 2.3. We consider the path graph $P_6 = (e_1, e_2, e_3, e_4, e_5)$. The partition $\pi = \{\{e_1, e_5\}, \{e_2\}, \{e_3\}, \{e_4\}\}$ is an *ec*-partition of P_6. No set of π is an edge dominating set, but the subset $\{e_1, e_5\}$ and $\{e_2\}$ together form an edge coalition, the subset $\{e_3\}$ and $\{e_1, e_5\}$ togetherform an edge coalition, and the subset $\{e_4\}$ and $\{e_1, e_5\}$ also form an edge coalition. Therefore, each set establishes an edge coalition with at least one other set. To show that this is also an $EC(P_6)$-partition, we note that any larger partition of $E(P_6)$ would necessarily be of order 5 with each edge residing in a singleton set (that is called the singleton edge partition of P_6). But we know that no edge dominating set of P_6 contains the edge e_3 along with one other edge, thus, the singleton edge partition of P_6 is not an *ec*-partition. Therefore, $EC(P_6) = 4$ and π is an $EC(P_6)$-partition.

Proposition 2.4. *I) For any star $K_{1,n}$, $EC(K_{1,n}) = n$.*
II) For any path P_n,

$$EC(P_n) = \begin{cases} n-1 & if\ n \leq 5 \\ 4 & if\ n = 6 \\ 5 & if\ 7 \leq n \leq 10 \\ 6 & if\ n \geq 11 \end{cases}$$

III) For cycle C_n

$$EC(C_n) = \begin{cases} n & if\ n \leq 6 \\ 5 & if\ n = 7 \\ 6 & if\ n \geq 8 \end{cases}$$

IV) For any double star $S_{n,m}$, the edge coalition number satisfies $EC(S_{n,m}) = n + m + 1$.

Proposition 2.5. *For any complete graph K_n of even order, $EC(G) \geq 2(n-1)$. For the complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$, the edge coalition number satisfies the inequality: $EC(K_{r,s}) \geq 2s$.*

3. Main results

3.1. Bounds on $EC(G)$.

NOTATION 3.1. If G is a graph with no isolated edges and no full edges, then $3 \leq 2\gamma'(G) - 1 \leq EC(G)$.

THEOREM 3.2. *If G is a graph with no full edge and minimum degree $\delta(G) \geq 1$, then $1 + \delta(G) \leq EC(G)$.*

PROOF. Let G be a graph of order n with $1 \leq \delta(G) \leq n-2$, and let v be a vertex in G with degree $deg(v) = \delta(G) = k$. Let $N_E(v) = \{e_1, e_2, \cdots, e_k\}$ as the set of edges incident to v. Since G has no full edge, $E(G) \setminus N[e_1] \neq \emptyset$. We define a partition π of the edges of G as follows: $E_i = \{e_i\}$, for $1 \leq i \leq k$ and $E_{k+1} = E(G) \setminus N[e_1]$. To see that π is an *ec*-partition of G, observe that E_{k+1} does not edge dominate e_1. Moreover, since G has no full edge, no singleton set edge dominates G. Hence, no set in π is an edge-dominating set of G. We now proceed by considering two cases.
i) If $k = \delta(G) = 1$, then $\pi = \{E_1, E_2\}$. It is clear that π forms an *ec*-partition.
ii) If $k = \delta(G) > 1$, then π is still an*ec*-partition, because, $E_1 \cup E_{k+1}$ constitutes an edge-dominating set of G. Moreover, for each $2 \leq i \leq k$, the set $E_i \cup E_{k+1}$, is also an edge-dominating set of G. Since, every edge in E_k is adjacent to at least one edge in E_{k+1}, assume, for contradiction, that there exists an edge e_j in E_i that is not adjacent to any edge in E_{k+1}. In this case, e_j would be a pendant edge, meaning there exists a vertex v_j incident to this edge with $deg(v_i) = 1$, which contradicts the assumption that $k = \delta(G) > 1$. □

3.2. Graphs G with small $EC(G)$.

THEOREM 3.3. *Let G be a graph. Then*
1. *$EC(G) = 1$ if and only if $G = K_2$.*
2. *$EC(G) = 2$ if and only if $G \in \{P_3, \overline{C_4}\}$.*

PROOF. (1) It is straightforward.
(2) If $G = \overline{C_4}$ or $G = P_3$, then clearly edge singleton sets will form the only edge coalition partition of graph G and so we have $EC(G) = 2$. Conversely, assume that $EC(G) = 2$. By (1), G must have order $n \geq 3$. If G has three or four vertices, then $G = P_3$ or $G = \overline{C_4}$ respectively. This follows because any graph with at least four vertices, other than $G = \overline{C_4}$ has an *ec*-coalition with three sets. Therefore, the result holds. □

THEOREM 3.4. *Let G be a connected graph. Then $EC(G) = 3$ if and only if $G \in \{C_3, P_4, K_{1,3}\}$.*

3.3. Graphs G where $EC(G)$ equals the size of G. In this section, we study the graphs G in which the edge coalition number $EC(G)$ equals the size of G.

Unicyclic graphs. Let G be a unicyclic connected graph of order n and size m, then $m = n$. In this section, we examine the unicyclic graph G in which $EC(G) = m$.
Let Θ be a family of unicyclic graphs constructed as:
(a) Cycle C_n where $3 \leq n \leq 6$.
(b) Graphs G obtained from C_3 by attaching at least one star, where some vertex of C_3 is identified with the center of each star, such as the graphs G_5, G_6, G_7 shown in Figure 1.
(c) Graphs G obtained from C_3 and a double star by identifying one vertex of C_3 with one of the centers of the double star, such as graph G_8 in Figure 1.
(d) Graphs G obtained from C_4 by attaching one or two stars, where the center of each

star is identified with some vertex of C_4, such as the graphs G_9, G_{10}, G_{11} in Figure 1.
(e) Graphs G obtained from C_5 by attaching a star, where one vertex of C_5 is identified with the center of the star, such as graph G_{12} in Figure 1.

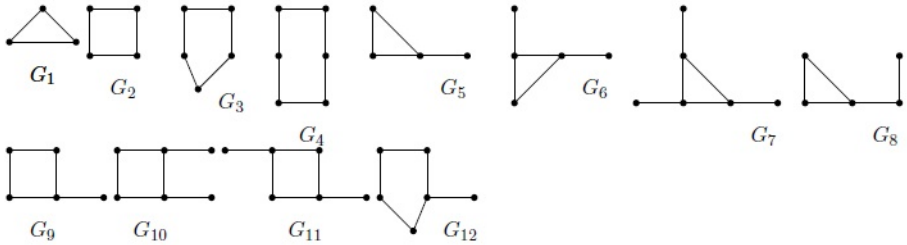

FIGURE 1. graphs that are in Θ

THEOREM 3.5. *For any unicyclic connected graph G of size m, $EC(G) = n = m$ if and only if $G \in \Theta$.*

Trees. For any tree T, it is well known that the size of T is $m = n - 1$ where n is the order of T.

THEOREM 3.6. *For every tree T of order n, and size m, $EC(T) = m = n - 1$ if and only if $T \in \Phi$ where Φ is a family of trees with diameter at most 3, $d(T) \leq 3$, or trees with diameter 4 such that the middle vertex of any longest path has degree 2, as shown in Figure 2.*

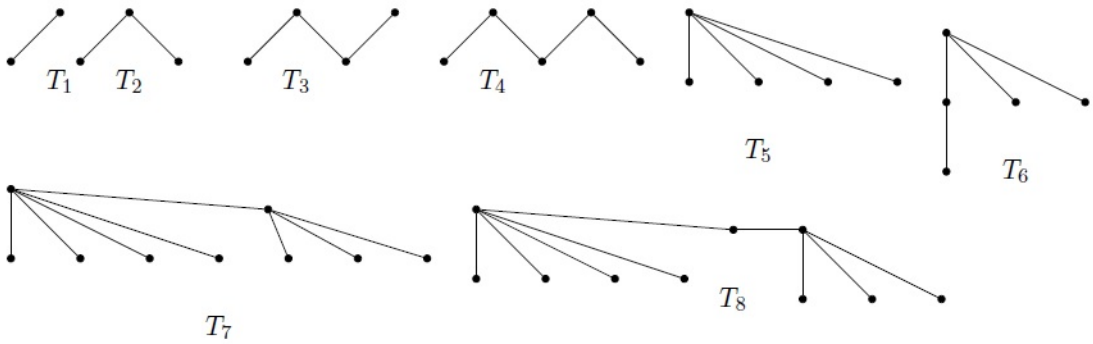

FIGURE 2. Some trees that are in Φ

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Perfect Star Packings in One Type of (2, 6)-Fullerene Graphs

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Abstract: A (2,6)-fullerene graph is a 2-connected, 3-regular and planar graph that only has faces with lengths of two or six. The number of faces of length two in these graphs is equal to three, according to Euler's formula. A perfect star packing in a graph G is a spanning subgraph of G in which each component of it is isomorphic to the star graph $K_{1,3}$. In this presentation, we investigate the existence of perfect star packings in a specific type of (2,6)-fullerene graph.

Keywords: (2,6)-fullerene graphs, Perfect star packing.

1. Introduction

A (k,6)-Fullerene graphs is a cubic and planar graphs whose faces each have k or six edges. Došlic [1], has proven that for $k = 2, 3, 4$ and 5, (k,6)-fullerene graphs exist. A (2,6)-fullerene graph is a cubic, planar and 2-connected graph in which the number of 2-lengths in it is equal to three. A perfect star packing in a graph G is a spanning subgraph of G whose each component is isomorphic to $K_{1,3}$. Several studies have addressed the topic of perfect packings of fullerene graphs. In [2, 3], the authors investigated the existence of perfect star packings and calculated the shortest size of these packings in (k,6)-fullerene graphs for $k = 3, 4$ and, 5. The structure of (2,6)-fullerene graphs is investigated by yang et al. [5]. In addition, the existence of perfect star packings in three other types of (2,6)-fullerene graphs is studied by the author of this presentation [4]. In this paper, considering the structure of one type of (2,6)-fullerene graphs, we investigate the existence of perfect star packing in this type of graphs. Furthermore we present a perfect star packing for these types of graphs.

2. Main results

In this section, according to the structure of one type of (2,6)-fullerene graphs, we investigate the existence of perfect star packing in this type of graphs and introduce a perfect star packing for it.

Definition 2.1. Let G be a (2,6)-fullerene graph. A fragment H of G is a subgraph of G consisting of a cycle with its interior and every inner face of H is also a face of G [5].

[1] . Corresponding Author

We denote the number of hexagons of the fragment H by G_H.

Proposition 2.2. [5]. In every (2,6)-fullerene graph, there is a fragment, say K_s, which $G_{K_s} = 2s^2 + 3s, s \in \mathbb{Z}$.

First, we consider the fragment G_s. If G_0 is a cycle of length two, then G_s, with s layers of hexagons, is shown in Figure 1.

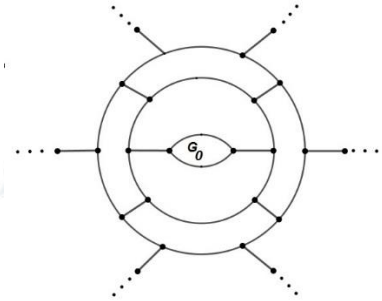

Figure 1. The fragment G_s

If G_1 and G_2 are two fragments with s layers, as shown in Figure 2, then there are $4s + 2$ vertices in the sth layer of these fragments. Suppose, $e_i = u_i v_i$, where i is an odd number, (see Figure 2), then we have the number of s hexagons, $h_1, h_2, \cdots, h_s$, in G_s. We refer to this type of (2,6)-fullerene graphs, as (2,6)-fullerene graphs of type G_s.

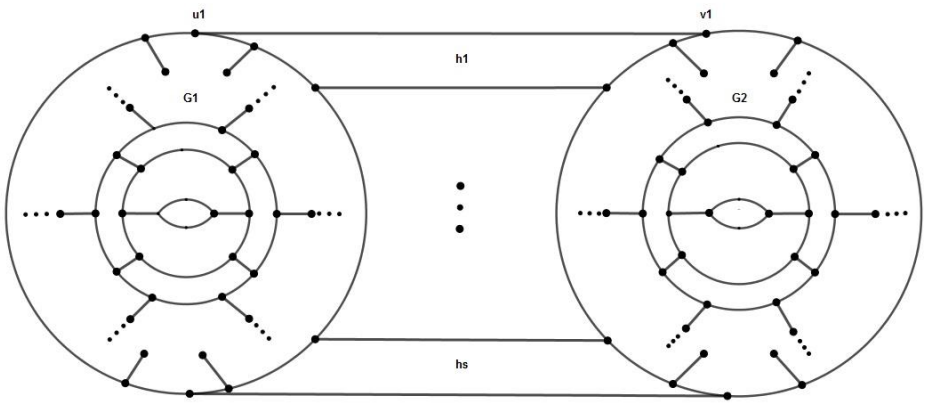

Figure 2. A fragment of (2,6)-fullerene graph of type G_s

Therefore, (2,6)-fullerene graphs of type G_s consist of an initial fragment including two fragments G_1 and G_2. In the next Theorem, we create a perfect star packing in these graphs.

Theorem 2.3. Let G be a fullerene graph of type G_s. Then

a) G has a perfect star packing, if s is odd.
b) G does not have a perfect star packing, if s is even.

Proof. First, suppose that the number of hexagonal layers in the fragments G_1 and G_2 is an odd number like s. Then s layers are formed around the initial fragment. The first layer has $4s + 2$ vertices. In the second layer, we will have $4s - 2$ vertices. Similarly, in the $\boldsymbol{s}$th layer, the vertices of a hexagon form six vertices. Finally, two vertices are added to them. Therefore, the number

of vertices of a (2,6)-fullerene graph of type G_s, that has s layers in the initial fragment, is equal to

$$2(2s^2+4s+2)+2s^2+4s+2=6s^2+12s+6=4k+2(s^2+1), k\in\mathbb{Z}$$

This value is a multiple of four if s is an odd number. Therefore, G does not have a perfect star packing if s is even.

Now, suppose s is an odd number. In this case, we show that G has a perfect star packing. For this purpose, we create a packing for fragments $G_1$and G_2 as follows. In the first layer, we cover the edges as shown in Figure 3.

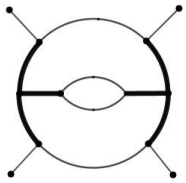

Figure 3. Packing in the first layer

There are two hexagons in the first layer. By doing this, the initial fragment of the graph G is covered. Now, we consider vertices as the external layers. We start from the outermost layer. For this layer, we must have the following packing, as shown in Figure 4.

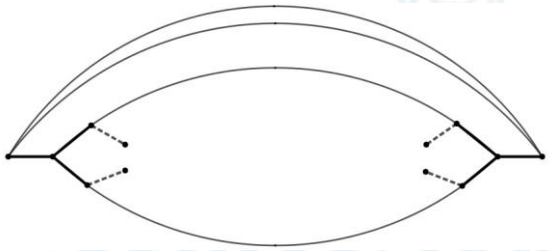

Figure 4. Packing in the outermost layer

An even number of layers remain. We cover the vertices of each pair of layers, as shown in Figure 5.

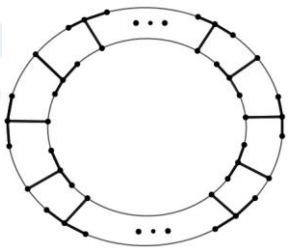

Figure 5. Packing in the pair of layers

Therefore, we have a perfect star packing in G. ■

3. Conclusion

In this presentation, we investigated the existence of a perfect star packing in one type of (2,6)-fullerene graphs. Using the structure of this class of graphs, we showed that if the number of hexagonal layers is odd, then there is a perfect star packing. For future research, one could investigate the existence of perfect star packing for all types of (2,6)-fullerene graphs.

Acknowledgement

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Degree-based topological indices of sedative drugs with M−polynomial

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Abstract. Naproxen and aspirin are sedative drugs. In this paper we compute some degree-based topological indices by using M−polynomial for these drugs. The scientists use these computations for analgesics of drugs.

Keywords: Topological index, Sedative drug, M−polynomial

AMS Mathematics Subject Classification [2020]: 05C05, 05C07

1. Introduction

Let G be a chemical graph that atoms are vertex set $V(G)$ and the bonds between them are edge set $E(G)$. For vertices $u, v \in V(G)$, the weight of edge uv, $w(uv)$ is number of molecular bonds between atoms. Degree of a vertex v is $d_v = \sum_{uv \in E(G)} w(uv)$. For more information refer the book [3]
Recently, one of the interesting subjects for authors compute numerical calculations related to drug graphs [2, 4, 5]. This numerical calculations are called topological indices. Some of authors calculate the topological indices by use M-polynomial and NM-polynomial [1]. In this paper, we use differentiation, integral and some other mathematic operations 1,2 and 3 for computing the topological indices.

2. Main results

Let G be a chemical weighted graph, $\kappa_{ij} = \{|E_{ij}||d_u = i, d_v = j\}$, $\delta(u) = \sum_{v \in N_G(u)} d_v$ and $\kappa^*_{ij} = \{|\check{N}E_{ij}|\,|\delta(u) = i, \delta(v) = j\}$.
In 2000 $M-$polynomail is defined as

$$M(G, x, y) = \sum_{i \leq j} \kappa_{ij} x^i y^j.$$

For computing the topological indices from $M(G, x, y)$ we need some functions that they are defining as follows:

$$n^2(M(G, x, y)) = \sum_{i \leq j} \kappa^2_{ij} x^i y^j, \quad sqrt(M(G, x, y)) = \sum_{i \leq j} \sqrt{\kappa_{ij}} x^i y^j, \tag{1}$$

TABLE 1. Definition of topological indices and formula for calculating them from M-polynomial.

Topological Indices	mathematical define	Derivation from $M(G,x,y)$ or $NM(G,x,y)$
General sum-connectivity index ($\chi_\alpha(G)$)	$\sum_{uv\in E(G)}(d_u+d_v)^\alpha$	$d_x^\alpha I(f(x,y))\|_{x=1}$
First Zagreb index ($M_1(G)$)	$\sum_{uv\in E(G)} d_u+d_v$	$(d_x+d_y)(f(x,y))\|_{x=y=1}$
Second Zagreb index ($M_2(G)$)	$\sum_{uv\in E(G)} d_ud_v$	$d_xd_y(f(x,y))\|_{x=y=1}$
Forgotten index ($F(G)$)	$\sum_{uv\in E(G)}(d_u^2+d_v^2)$	$(d_x^2+d_y^2)(f(x,y))\|_{x=y=1}$
VL-index ($VL(G)$)	$\sum_{uv\in E(G)}(d_u+d_v+d_ud_v)$	$(d_x+d_y+d_xd_y)(f(x,y))\|_{x=y=1}$
Atom bond connectivity index ($ABC(G)$)	$\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_ud_v}}$	$sqrt(d_x+d_y)\Phi_{-2}E_xE_yn^2(f(x,y))\|_{x=1}$
Sum-connectivity index ($SCI(G)$)	$\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u+d_v}}$	$sqrtE_xn^2I(f(x,y))\|_{x=1}$
Randic index ($R(G)$)	$\sum_{uv\in E(G)}\frac{1}{\sqrt{d_ud_v}}$	$sqrtE_yE_xn^2(f(x,y))\|_{x=y=1}$
First geometric index ($GA(G)$)	$\sum_{uv\in E(G)}\frac{2\sqrt{d_ud_v}}{d_u+d_v}$	$2E_xIsqrtd_xd_yn^2(f(x,y))\|_{x=1}$
Sombor index ($SO(G)$)	$\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$	$sqrt(d_x^2+d_y^2)n^2(f(x,y))\|_{x=y=1}$

$$d_x = x\Big(\frac{\partial(f(x,y))}{\partial(x)}\Big), d_y = y\Big(\frac{\partial(f(x,y))}{\partial(y)}\Big), E_x = \int_0^x \frac{f(t,y)}{t}dt, E_y = \int_0^y \frac{f(x,t)}{t}dt \tag{2}$$

and

$$\Phi_k(f(x,y)) = x^k f(x,y),\ \ I(f(x,y)) = f(x,x). \tag{3}$$

There are more degree based topological indices. In this paper we study some of them that the definition of each index and the computation of them by using M-polynomial are in table 1.

There are several brain anticancer drugs. In this paper, we study Naproxen $C_{14}H_{14}O_3$ and Aspirin $C_9H_8O_4$. The structure of these drugs are in figure 1.

THEOREM 2.1. *If G_1 is the chemical graph of Naproxen $C_{14}H_{14}O_3$, then*

$$M(G_1,x,y) = xy^2 + 13xy^4 + 4x^2y^4 + 14x^4y^4.$$

PROOF. From Figure 1, we see that $|V| = 31$ and $|E| = 32$ and also $|E_{1,2}| = 1$, $|E_{1,4}| = 13$, $|E_{2,4}| = 4$ and $|E_{4,4}| = 14$ Therefor

$$M(G_1,x,y) = \sum_{i\le j}\tau_{ij}x^iy^j = xy^2 + 13xy^4 + 4x^2y^4 + 14x^4y^4.$$

□

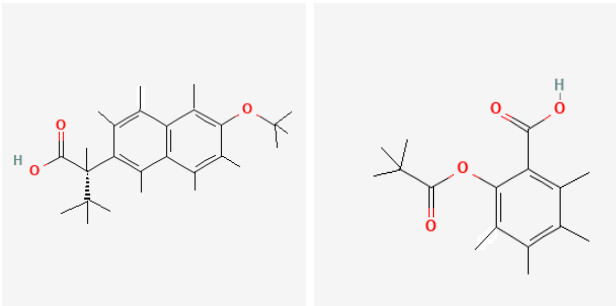

FIGURE 1. Mulecules of Naproxen and Aspirin

THEOREM 2.2. *Let G_1 be the chemical graph of Naproxen $C_{14}H_{14}O_3$. Some topological indices of G_1 are as:*

(1) $M_1(G_1) = 204$,
(2) $M_2(G_1) = 310$,
(3) $F(G_1) = 754$,
(4) $VL(G_1) = 514$,
(5) $ABC(G_1) \simeq 23.37$,
(6) $SCI(G_1) \simeq 12.9$,
(7) $R(G_1) \simeq 12.12$,
(8) $GA(G_1) \simeq 29.11$,
(9) $SO(G_1) \simeq 152.92$.

PROOF. From theorem 2.1, we obtain the following equations.

$I(M(G_1,x,y)) = x^3 + 13x^5 + 4x^6 + 14x^8$,
$d_x^\alpha I(M(G_1,x,y)) = 3^\alpha x^3 + 13 \times 5^\alpha x^5 + 4 \times 6^\alpha x^6 + 14 \times 8^\alpha x^8$,
$(d_x + d_y)(M(G_1,x,y)) = 3xy^2 + 65xy^4 + 24x^2y^4 + 112x^4y^4$,
$d_x d_y(M(G_1,x,y)) = 2xy^2 + 52xy^4 + 32x^2y^4 + 22x^4y^4$,
$n^2(M(G_1,x,y)) = xy^2 + 169xy^4 + 16x^2y^4 + 196x^4y^4$,
$sqrt(d_x + d_y)\Phi_{-2}E_xE_yn^2(M(G_1,x,y)) = \frac{1}{\sqrt{2}}x_{-1}y^2 + \frac{13}{2}\sqrt{3}x^{-1}y^4 + 2\sqrt{2}y^4 + \frac{7}{2}\sqrt{6}x^2y^4$,
$sqrtE_xn^2I(M(G_1,x,y)) = \frac{1}{\sqrt{3}}x^3 + \frac{13}{\sqrt{5}}x^5 + \frac{4}{\sqrt{6}}x^6 + \frac{7}{\sqrt{2}}x^8$,
$sqrtE_yE_xn^2(M(G_1,x,y)) = \frac{1}{\sqrt{2}}xy^2 + \frac{13}{2}xy^4 + \sqrt{2}x^2y^4 + \frac{7}{2}x^4y^4$,
$E_xIsqrtd_xd_yn^2(M(G_1,x,y)) = \frac{\sqrt{2}}{3}x^3 + \frac{26}{5}x^5 + 4\frac{\sqrt{2}}{3}x^6 + 7x^8$,

Now we calculate the topological indices. (Use upper results and table 1.)

□

THEOREM 2.3. *If G_2 is the chemical graph of Naproxen $C_{14}H_{14}O_3$, then*

$$M(G_2,x,y) = xy^2 + 7xy^4 + 5x^2y^4 + 8x^4y^4.$$

PROOF. From Figure 1, we see that $|V| = 21$ and $|E| = 21$ and also $|E_{1,2}| = 1$, $|E_{1,4}| = 7$, $|E_{2,4}| = 5$ and $|E_{4,4}| = 8$ Therefor

$$M(G_2,x,y) = \sum_{i \le j} \tau_{ij} x^i y^j = xy^2 + 7xy^4 + 5x^2y^4 + 8x^4y^4.$$

□

THEOREM 2.4. *Let G_2 be the chemical graph of Naproxen $C_{14}H_{14}O_3$. Some topological indices of G_2 are as:*

(1) $M_1(G_2) = 132$,
(2) $M_2(G_2) = 198$,
(3) $F(G_2) = 480$,
(4) $VL(G_2) = 330$,
(5) $ABC(G_2) \simeq 15.2$,
(6) $SCI(G_2) \simeq 8.58$,
(7) $R(G_2) \simeq 7.97$,
(8) $GA(G_2) \simeq 8$,
(9) $SO(G_2) \simeq 98.71$.

PROOF. From theorem 2.1, we obtain the following equations.

$$I(M(G_2,x,y)) = x^3 + 7x^5 + 5x^6 + 8x^8,$$
$$d_x^{\alpha} I(M(G_2,x,y)) = 3^{\alpha}x^3 + 7 \times 5^{\alpha}x^5 + 5 \times 6^{\alpha}x^6 + \times 8^{\alpha+1}x^8,$$
$$(d_x + d_y)(M(G_2,x,y)) = 3xy^2 + 35xy^4 + 30x^2y^4 + 64x^4y^4,$$
$$d_x d_y(M(G_2,x,y)) = 2xy^2 + 28xy^4 + 40x^2y^4 + 128x^4y^4,$$
$$n^2(M(G_2,x,y)) = xy^2 + 49xy^4 + 25x^2y^4 + 64x^4y^4,$$
$$sqrt(d_x + d_y)\Phi_{-2}E_xE_yn^2(M(G_2,x,y)) = \sqrt{\tfrac{1}{2}}x^{-1}y^2 + \tfrac{7}{2}\sqrt{3}x^{-1}y^4 + \tfrac{5}{\sqrt{2}}y^4 + 2\sqrt{6}x^2y^4,$$
$$sqrtE_xn^2I(M(G_2,x,y)) = \tfrac{1}{\sqrt{3}}x^3 + \tfrac{7}{\sqrt{5}}x^5 + \tfrac{5}{\sqrt{6}}x^6 + 2\sqrt{2}x^8,$$
$$sqrtE_yE_xn^2(M(G_2,x,y)) = \tfrac{1}{\sqrt{2}}xy^2 + \tfrac{7}{2}xy^4 + \tfrac{5}{2\sqrt{2}}x^2y^4 + 2x^4y^4,$$
$$E_xIsqrtd_xd_yn^2(M(G_2,x,y)) = \tfrac{\sqrt{2}}{3}x^3 + \tfrac{2\sqrt{7}}{5}x^5 + \tfrac{\sqrt{10}}{3}x^6 + \sqrt{2}x^8,$$

Now we can calculate the topological indices . (Use upper results and table 1.) □

3. Conclusion

The computations in this paper are useful for pharmacists and we hope every one is healthy and safe and no one needs medication.

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NL-coloring of of Vertex-Deleted Subgraphs

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Abstract. A k-coloring of a simple graph G partitions its vertex set $V(G)$ into k disjoint independent sets. Such a coloring is a neighbor-locating coloring if any two vertices within the same color class are distinguished by their external neighborhoods; the minimum such k is the NL-chromatic number, $\chi_{NL}(G)$. This paper investigates the effect of vertex deletion on this parameter, establishing bounds and exact values for $\chi_{NL}(G-u)$. To quantify this impact, we introduce the Neighbor-Locating Stability Index, $\rho_{NL}(G)$, as a measure of a graph's resilience to structural perturbations. We use this index to classify graph families as absolutely stable, bounded stable, or unstable, analyzing the structural sensitivity of NL-colorings.

Keywords: Coloring, neighbor-locating coloring, vertex deletion.

AMS Mathematics Subject Classification [2020]: 05C15, 05C76.

1. Introduction

Vertex coloring is a fundamental and widely applicable concept in graph theory, primarily used to model non-interference constraints within various systems. Its applications range from the optimal allocation of resources in exam scheduling and radio frequency assignment to compiler optimization techniques like register allocation. Despite the inherent NP-hardness of determining the chromatic number ($\chi(G)$), deep investigations into this area have led to the definition of more sophisticated generalizations, such as the neighbor-locating chromatic number ($\chi_{\mathrm{NL}}(G)$). These generalizations address subtle requirements, particularly the distinguishability of elements in complex networks, opening new avenues in both theory and practical application.

Let $G = (V, E)$ be a simple, finite, and undirected graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The open neighborhood of a vertex v, denoted by $N(v)$, is the set of all vertices directly connected to v. The closed neighborhood of v, denoted by $N[v]$, is the union of its open neighborhood and v itself, i.e., $N[v] = N(v) \cup \{v\}$. The minimum degree of G, denoted by $\delta(G)$, is the smallest number of neighbors any vertex in G has, while the maximum degree of G, denoted by $\Delta(G)$, is the largest. For terminologies and notations not defined here, we refer the reader to [6].

*Speaker.

A proper k-coloring of a graph G (where $k \in \mathbb{N}$) is a function that assigns one of k colors to each vertex such that no two adjacent vertices have the same color. The minimum value of k for which such a coloring exists is called the chromatic number of G, denoted by $\chi(G)$. The resulting sets of vertices with the same color are called color classes, typically denoted by $\pi = \{S_1, S_2, \ldots, S_k\}$. The color-degree of a vertex v is the number of distinct colors appearing among its neighbors.

A k-coloring $\pi = \{S_1, S_2, \ldots, S_k\}$ is defined as a neighbor-locating (NL) coloring if for any two distinct vertices u and v within the same color class S_i, their neighborhood color sets $C(N(u))$ and $C(N(v))$ are distinct. The minimum value of k for which such a coloring exists is called the neighbor-locating chromatic number of G, denoted by $\chi_{NL}(G)$. This concept has been extensively studied in various works, including [1, 2, 5].

In this paper, we initiate an investigation into the impact of vertex deletion on the neighbor-locating chromatic number. Specifically, we explore how the removal of a vertex u from a graph G affects its NL-chromatic number. We examine the relationship between $\chi_{NL}(G-u)$ and $\chi_{NL}(G)$ for various graph families. We aim to determine precise values for $\chi_{NL}(G-u)$ in certain cases and establish tight bounds on this parameter, shedding light on the robustness and sensitivity of NL-coloring to minor structural alterations. We begin by establishing tight bounds for the case of removing a leaf vertex. This initial result demonstrates the robustness of the NL-chromatic number to this simple form of vertex deletion. Subsequently, we generalize our analysis by providing a general upper bound for the removal of a vertex with an arbitrary degree. This more comprehensive result accounts for how the local properties of the deleted vertex influence the overall graph coloring.

This investigation reveals that NL-coloring exhibits varying degrees of resilience to vertex removal. To formally quantify this robustness, we introduce a novel parameter, the Neighbor-Locating Stability Index. For a given graph G and vertex u, the local stability index is defined as the signed difference:

$$\rho_{NL}(G, u) := \chi_{NL}(G-u) - \chi_{NL}(G)$$

Furthermore, we define the maximum stability index of the graph G as the largest possible value of this difference:

$$\rho_{NL}(G) := \max_{u \in V(G)} \{\rho_{NL}(G, u)\}$$

This index is more than a simple numerical result; it provides a new analytical framework for classifying graphs and their families based on their structural stability. By leveraging this index, we provide a comprehensive analysis of the stability of various graph families, including paths, cycles, and trees. We classify these families as absolutely stable, bounded stable, or unstable based on the asymptotic behavior of their stability index. This work not only expands the study of neighbor-locating coloring but also establishes a new paradigm for evaluating graph robustness against local perturbations.

Definition 1.1. For a k-coloring $\pi = \{S_1, S_2, \ldots, S_k\}$ of a graph G and a vertex $v \in V(G)$, the color-degree of v is defined to be the number of distinct colors present in the neighborhood of v, i.e., $|\{j : N(v) \cap S_j \neq \emptyset\}|$.

2. Main results

Proposition 2.1 ([4]). *Let $k \geq 4$ be an integer. There exist a graph G with diameter $d = \ell(k)$ for which $\chi_{NL}(G) = k$ and $\chi_{NL}(P_{\ell(k)+1}) = k+1$.*

Proposition 2.2 ([4]). *Let T be a tree and $\chi_{NL}(T) = k$. Let v be a vertex of maximum degree $\Delta(T)$. If $\Delta(T) \geq (k-1)^2 + 1$, then the number of vertices of degree 1 or 2 in $N(v)$ is at least and at most*

$$(k-1)^2 - (k-3) \text{ and } (k-1)^2$$

Theorem 2.3. *Let G be a non-trivial graph and let u be a vertex with $\deg(u) = 1$. If $\chi_{NL}(G) = k$, then the following inequality holds:*

$$k - 1 \leq \chi_{NL}(G - u) \leq k + 1$$

Theorem 2.4. *Let G be a non-trivial graph. Let $u \in V(G)$ be a vertex with $\deg(u) = m \geq 2$. Let c be an optimal NL-coloring of G such that $\chi_{NL}(G) = k$ and $c(u) = i$. Suppose that $l = |\{v \in N(u) \mid \deg_G(v) = 1\}|$ is the number of leaf neighbors of u in G and $r = |\{v \in N(u) \mid i \notin \{c(w) \mid w \in N_{G-u}(v)\}\}|$ is the number of neighbors whose neighborhood color sets in $G - u$ do not contain color i. Then the following inequality holds:*

$$\chi_{NL}(G - u) \leq \begin{cases} k + r & if\ l \leq k \\ r + l & if\ l > k \end{cases}$$

Theorem 2.5. *For any path graph P_n with $n \geq 3$, let $\chi_{NL}(P_n) = k$. For any vertex $u \in V(P_n)$, the Neighbor-Locating chromatic number of the resulting subgraph $P_n - u$ satisfies:*

$$\chi_{NL}(P_n) - 1 \leq \chi_{NL}(P_n - u) \leq \chi_{NL}(P_n) + 1$$

Proposition 2.6. *Let $G = K_{n_1,\ldots,n_m}$ be a complete m-partite graph with partitions $V_1, \ldots, V_m$, where the partition sizes are ordered such that $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. Let $n = |V(G)| = \sum_{i=1}^{m} n_i$. When an arbitrary vertex u is removed, the Neighbor-Locating chromatic number of the resulting subgraph satisfies:*

$$\chi_{NL}(K_{n_1,\ldots,n_m} - u) = \chi_{NL}(K_{n_1,\ldots,n_m}) - 1 = n - 1$$

Proposition 2.7. *If u is a vertex of the star graph $K_{1,n-1}$, and let $n = |V(G)|$, the Neighbor-Locating chromatic number of the resulting subgraph satisfies:*

$$\chi_{NL}(K_{1,n-1} - u) = n - 1$$

Proposition 2.8. *For any complete graph K_n where $n \geq 2$, if u is an arbitrary vertex, then the resulting subgraph $K_n - u$ satisfies:*

$$\chi_{NL}(K_n - u) = \chi_{NL}(K_n) - 1 = n - 1$$

Proposition 2.9. *Let K_n be a complete graph on $n \geq 3$ vertices. For any arbitrary vertex $u \in V(K_n)$, we have, $\rho_{NL}(K_n) = -1$.*

Proposition 2.10. *Let $G = K_{1,n-1}$ be a star graph on $n \geq 3$ vertices. For any arbitrary vertex $u \in V(G)$, the deletion index is a constant value:*

$$\rho_{NL}(G, u) = \chi_{NL}(G - u) - \chi_{NL}(G) = -1$$

Proposition 2.11. *Let $G = K_{n_1,\ldots,n_m}$ be a complete m-partite graph with partitions $V_1, \ldots, V_m$, where the partition sizes are ordered such that $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. Let $n = |V(G)| = \sum_{i=1}^{m} n_i$. For any arbitrary vertex $u \in V(G)$, the deletion index is a constant value:*

$$\rho_{NL}(G, u) = \chi_{NL}(G - u) - \chi_{NL}(G) = (n - 1) - n = -1$$

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Fuzzy E-Banhatti Index and Its Application to Autism Drugs

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Abstract. This paper introduces the fuzzy E–Banhatti index for fuzzy molecular graphs. Upper and lower bounds are derived for fuzzy paths and cycles, showing how vertex and edge fuzziness affects the index. Fuzzy molecular graphs of autism-related drugs are constructed and their indices calculated. The indices are correlated with physicochemical properties using QSPR analysis. Results indicate the fuzzy E–Banhatti index is a useful predictor of molecular properties, bridging fuzzy graph theory and computational chemistry.

Keywords: Fuzzy E–Banhatti index, Fuzzy molecular graphs, Autism drugs, Topological indices, QSPR analysis

AMS Mathematics Subject Classification [2020]: 05C10, 05C90, 05C92.

1. Introduction

Topological indices (TIs) are numerical descriptors that quantify structural properties of molecular graphs, linking molecular structure to physicochemical and biological properties (PMC). They are widely used in QSPR and QSAR analyses to predict chemical behavior in biological systems [1, 5]. Banhatti-type indices, in particular, consider vertex degrees and adjacency, providing insight into molecular connectivity and stability [4]. The E–Banhatti index captures interactions between adjacent vertices, offering a detailed view of molecular topology. Real-world molecular systems often involve uncertainty due to structural flexibility or measurement errors. Fuzzy graph theory addresses this by assigning membership values between 0 and 1 to vertices and edges, accommodating these uncertainties [2]. In this study, we extend the E–Banhatti index to fuzzy graphs and derive upper and lower bounds for fuzzy paths, cycles, and chain graphs. To illustrate its application, fuzzy molecular graphs of autism-related drugs are constructed, and their fuzzy E–Banhatti indices are computed. A QSPR analysis then explores correlations between these indices and selected physicochemical properties. The results demonstrate that

*Speaker.

the fuzzy E–Banhatti index can effectively capture structural variations under uncertainty and serve as a predictive tool in molecular modeling and drug design.

1.1. Preliminaries. Here some necessary definitions required to develop our results are given as following which most of them can be found in ([3]). For any universal set X, with a membership function $\mu : X \to [0,1]$, the pair $S = (X, \mu)$ is named a fuzzy set S.

Definition 1.1. A fuzzy graph is defined as a triple $G = (V, \sigma, \mu)$, where V is a non-empty set, and $\mu : V \times V \to [0,1]$ and $\sigma : V \to [0,1]$ are two mapping functions.

$$\mu(x,y) \leq \sigma(x) \wedge \sigma(y) \text{ for all } x, y \in V.$$

Definition 1.2. A sequence of different vertices $v_1, v_2, \cdots, v_n$ that $\mu(v_i v_{i+1}) > 0; i = 1, ..., n$ is named a path P of the lenth n in a fuzzy graph $G = (V, \sigma, \mu)$ and the successive pairs $(v_i v_{i+1})$ are considered as edges of the path.

Definition 1.3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph, then:

(i) if $(Supp(\sigma), Supp(\mu))$ be a cycle, then G will be named a cycle.
(ii) if $\nexists$ unique $uv \in Supp(\mu)$ such that $\mu(uv) = \wedge\{\mu(xy) | xy \in Supp(\mu)\}$, and $(Supp(\sigma), Supp(\mu))$ be a cycle, then G will be named a fuzzy cycle.

2. The fuzzified E-Banhatti indices

Definition 2.1. For a fuzzy graph $G = (V, \sigma, \mu)$, the degree of an edge $e = uv \in E(G)$ is defined as $d(e) = d(u) + d(v) - 2\mu(uv)$ and the Banhatti degree of a vertex $u \in V(G)$ is defined as follows:

$$B_u = \frac{d(e)}{n - d(u)} \tag{1}$$

where n is the number of vertices of G.

According to definition 2.1 we can define the E-Banhatti indices for a fuzzy graph as Table 1:

Table 1. E-Banhatti topological indices of FG.

E-Banhatti Index name	**Definition**
First E-Banhatti index	$EB_1 = \sum_{uv \in E(G)} (\sigma(u)B_u + \sigma(v)B_v)$
Second E-Banhatti index	$EB_2 = \sum_{uv \in E(G)} (\sigma(u)B_u . \sigma(v)B_v)$
First modified E-Banhatti indices	$MEB_1(G) = \sum_{uv \in E(G)} \frac{1}{\sigma(u)B_u + \sigma(v)B_v}$
Second modified E-Banhatti indices	$MEB_2(G) = \sum_{uv \in E(G)} \frac{1}{\sigma(u)B_u . \sigma(v)B_v}$

Theorem 2.2. *For a fuzzy path $P = (\sigma, \mu) : (v_1, v_2, \cdots, v_n)$, we have:*

$$EB_1(P) \leq \sigma_{max}\left(\frac{2\mu_{max}}{n - \mu_{max}} + \frac{(4n-10)\mu_{max}}{n - 2\mu_{max}}\right).$$

Proof. As P is a path, $d(v_1) = \mu_1$, $d(v_n) = \mu_{n-1}$ and $d(v_i) = \mu_{i-1} + \mu_i$ for $i = 2, 3, ..., n-1$, where $\sigma(v_i) = \sigma_i$ and $\mu(v_i v_{i+1}) = \mu_i$ for $i = 1, 2, 3, ..., n$. Therefore

$$\begin{aligned}
EB_1(P) =&(\sigma_1 \frac{\mu_2}{n-\mu_1} + \sigma_2 \frac{\mu_2}{n-(\mu_1+\mu_2)}) + \sum_{i=2}^{n-2}(\sigma_i \frac{\mu_{i-1}+\mu_{i+1}}{n-(\mu_{i-1}+\mu_i)}) + (\sigma_{i+1} \frac{\mu_{i-1}+\mu_{i+1}}{n-(\mu_i+\mu_{i+1})}) \\
&+(\sigma_{n-1} \frac{\mu_{n-2}}{n-(\mu_{n-2}+\mu_{n-1})} + \sigma_n \frac{\mu_{n-2}}{n-\mu_{n-1}}) \\
&\leq \sigma_{max} \frac{\mu_{max}}{n-\mu_{max}} + \sigma_{max} \frac{\mu_{max}}{n-2\mu_{max}} + (n-3)(2\sigma_{max} \frac{2\mu_{max}}{n-2\mu_{max}}) \\
&+\sigma_{max} \frac{\mu_{max}}{n-\mu_{max}} + \sigma_{max} \frac{\mu_{max}}{n-2\mu_{max}} \\
&=\sigma_{max}(\frac{2\mu_{max}}{n-\mu_{max}} + \frac{(4n-10)\mu_{max}}{n-2\mu_{max}}).
\end{aligned}$$

□

Corollary 2.3. *Let $P = (\sigma, \mu) : (v_1, v_2, \cdots, v_n)$ be a fuzzy path. Then*

(1) $EB_2(P) \leq \sigma_{max}^2 \times \frac{\mu_{max}^2}{n-2\mu_{max}}(\frac{2}{n-\mu_{max}} + \frac{4(n-3)}{n-2\mu_{max}})$

(2) $MEB_1(P) \geq \frac{2(n-\mu_{max})(n-2\mu_{max})}{\sigma_{max}\mu_{max}(2n-3\mu_{max})} + \frac{(n-3)(n-2\mu_{max})}{4\sigma_{max}\mu_{max}}$

(3) $MEB_2(P) \geq \frac{2(n-\mu_{max})(n-2\mu_{max})}{\sigma_{max}^2\mu_{max}^2} + \frac{(n-3)(n-2\mu_{max})^2}{4\sigma_{max}^2\mu_{max}^2}$.

Theorem 2.4. *For a fuzzy cycle $C = (\sigma, \mu) : (v_1, v_2, \cdots, v_n)$, we have*

$$EB_1(C) \leq \frac{4n\sigma_{max}\mu_{max}}{n-2\mu_{max}}.$$

Proof. As $C(v_1, v_2, \cdots, v_n)$ is a cycle, then $d(v_1) = \mu_1 + \mu_n$ and $d(v_i) = \mu_{i-1} + \mu_i$ for $i = 2, 3, ..., n$, where $\sigma(v_i) = \sigma_i$ and $\mu(v_i v_{i+1}) = \mu_i$ for $i = 1, 2, 3, ..., n$. Therefore

$$\begin{aligned}
EB_1(C) =&(\sigma_1 \frac{\mu_n+\mu_2}{n-(\mu_1+\mu_n)} + \sigma_2 \frac{\mu_n+\mu_2}{n-(\mu_1+\mu_2)}) + \sum_{i=2}^{n-1}(\sigma_i \frac{\mu_{i-1}+\mu_{i+1}}{n-(\mu_{i-1}+\mu_i)} + \sigma_{i+1} \frac{\mu_{i-1}+\mu_{i+1}}{n-(\mu_i+\mu_{i+1})}) \\
&+(\sigma_n \frac{\mu_{n-1}+\mu_1}{n-(\mu_{n-1}+\mu_n)} + \sigma_1 \frac{\mu_{n-1}+\mu_1}{n-(\mu_n+\mu_1)}) \\
&\leq 4\sigma_{max} \frac{2\mu_{max}}{n-2\mu_{max}} + 2(n-2)\sigma_{max} \frac{2\mu_{max}}{n-2\mu_{max}} \\
&=\frac{4n\sigma_{max}\mu_{max}}{n-2\mu_{max}}.
\end{aligned}$$

□

Corollary 2.5. *For a fuzzy cycle $C = (\sigma, \mu) : (v_1, v_2, \cdots, v_n)$, we have*

(1) $EB_2(C) \leq (\frac{2\sigma_{max}\mu_{max}}{n-2\mu_{max}})^2(n+1)$

(2) $MEB_1(C) \geq \frac{n(n-2\mu_{max})}{4\sigma_{max}\mu_{max}}$

(3) $MEB_2(C) \geq \frac{n(n-2\mu_{max}^2)}{4\sigma_{max}^2\mu_{max}^2}$.

3. Results

In this study, six Autism-related drugs (Aripiprazole, Haloperidol, Sertraline, Risperidone, and Venlafaxine, Ziprasidone) were analyzed using fuzzy molecular graphs to compute their E-Banhatti and modified E-Banhatti (MEB_1, MEB_2) indices. These indices were derived by fuzzifying atomic masses and bond lengths according to:

$$\sigma(u) = \frac{Atomic\ mass\ of\ u}{Max\ atomic\ mass}, \quad \mu(uv) = \frac{Bond\ length\ of\ uv}{Max\ atomic\ mass\ of\ u, v}.$$

The computed topological indices were correlated with six physicochemical properties: boiling point (BP), electronegativity (EN), flash point (FP), molar refractivity (MR), polarizability (PO), and molar volume (MV).

Linear regression models of the form $P = B + A(TI)$ were fitted, where P denotes a physical property, and TI represents a topological index.(For example:

$BP = 0.060[MEB_1(G)] + 303.943$,

$EN = 0.007[MEB_1(G)] + 57.188$ and

$FP = 0.0001[MEB_2(G)] + 174.382$)

The regression coefficients and significance levels confirmed that all relationships were statistically meaningful ($p < 0.05$).

Among the indices, MEB_1 and MEB_2 showed the strongest correlations with MR and PO, with $r = 0.987$ and $r = 0.986$, respectively.

Table 2. Correlation coefficients between E-Banhatti indices and physical properties of Autism drugs.

E-Banhatti indices	BP	EN	FP	MR	PO	MV
EB_1	-0.569	-0.517	-0.570	-0.590	-0.587	-0.333
EB_2	-0.805	-0.751	-0.805	-0.810	-0.808	-0.613
MEB_1	**0.964**	**0.917**	**0.964**	**0.987**	**0.987**	**0.826**
MEB_2	**0.937**	**0.881**	**0.937**	**0.986**	**0.986**	0.801

4. Conclusion

The obtained results reveal a strong quantitative structure–property relationship (QSPR), suggesting that fuzzy-based topological indices can serve as reliable predictors for the physicochemical behavior of Autism drugs. This highlights their potential in drug design and optimization studies.

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K−distance degree index for Dendrimer

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Abstract. Dendrimers are highly branched macromolecules that exhibit uniform diffusion with a uniform and controlled size, abundant surface functional properties, and low viscosity in solution. We calculate the N_k for these graphs.

Keywords: k-distance degree index, graph, Dendrimer

AMS Mathematics Subject Classification [2020]: 05C92, 05D40.

1. Introduction

Dendrimers are highly branched macromolecules that exhibit uniform diffusion with a uniform and controlled size, abundant surface functional properties, and low viscosity in solution. These properties make them ideal nanocarriers for biomedical applications.
All over this paper, all the graphs are simple, finite and connected. A graph $G = (V, E)$ is with the vertex set V of order $|V| = n$ and the edge set E of size $|E| = m$. For two vertex w and t of G, $d(w, t)$ is the size of the nearest path between them. For vertex $x \in V$ and $k \in \mathbb{Z}$, the open k-neighborhood of x in G, $N_k(x/G)$ or $N_k(x)$, is $N_k(x) = \{y \in V(G) : d(x, y) = k\}$ and the closed k-neighborhood of x in G, $N_k[x/G]$, is $N_k[x/G] = N_k(x/G) \cup \{x\}$. The k-degree of a vertex x in a graph G, $d_k(x/G)$ or $d_k(x)$, is $d_k(x/G) = |N_k(x/G)|$. For more definitions or symbols not indication here, see the book [2].
Recently, one of the interesting subjects for authors compute numerical calculations related to drug graphs [1, 3, 4]. This numerical calculations are called topological indices.
In 2018, Naji et al. [5] have been introduced the k-distance degree index or N_k-index as

$$N_k(G) = \sum_{k=1}^{diam(G)} \Big(\sum_{x \in V(G)} d_k(x) \Big) k.$$

*Speaker.

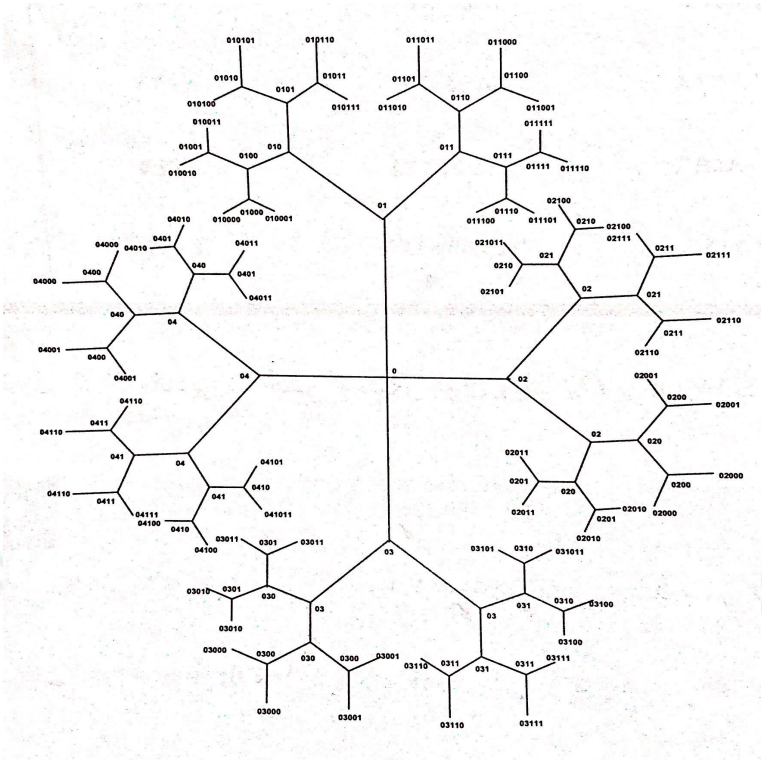

FIGURE 1. The graph of Dendrimer

2. Main results

Suppose that G is the graph of a Dendrimer on the figure 1. Dendrimers are tree (the connected graph without cycle) and we can rooted it. Let G is rooted at the central vertex and the other vertices are the levels of it, therefor we have 64 vertex of degree 1.

THEOREM 2.1. *Let G be a Dendrimer on the figure 1, then $N_k(G) = 113948$.*

PROOF. According to the figure 1, $|V(G)| = 125$, $|E(G)| = 124$ and $diam(G) = 10$. We calculate the k-distance degree index for $k = 1 \quad to \quad 10$.

$k = 1, \qquad \sum_{x\in V(G)} d_1(x)) = 248,$
$k = 2, \qquad \sum_{x\in V(G)} d_2(x)) = 372,$
$k = 3, \qquad \sum_{x\in V(G)} d_3(x)) = 496,$
$k = 4, \qquad \sum_{x\in V(G)} d_4(x)) = 752,$
$k = 5, \qquad \sum_{x\in V(G)} d_5(x)) = 928,$
$k = 6, \qquad \sum_{x\in V(G)} d_6(x)) = 1408,$
$k = 7, \qquad \sum_{x\in V(G)} d_7(x)) = 2068,$
$k = 8, \qquad \sum_{x\in V(G)} d_8(x)) = 2816,$
$k = 9, \qquad \sum_{x\in V(G)} d_9(x)) = 3072,$
$k = 10, \qquad \sum_{x\in V(G)} d_{10}(x)) = 3072.$

Therefor, $N_k(G) = \sum_{k=1}^{diam(G)} (\sum_{x\in V(G)} d_k(x))k = 113948.$

□

3. Conclusion

The computations in this paper are useful for pharmacists and we hope every one is healthy and safe and no one needs medication.

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Doubly resolving set of a class of graphs on dihedral group

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Abstract. A set S of vertices of graph Γ is a doubly resolving set for Γ, if every two distinct vertices of Γ are doubly resolved by some two vertices of S. The minimal doubly resolving set of vertices of graph Γ is a doubly resolving set with the minimum cardinality and is denoted by $\psi(\Gamma)$. In this paper, we consider the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$, where $\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ is the dihedral group of order $2n$ $(n \geq 4)$, and $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$, and we show that the minimum cardinality of a doubly resolving set of this class of graphs is n in some cases.

Keywords: Cayley graph, Resolving set, Doubly resolving set

AMS Mathematics Subject Classification [2020]: 05C12

1. Introduction

In this paper we consider finite, simple, and connected graphs. The vertex and edge sets of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The distance $d_\Gamma(u, v)$ between two vertices u and v in a connected graph Γ is the length of the shortest path from u to v, is denoted by $d_\Gamma(u, v)$, or simply $d(u, v)$. The size of the largest clique in the graph Γ is denoted by $\omega(\Gamma)$. A vertex $x \in V(\Gamma)$ is said to resolve a pair $u, v \in V(\Gamma)$ if $d_\Gamma(u, x) \neq d_\Gamma(v, x)$. For an ordered subset $S = \{s_1, s_2, ..., s_k\}$ of vertices in a connected graph Γ and a vertex v of Γ, the metric representation of v with respect to S is the k-vector $r(v|S) = (d(v, s_1), d(v, s_2), ..., d(v, s_k))$. If every pair of distinct vertices of Γ have different metric representations then the ordered set S is called a resolving set of Γ. Indeed, the set S is called a resolving set for Γ if $r(u|S) = r(v|S)$ implies that $u = v$ for all pairs u, v of vertices of Γ. If the set S is as small as possible, then it is called a metric basis of the graph Γ. We recall that the metric dimension of Γ, denoted by $dim(\Gamma)$ is defined as the minimum cardinality of a resolving set for Γ. If $dim(\Gamma) = k$, then Γ is said to be k-dimensional. Chartrand et. al. [3] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well known families of graphs such as trees, paths, and complete graphs. Bounds on $dim(\Gamma)$ are presented in

*Speaker.

terms of the order and the diameter of Γ. The concept of resolving set and that of metric dimension date back to the 1950s. They were defined by Bluementhal [1] in the context of metric space. These notions were introduced to graph theory by Harary and Melter [4] and Slater [6] in the 1970s. These concepts have different applications in the areas of network discovery and verification, robot navigation, and combinatorical optimization. Notice, for each connected graph Γ and each ordered set $S = \{s_1, s_2, ..., s_k\}$ of vertices of Γ, that the ith coordinate of $r(s_i|S)$ is 0 and that the ith coordinate of all other vertex representations is positive. Thus, certainly $r(u|S) = r(v|S)$ implies that $u = v$ for $u \in S$. Therefore, when testing whether an ordered subset S of $V(\Gamma)$ is a resolving set for Γ, we need only be concerned with the vertices of $V(\Gamma) - S$.

Cáceres et al. [2] define the notion of a doubly resolving set as follows. Vertices x, y of the graph Γ of order at least 2, are said to doubly resolve vertices u, v of Γ if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set $S = \{s_1, s_2, ..., s_l\}$ of vertices of Γ is a doubly resolving set of Γ if every two distinct vertices of Γ are doubly resolved by some two vertices of S. The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of minimum doubly resolving set is denoted by $\psi(\Gamma)$. The minimal doubly resolving sets for Hamming and Prism graphs has been obtained. Since if x, y doubly resolve u, v, then $d(u, x) - d(v, x) \neq 0$ or $d(u, y) - d(v, y) \neq 0$, and hence x or y resolve u, v. Therefore, a doubly resolving set is also a resolving set and $dim(\Gamma) \leq \psi(\Gamma)$.

If G is a finite group and a subset Ω of G is closed under taking inverses and does not contain the identity, then the Cayley graph $Cay(G, \Omega)$ is the graph with vertex set G and the edge set $E(Cay(G, \Omega)) = \{\{x, y\} \mid x^{-1}y \in \Omega\}$.

Let $\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ be the dihedral group of order $2n$ $(n \geq 4)$. The minimum cardinality of a doubly resolving set of the Cayley graph $Cay(\mathbb{D}_{2n}, \Psi)$, where $\mathbb{D}_{2n}$ is the dihedral group of order $2n$, and $\Psi = \{ab, a^2b, ..., a^{n-1}b, b\} \cup \{a^{\frac{n}{2}}\}$ is an inverse closed subset of $\mathbb{D}_{2n} - \{1\}$, is computed, see [5]. Now, let $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$. In this paper, we show that if n is an even integer and $(k = \frac{n}{2},\ m = \frac{n}{2} - 1\ )$, then the minimum cardinality of a doubly resolving set of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is n. Moreover, we show that if n is an odd integer and $(k = m = [\frac{n}{2}])$, then the minimum cardinality of a doubly resolving set of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is n.

2. Main results

Theorem 2.1. *Let $\Gamma = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n} (n \geq 4)$, where Ω_k and S_m which are defined as before. If n is an even integer and $(k = \frac{n}{2},\ m = \frac{n}{2} - 1\ )$, then the minimum cardinality of a doubly resolving set of Γ is n.*

Proof. Let $V(\Gamma) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of Γ. We know that, if n is an even integer and $k = \frac{n}{2}$, $m = \frac{n}{2} - 1$, then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\}$, and $S_m = \{a, a^2, ..., a^n\} - \{a^{\frac{n}{2}}, a^n\}$. Also, we can verify that the diameter of Γ is 2, and hence the distance of any two vertices of Γ is 1 or 2. We can verify that, any clique of size n is a minimal resolving set of Γ and hence $dim(\Gamma) = n$. Especially, it is well known that $dim(\Gamma) \leq \psi(\Gamma)$. Now, we show that any clique in Γ of size n is a doubly resolving set of Γ. For this purpose, let S be an arbitrary clique in Γ of the size n. It is sufficient to show that for two vertices u and v of Γ there are vertices $x, y \in S$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Consider two vertices u and v of Γ. In the following

cases, we show that the minimum cardinality of a doubly resolving set of Γ is n.

Case 1. If $u, v \in S$, then $d(u, v) = 1$. Now, by considering $x = u \in S$, and $y = v \in S$, we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$, because $d(u, x) - d(u, y) < 0$ and $d(v, x) - d(v, y) > 0$.

Case 2. Let $u \in S$ and $v \notin S$, hence $d(u, v) = 1$ or $d(u, v) = 2$. If $d(u, v) = 1$, then there are $x, y \in S$ so that u is adjacent to x, y and v is adjacent to exactly one of two elements x, y. Hence $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Now, let $d(u, v) = 2$ then by considering $x = u \in S$, and $u \neq y \in S$ we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 3. Finally, let $u \notin S$ and $v \notin S$, hence $d(u, v) = 1$. On the other hand, we know that for every vertex $u \in V(\Gamma) - S$, there is exactly one $x \in S$ such that $d(u, x) = 2$. Also for every vertex $v \in V(\Gamma) - S$, there is exactly one $y \in S$ such that $d(v, y) = 2$. Hence, $1 = 2 - 1 = d(u, x) - d(u, y) \neq d(v, x) - d(v, y) = 1 - 2 = -1$.

From the above cases, we conclude that the minimum cardinality of a doubly resolving set of Γ is n.

□

Theorem 2.2. *Let $\Gamma = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n} (n \geq 4)$, where Ω_k and S_m which are defined as before. If n is an odd integer and $(k = m = [\frac{n}{2}])$, then the minimum cardinality of a doubly resolving set of Γ is n.*

Proof. Let $V(\Gamma) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of Γ. We know that, if n is an odd integer and $k = m = [\frac{n}{2}]$ then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\} - \{a^kb\}$, and $S_m = \{a, a^2, ..., a^{n-1}\}$. Now, by similar way is done in the proof of Theorem 2.1, we can show that the minimum cardinality of a doubly resolving set of Γ is n. □

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Existence and fundamental properties of dominating induced matchings

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Abstract. A dominating induced matching is a subset of edges in a graph such that every edge is either in this subset or shares exactly one endpoint with an edge of it. The concept lies at the intersection of domination and matching theory. This paper introduces the fundamental notions related to dominating induced matchings, explores their existence in several graph classes. Also, we investigate the relation between the size and order of a graph with dominating induced matching number.

Keywords: domination, induced matching, dominating induced matching

AMS Mathematics Subject Classification [2020]: 05C07, 05C12

1. Introduction

We consider finite, simple, and undirected graphs. For a graph $G = (V, E)$, the order and size are denoted by $n = |V|$ and $m = |E|$, respectively. For x, $y \in V(G)$ with $x \neq y$, $\mathrm{d(x, y)}$ denotes the length of a shortest path from x to y. A graph G is called *connected* if there is a path between each pair x and y in $V(G)$. The degree of a vertex v is denoted by $d(v)$, with $\delta(G)$ and $\Delta(G)$ representing the minimum and maximum degrees.

We write K_n for the *complete graph*, P_n for a path and C_n for a *cycle* of order n. A generalized Petersen graph $P(n, k)$ is a graph with vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i = 0, \ldots, n-1\}$ where subscripts are to be read modulo n and $k < n/2$. A set D of vertices in a graph G is called a *dominating set* if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A subset $D \subseteq E$ is an *edge dominating set* if each edge in E is either in D or is adjacent to an edge in D. An edge dominating set D is called a *minimal edge dominating set* if no proper subset F of D is an edge dominating set [5].

A subset $M \subseteq E(G)$ is a *matching* if no two edges in M share a common endpoint. A matching is *maximal* if it cannot be enlarged by adding another edge, and *maximum* if it has the largest possible size, denoted by $\nu(G)$. A graph has a *perfect matching* if $\nu(G) = n/2$.

A *dominating induced matching* (DIM) in G is a matching M satisfying the following two conditions:

(1) M is *induced*: no two edges of M are joined by another edge of G;

(2) every edge of G is either in M or adjacent to exactly one edge of M.

If a graph G admits at least one DIM, then all such matchings have the same cardinality [1]. This common size is called the *dominating induced matching number* of G, written $\dim(G)$.

Domination and matching are two of the most central concepts in graph theory. The interplay of these ideas has led to a wide spectrum of parameters with combinatorial and algorithmic significance. A dominating induced matching (DIM) combines these two notions. Roughly speaking, a DIM represents a matching that dominates all edges of the graph [4]. In this paper, we investigate some basic properties of dominating induced matchings. First we consider the problem of their existence and construct the graphs with a prescribed DIM size. Then we give the relation between the size and order of a graph with $\dim(G)$.

2. Main Results

Not all graphs possess a DIM. For instance, if every edge of a graph lies on a 4-cycle, the graph cannot have a DIM. This observation excludes entire families such as prisms and hypercubes [2]. On the other hand, paths P_n and cycles C_{3k} are classical examples of graphs with DIMs:

$$\dim(P_n) = \left\lceil \frac{n-1}{3} \right\rceil, \qquad \dim(C_{3k}) = k.$$

The following result, due to Akbari *et al.* [1], establishes the uniformity of DIM size.

Theorem 2.1. *Let G be a graph. If D_1 and D_2 are two DIMs of G, then $|D_1| = |D_2|$.*

In the following lemma consider $P(n,2)$, for $n \overset{5}{\equiv} 0$.

Lemma 2.2. *For the Petersen graph $P(n,2)$, $n \overset{5}{\equiv} 0$, we have* $\dim(\mathrm{P(n,2)}) = \dfrac{3\mathrm{n}}{5}$

Proof. Let $n = 5t$, $t \geq 1$. Index the inner and outer vertices of the Petersen graph $P(n,2)$ by $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ respectively and select the edges $\{v_i v_{i+1} \mid i \overset{5}{\equiv} 3\}$, $\{v_i u_i \mid i \overset{5}{\equiv} 1\}$ and $\{u_i u_{i+2} \mid i \overset{5}{\equiv} 0\}$. It is easy to see that the selected edges are a DIM for $P(n,2)$, $n \overset{5}{\equiv} 0$. This verification in conjunction with Theorem 2.1 yields the assertion. □

We construct the graphs with a prescribed DIM size.

Lemma 2.3. *For any integers n and k satisfying $1 \leq k < n/2$, there exists a connected graph G of order n such that* $\dim(G) = k$.

Proof. Take a matching A of k independent edges and an independent set B of size $n - 2k$. Connect each vertex of B to endpoints of edges in A so that the resulting graph is connected. By construction, the edges of A form a dominating induced matching. □

Some of the well-known graphs such as complete graphs except K_3, complete bipartite graphs and wheel graphs do not have DIM. Therefore, a question arises here that how many edges are needed to be removed such that the resulting graph contains a DIM. In the following theorem we investigate this problem for complete graphs.

Theorem 2.4. *The least number of edges needed to be removed from a complete graph K_n so that it has a DIM of size k is $\dfrac{n(n-1)}{2} - (k(2n-4k+1))$. Moreover, if $k = \lfloor \frac{n}{4} \rfloor$, then the minimum number of edges need to be deleted.*

Proof. Let us put the DIM vertices in one side and the non-DIM vertices in other side and enumerate the edges. There are at most $2k(n-2k)+k$ edges. Therefore, the least number of edges needed to be deleted from a complete graph so that the remaining graph has a DIM of size k is

$$\frac{n(n-1)}{2} - (2k(n-2k)+k).$$

For the last part, we need to maximize $f(k) = 2k(n-2k)+k$. The maximum occurs at $k = \frac{2n+1}{8} = \lfloor \frac{n}{4} \rfloor$ which is a minimum for the total statement. □

Here we investigate the relation between the size and order of graph with dim(G). For every $e \in E(G)$, let E_e be the set of edges, which dominates by e. If G has a DIM, then

$$|E(G)| = |\cup_{e \in DIM} E_e| = \sum_{e \in DIM} |E_e| \geq (2\delta - 1)\mathrm{dim(G)}.$$

So we have the following result.

Lemma 2.5. *[3] Let G be a graph of order n with a DIM. Then* $\mathrm{dim(G)} \leq \frac{|\mathrm{E(G)}|}{2\delta-1}$. *Moreover, if G is a k-regular graph, then* $\mathrm{dim(G)} = \frac{\mathrm{nk}}{\mathrm{4k-2}}$.

Lemma 2.5 helps us to characterize the graphs which have or do not have a DIM. Consider healthy spider S_t, $t \geq 1$, with vertex set $\{v, v_1, v_2, \ldots, v_t, v'_1, \ldots, v'_t\}$. Let F be the family of all graphs G with vertex set $\{v, v_1, v_2, \ldots, v_t, v'_1, \ldots, v'_t\}$ and $E(G) = E(S_t) \cup \{vv'_i | \text{for some } i, 1 \leq i \leq t\}$.

Theorem 2.6. *Let G be a connected graph of order n with a DIM. The following statements hold:*

(1) $\mathrm{dim(G)} = \lfloor \frac{\mathrm{n}}{2} \rfloor$ *if and only if n is odd and G is a healthy spider or* $K_1 \vee \cup_{i=1}^{\lfloor \frac{n}{2} \rfloor} K_2$ *or* $G \in \mathrm{F}$.

(2) $\mathrm{dim(G)} = 1$ *if and only if G is a star or a double star or one of the following graphs:*

a) *an edge $e = xy$ for which x and y have some common neighbors, or*

b) *an edge $e = xy$ for which x and y are support vertices and have some common neighbors.*

Proof. Suppose that $\mathrm{dim(G)} = \lfloor \frac{\mathrm{n}}{2} \rfloor$ and let $DIM = \{e_1, e_2, \ldots, e_{\lfloor \frac{n}{2} \rfloor}\}$. Since there is no edge between the e_i, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, and G is a connected graph, thus n is odd. Without loss of generality set $DIM = \{e_1 = v_1v_2, e_2 = v_3v_4, \ldots, e_{\lfloor \frac{n}{2} \rfloor} = v_{n-2}v_{n-1}\}$. Hence $v_n \notin \{v_1, v_2, \ldots, v_{n-1}\}$. Three cases may be happen. Assume first that v_n is adjacent to each v_i, $1 \leq i \leq n-1$, then G is $K_1 \vee \cup_{i=1}^{\lfloor \frac{n}{2} \rfloor} K_2$. So without loss of generality, suppose that v_n is adjacent to v_i, $1 \leq i \leq n-1$ and i is odd, then G is a healthy spider. Finally, it is possible that v_n is adjacent to two ends of some DIM's edges and also is adjacent to one end of some DIM's edges. Therefore $G \in \mathrm{F}$. This completes the proof of (1) since the converse assertion in (1) is obvious. (2) is straightforward. □

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Some relations between dominating induced matchings and other graph parameters

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Abstract. Dominating induced matchings play a central role in bridging several well-known graph parameters including domination, independence, and edge domination. In this paper, we explore quantitative relations among these parameters and attempt to relate the induced matching number of G to the independence number of it's line graph.

Keywords: dominating induced matching, domination number, line graph

AMS Mathematics Subject Classification [2020]: 05C07, 05C12

1. Introduction

The theory of graph domination and matching continues to reveal deep interconnections between apparently distinct combinatorial parameters. Among them, the *dominating induced matching* (DIM) forms a particularly interesting hybrid structure. A DIM requires both disjointness (as in matchings) and local coverage (as in domination), making it an ideal object for analyzing the interaction of adjacency restrictions.

We consider simple graph G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* and *size* of a graph G are denoted by $n = n(G)$ and $m = m(G)$, respectively. For x, $y \in V(G)$ with $x \neq y$, $\mathrm{d(x,y)}$ denotes the length of a shortest path from x to y. A graph G is called *connected* if there is a path between each pair x and y in $V(G)$. The *degree* of a vertex $v \in V$ is $d_G(v) = d(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta_G$ and $\Delta = \Delta_G$, respectively. We write K_n for the *complete graph*, P_n for a path and C_n for a *cycle* of order n.

A set D of vertices in a graph G is called a *dominating set* if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A set D of vertices in a graph G is called a *total dominating set* if every vertex $v \in V(G)$ is adjacent to an element of D. The *domination number* of a graph G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in G. Respectively, the *total domination number* of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G.

An *independent set* is a subset of vertices in a graph such that no two vertices in it are adjacent. A *maximum independent set* is an independent set of largest possible size for a given graph G. This size is called the *independence number* of G, and denoted by $\alpha = \alpha(G)$. An *independent dominating* set is a set that is both dominating and

independent. The *independent domination number* of G, denoted by $i(G)$, is the minimum size of an independent dominating set [**?**].

A subset $D \subseteq E$ is an *edge dominating set* if each edge in E is either in D or is adjacent to an edge in D. An edge dominating set D is called a *minimal edge dominating set* if no proper subset F of D is an edge dominating set. The *edge domination number* $\gamma'(G)$ is the minimum cardinality among all minimal edge dominating sets.

A matching M is a *dominating induced matching* (DIM) of a graph if every edge is either in M or has a common end-vertex with exactly one edge in M. The number of members of the two dominating induced matchings of G is the same [**?**]. The dominating induced matching number of G, denoted by dim(G), is the size of a dominating induced matching of G. In this paper, we study dominating induced matching with combinatorial aspects and investigate the relations between it and other parameters of graph.

2. Main Results

In this paper, we attempt to relate the DIM to other parameters of a graph G. The *line graph* $L(G)$ has as vertices the edges of G, and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G are incident. Hence, edge properties of G often translate naturally into vertex properties of $L(G)$. At first, we prove that the induced matching number of G is equal to the independence number of $L(G)$ and $\gamma'(G)$.

THEOREM 2.1. *Let G be a graph and $S \subseteq E$ a DIM for G. Then the corresponding vertices of S form a minimum independent dominating set in $L(G)$.*

PROOF. Assume first that $S = \{e_1, \ldots, e_t\}$, $t = \dim(\mathrm{G})$. Since S is a DIM, e_i's are independent in $L(G)$ and every arbitrary $e \in E(G) \setminus S$ is dominated by exactly one vertex e_i, $1 \leq i \leq t$, in $L(G)$. So S is an independent dominating set in $L(G)$. Now, we show the minimality of S. Assume that $A \subseteq V(L(G))$ be an independent dominating set such that $|A| < |S|$. Let $A' = A \setminus S$ and $S' = S \setminus A$. Since S is a DIM for G, $|A'| = |A \setminus S| \geq 1$. On the other hand, we have

$$|S'| = |S \setminus A| = |S| - |A \cap S| > |A| - |A \cap S| = |A'|.$$

Therefore, there exist at least two vertices e_p and e_k of S' which are dominated by a vertex e_j from A', which is a contradiction. □

LEMMA 2.2. *Let G be a graph. Then there exists a $\gamma'(G)$-set with no edges of it incident to each other.*

PROOF. Suppose that S is a $\gamma'(G)$-set with e_1 and e_2 incident to each other in S. According to the minimality of S, e_1 has an incident edge e_3 which is not incident to e_2. Then $(S \setminus \{e_1\}) \cup \{e_3\}$ is a $\gamma'(G)$-set. Repeating the process, we obtain a $\gamma'(G)$-set with the desired property since each of the edges in S has an incident edge which does not meet any other edges of S. □

THEOREM 2.3. *Let G be a graph. Then $i(L(G)) = \gamma'(G)$.*

PROOF. One has $\gamma'(G) \leq i(L(G))$ since every independent dominating set for $L(G)$ is an edge dominating set for G. Assume that S is a $\gamma'(G)$-set. By Lemma **??**, one can assume that no edges of S incident to each other. Thus the corresponding vertices of the elements of S in $L(G)$ form an independent dominating set for $L(G)$. Hence, $i(L(G)) \leq \gamma'(G)$ and we are done. □

We can state now the following main theorem.

Corollary 2.4. *Let G be a graph with a DIM. Then*

$$i(L(G)) = \mathrm{dim(G)} = \gamma'(\mathrm{G}).$$

Proof. It follows immediately that each DIM for a graph G is an edge dominating set for it and so $\gamma'(G) \leq \mathrm{dim(G)}$. By Theorems **??** and **??**, one has

$$\mathrm{dim(G)} \leq \mathrm{i(L(G))} = \gamma'(\mathrm{G}) \leq \mathrm{dim(G)}.$$

□

Using Corollary **??**, we can state the following result. The lower bound arises from the necessity of covering all edges through disjoint dominating units, while the upper bound results from the independence condition in $L(G)$.

Corollary 2.5. *Let G be a graph with a DIM. Then*

$$\frac{n - \alpha(G)}{2} \leq \mathrm{dim(G)} \leq \alpha(\mathrm{L(G)}).$$

The following proposition establishes a bridge between three major domination-related parameters.

Proposition 2.6. *Let G be a graph with a DIM. Then $\gamma(G) \leq \gamma_t(G) \leq 2\mathrm{dim(G)}$.*

Proof. The vertices corresponding to DIM's edges are total dominating set for G. So the assertion holds. □

The distance between $2\mathrm{dim(G)}$ and $\gamma_t(G)$ can be arbitrarily high or low. Consider the healthy spider S_t with t leaves. Then $2\mathrm{dim(G)} - \gamma_\mathrm{t}(\mathrm{G}) = \mathrm{t} - 1$. It shows that $2\mathrm{dim(G)} - \gamma_\mathrm{t}(\mathrm{G})$ can be arbitrary large. Also, it is possible that $2\mathrm{dim(G)} = \gamma_\mathrm{t}(\mathrm{G})$. To see this, consider P_{3k+1}, where k is a positive integer.

Theorem 2.7. *[?] If G does not have an induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$.*

Proposition 2.8. *[?] Let G be a graph such that G and $L(G)$ have a DIM. Then $\mathrm{dim(G)} \leq 2\mathrm{dim(L(G))}$.*

Proposition 2.9. *[?] Let G be a graph with no isolated vertices which has a DIM. Then $n \leq \beta + \beta' \leq n + \mathrm{dim(G)}$, the lower bound is sharp.*

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The relationship between the Sombor index and other topological indices

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Abstract. Gutman recently introduced a new vertex-degree-based topological index called the Sombor index. It is a vertex-degree-based topological index and is denoted by Sombor index, $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_u is the degree of vertex u in G. In this paper, we obtain some result on the bounds of the Sombor index in terms of some other topological indices.

Keywords: Sombor index, first Zagreb index, forgotten index, sum-connectivity index, collaboration index

AMS Mathematics Subject Classification [2020]: 05C05, 05C09

1. Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The degree of a vertex $u \in V(G)$ is denoted by d_u and it is defined as the number of edges incident with u and the meaning of size and order is equal to $|E(G)| = m$ and $|V(G)| = n$, respectively. The Δ and δ stand for the maximum and minimum vertex degree of graph G, respectively. Recently, Gutman devised a topological index under the name Sombor index. For a graph G, its Sombor index is defined [2] as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

The first Zagreb index $M_1(G)$ can be also represented [6] as follows

$$M_1(G) = \sum_{uv \in E(G)} d_u + d_v.$$

For a graph G, its forgotten topological index can be expressed [1] as

$$F(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2).$$

*Speaker.

Note that because of the identity $\sum_{uv\in E(G)} d_u^\alpha + d_v^\alpha = \sum_{u\in V(G)} d_u^{\alpha+1}$, the first Zagreb, forgotten indices are equal to $\sum_{u\in V(G)} d_u^2$ and $F_3(G) = \sum_{u\in V(G)} d_u^3$, respectively. The sum-connectivity index $\chi(G)$ [2] and the collaboration index [4] are defined as follows:

$$\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u + d_v}} \quad , \quad CI(G) = \sum_{uv\in E(G)} \frac{1}{d_u + d_v}.$$

For any graph G of size m, the inverse sum indeg index ISI defined as

$$ISI(G) = \sum_{uv\in E(G)} \frac{1}{\frac{1}{d_u} + \frac{1}{d_v}} = \sum_{uv\in E(G)} \frac{d_u d_v}{d_u + d_v}$$

2. Main results

In this section, we establish some new upper and lower bounds for Sombor index of a connected graph G in terms of other topological indices.

2.1. Upper bounds of $SO(G)$.

Lemma 2.1. *[5]. Let $p = (p_i)$, $i = 1, 2, \cdots, n$, be nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \cdots, n$, positive real number sequence. Then for any real r, such that $r \leq 0$ or $r \geq 1$, holds*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r. \qquad (1)$$

If $0 \leq r \leq 1$, then the sense of (1) reverses. Equality holds if and only if either $r = 0$, or $r = 1$, or $p_1 = p_2 = \cdots = p_t = 0$, $a_{t+1} = \cdots = a_n$, for some t, $1 \leq t \leq n-1$ or $a_1 = a_2 = \cdots = a_n$.

Lemma 2.2. *(Cauchy-Schwarz inequality) Let a_k, b_k be real numbers for all $k = 1, 2, \cdots, m$ and $w_k \geq 0$. Then*

$$\left(\sum_{k=1}^{m} w_k a_k b_k\right)^2 \leq \left(\sum_{k=1}^{m} w_i a_k^2\right)\left(\sum_{k=1}^{m} w_k b_k^2\right).$$

Equality holds if and only if $k = 1, 2, \cdots, m$, there exists $\lambda \in \mathbb{R}$ such that $b_k = \lambda a_k$

Theorem 2.3. *Let G be a graph with m edges, then*

$$SO(G) \leq \sqrt{m \frac{\Delta}{\delta} F(G)}$$

.

Proof. Letting $a_i = \frac{\sqrt{d_u^2 + d_v^2}}{d_u}$ and $p_i = d_u$ in Lemma 2.1, with the sums running over the edges in G and $r = 2$, we have,

$$\left(\sum_{uv\in E(G)} d_u . \frac{\sqrt{d_u^2 + d_v^2}}{d_u}\right)^2 \leq \left(\sum_{uv\in E(G)} d_u\right) \sum_{uv\in E(G)} d_u . \left(\frac{\sqrt{d_u^2 + d_v^2}}{d_u}\right)^2.$$

Then

$$\left(\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}\right)^2 \leq \left(\sum_{uv\in E(G)} d_u\right)\sum_{uv\in E(G)}\frac{d_u^2+d_v^2}{d_u}$$

Therefore $SO^2(G) \leq \frac{m\Delta}{\delta}F(G)$, hence $SO(G) \leq \sqrt{\frac{m\Delta}{\delta}F(G)}$. □

THEOREM 2.4. *Let G be a graph with maximum degree Δ and minimum degree δ. Then*

$$SO(G) \leq \sqrt{(F_4(G)+\Delta^2 M_1(G))+CI(G)}.$$

PROOF. Letting $a_k = \sqrt{d_u^2+d_v^2}$ and $b_k = \frac{1}{d_u+d_v}$ in Lemma 2.2 and $w_k = d_u+d_v$, we have,

$$\left(\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}\right)^2 \leq \sum_{uv\in E(G)}(d_u+d_v)(d_u^2+d_v^2).\sum_{uv\in E(G)}\frac{d_u+d_v}{(d_u+d_v)^2}.$$

Then

$$\left(\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}\right)^2 \leq \sum_{uv\in E(G)}(d_u^3+d_v^3+d_u^2d_v+d_ud_v^2)\sum_{uv\in E(G)}\frac{1}{d_u+d_v}$$

Hence

$$\left(\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}\right)^2 \leq \left(\sum_{u\in V(G)} d_u^4+\sum_{uv\in E(G)} d_ud_v(d_u+dv)\right)\sum_{uv\in E(G)}\frac{1}{d_u+d_v}.$$

Note that $\sum_{uv\in E(G)} d_u^2d_v+d_ud_v^2 = \sum_{uv\in E(G)} d_ud_v(d_u+d_v) \leq \Delta^2 M_1(G)$, then

$$SO(G) \leq \sqrt{(F_4(G)+\Delta^2 M_1(G))+CI(G)}.$$

□

THEOREM 2.5. *Let G be a graph with m edges, then*

$$SO(G) \leq 2\Delta\sqrt{\chi(G).\Delta}.$$

PROOF. Letting $a_i = \sqrt{d_u^2+d_v^2}$ and $p_i = \frac{1}{d_u+d_v}$ in Lemma 2.1 and $r=2$, we have,

$$\left(\sum_{uv\in E(G)}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)^2 \leq \sum_{uv\in E(G)}\frac{1}{d_u+d_v}\sum_{uv\in E(G)}\frac{d_u^2+d_v^2}{d_u+d_v}.$$

Hence $\frac{1}{4\Delta^2}SO^2(G) \leq \chi(G)$, then $SO(G) \leq 2\Delta\sqrt{\chi(G).\Delta}$. □

2.2. Lower bounds of $SO(G)$.

LEMMA 2.6. *Let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers. Then*

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_1}+\cdots+\frac{1}{x_n}} \leq \sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1+x_2+\cdots+x_n}{n}$$

and equality is attained if $x_1 = x_2 = \cdots = x_n$.

THEOREM 2.7. *Let G be a graph on n vertices and m edges. Then*

$$\frac{\sqrt{2}}{2}M_1(G) \leq SO(G)$$

Moreover, the equality holds if and only if G is a regular graph.

PROOF. By Lemma 2.6, we have $\frac{d_u + d_v}{2} \leq \sqrt{\frac{d_u^2 + d_v^2}{2}}$.

Hence $\frac{1}{2}\sum_{uv\in E(G)} d_u + d_v \leq \frac{1}{\sqrt{2}}\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, so $\frac{\sqrt{2}}{2}M_1(G) \leq SO(G)$. □

THEOREM 2.8. *Let G be a graph with m edges, then*

$$\sqrt{8}ISI(G) \leq SO(G).$$

Moreover, the equality holds if and only if G is a regular graph.

PROOF. By Lemma 2.6, we have $\frac{2}{\frac{1}{d_u} + \frac{1}{d_v}} \leq \sqrt{\frac{d_u^2 + d_v^2}{2}}$.

Therefore $2\sum_{uv\in E(G)} \frac{d_u d_v}{d_u + d_v} \leq \frac{1}{\sqrt{2}}\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, hence $\sqrt{8}ISI(G) \leq SO(G)$. □

3. Conclusion

In this article, we found some bounds of the Sombor index in terms of order, size, and other topological indices. In particular, when the graph G is regular, the equality case is obtained. The results obtained in this article demonstrate that by employing various types of inequalities, different bounds for this index can be achieved.

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A GENERALIZATION OF INJECTIVE COLORING

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Abstract. An injective coloring of a given graph $G = (V, E)$ is a vertex coloring of G such that any two vertices with a common neighbor receive distinct colors. A 2-distance coloring of G is a proper vertex coloring of the graph G such that any pair of vertices at a distance of at most 2 receive different colors. A 2-distance injective coloring of a graph G is a vertex coloring in which any two vertices that share a common neighbor are assigned different colors, and any two vertices that share a common 2-distance neighbor are also assigned different colors. Equivalently, for every path $P_3 = ywz$ or $P_5 = xywzt$ in G, the vertices x, t, and w receive pairwise distinct colors, and y and z are assigned different colors. This new definition provides a generalization of the injective coloring and allows us to revisit the concept from a new perspective. We also investigate the 2-distance injective coloring of several families of graphs.

Keywords: Graph coloring, 2-distance coloring, injective coloring, 2-distance injective coloring

AMS Mathematics Subject Classification [2020]: 05C15

1. Introduction

There are several results related to injective coloring that review the usual coloring results in graph theory from a new point of view, in particular on the injective chromatic number of planar graphs [2–4, 6]. Furthermore, a number of significant conjectures concerning the injective coloring of planar graphs have been proposed and examined by various authors. In this context, we present one of them below.
In [3], Lužar *et al.* posed the following conjecture.

Conjecture [3] Let G be a planar graph with maximum degree Δ. Then,

(i) $\chi_i(G) \leq 5$ if $\Delta = 3$,
(ii) $\chi_i(G) \leq \Delta + 5$ if $4 \leq \Delta \leq 7$,
(iii) $\chi_i(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ otherwise.

*Speaker.

We believe that the idea of 2-distance injective coloring could shed new light on some well-known conjectures and may have potential applications, for instance, in coding theory; however, a detailed investigation of these aspects is beyond the scope of this paper.

The primary motivation for this generalization is to explore how 2-distance injective coloring is relates to injective coloring, usual coloring, and 2-distance coloring; which are discussed in the next section. At present, we formally define this concept.

Definition 1.1. A 2-*distance injective coloring* using k colors of a graph G is a function $f : V(G) \to \{1, 2, 3, \ldots, k\}$ such that any pair of vertices with a common neighbor receive different colors, and any two vertices with a common 2-distance neighbor also receive different colors. In other words, for a path $P_3 = ywz$ or a path $P_5 = xywzt$ in the graph, $f(x)$, $f(t)$, and $f(w)$ are mutually distinct, and $f(y) \neq f(z)$.

For a 2-distance injective k-coloring function f, the set of color classes $\{v \in V(G) \mid f(v) = i,\ 1 \leq i \leq k\}$ forms a 2-distance injective k-coloring class of G with color i. The smallest k such that G has a 2-distance injective k-coloring (or simply a 2-distance injective coloring if k is clear from the context) is called the 2-distance injective chromatic number, $\chi_{2i}(G)$, of G. It is worth noting that this type of coloring is not necessarily a proper coloring.

2. Basic results

Theorem 2.1. *If G is a simple graph with maximum degree Δ and maximum 2-distance degree $\Delta^{(2)}$, then:*

$$\Delta^{(2)} + 1 \leq \chi_{2i}(G) \leq \Delta(\Delta - 1)[1 + (\Delta - 1)^2] + 1.$$

These bounds are sharp.

Theorem 2.2. *For any tree T of order at least three,*

$$\chi_{2i}(T) = \Delta^{(2)} + 1.$$

3. Comparing 2-distance injective coloring with proper coloring, and 2-distance coloring

Proposition 3.1. *Let G be a graph where any three vertices on a path P_3 form a triangle, or any two of them are the endpoints of a path P_5. Then $\chi_2(G) \leq \chi_{2i}(G)$.*

Conversely, if the endpoints of every path P_5 in G are adjacent or share a common neighbor, then $\chi_{2i}(G) \leq \chi_2(G)$.

If G is a graph of order n and diameter at most 2, then $G^2 = K_n$. Thus, $\chi_2(G) = n$, and $\chi_2(G) \geq \chi_{2i}(G)$. This yields the following corollary:

Corollary 3.2. *For graphs with diameter at most 2, we have: $\chi(G^2) = \chi_2(G) \geq \chi_{2i}(G) = \chi(G^{2i})$.*

We state a result establishing conditions under which $\chi_2(G)$ and $\chi_{2i}(G)$ are equal.

Theorem 3.3. *Suppose G is a graph of order n and diameter 2. Then $\chi_2(G) = \chi_{2i}(G)$ if and only if every edge of G lies on a cycle C_3 or C_5.*

For graphs G with diameter at least three, we cannot directly compare $\chi_{2i}(G)$ and $\chi_2(G)$. In other words, we may find families of graphs G with diameter at least 3 for which $\chi_2(G) > \chi_{2i}(G)$, families of graphs for which $\chi_{2i}(G) > \chi_2(G)$ and families of graphs for which $\chi_{2i}(G) = \chi_2(G)$. Therefore, these two concepts ($\chi_{2i}(G)$ and $\chi_2(G)$) are generally not comparable.

Lemma 3.4. ([5] Proposition 10) *For any tree T, $\chi_2(T) = \Delta(T) + 1$.*

Example 3.5. Consider the trees in Figure 1, which have a diameter of 4. It is clear that, for the family of trees in Figure 1, $\chi_{2i}(T) = \Delta^{(2)}(T) + 1$.

Now, from Lemma 3.4, we have:

For the family of trees T in (I), $\Delta(T) > \Delta^{(2)}(T)$, so we have $\chi_2(T) > \chi_{2i}(T)$.

For the family of trees T in (II), $\Delta(T) = \Delta^{(2)}(T)$, so we have $\chi_2(T) = \chi_{2i}(T)$.

For the family of trees T in (III), $\Delta(T) < \Delta^{(2)}(T)$, so we have $\chi_2(T) < \chi_{2i}(T)$.

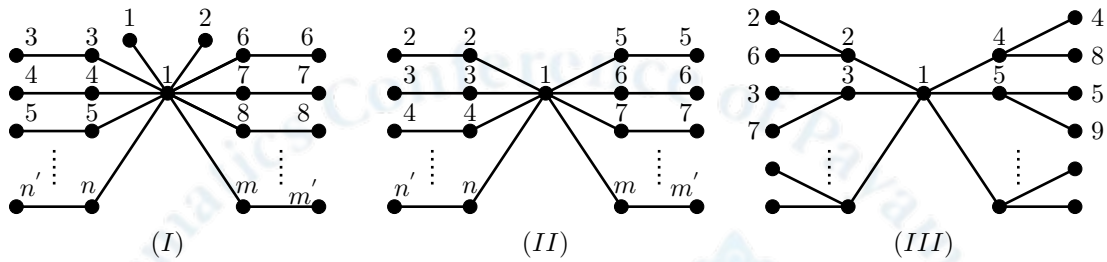

Figure 1. The 2-distance injective coloring of graphs G related to Example 3.5 for diameter 4

Lemma 3.6. ([1]) *Let G be a connected graph distinct from K_2. Then $\chi(G) \leq \chi_i(G)$.*

As an immediate consequence of Lemma 3.6, we have the following corollary.

Corollary 3.7. $\chi(G) \leq \chi_i(G) \leq \chi_{2i}(G)$.

The inequalities in Corollary 3.7 may or may not hold as equalities. Consider the following examples:

- For a complete graph K_n $(n \neq 2)$, we have $\chi(G) = \chi_i(G) = \chi_{2i}(G) = n$.
- For an odd cycle graph C_n, where $n \not\equiv 0 \pmod 3$, we have $\chi(C_n) = \chi_i(C_n) = 3 < 4 \leq \chi_{2i}(C_n)$.
- If $G = K_4 - e$ (a complete graph on four vertices with one edge removed), then $\chi(G) = 3 < 4 = \chi_i(G) = \chi_{2i}(G)$.
- For a spider graph G with $2n + 1$ vertices, obtained by subdividing one edge of a star S_{n+1} $(n \geq 3)$, we have $\chi(G) = 2 < n = \chi_i(G) < n + 1 = \chi_{2i}(G)$.

Proposition 3.8. *Let G be a graph where any two adjacent vertices are included in either a triangle or the endpoints of a path P_5. Then $\chi(G) \leq \chi_{2i}(G)$.*

Conversely, if the endpoints of every path P_3 or P_5 in G are adjacent, then $\chi_{2i}(G) \leq \chi(G)$.

4. 2-distance injective coloring of some standard graphs

Proposition 4.1. *For the path P_n and the cycle C_n, we have*

$$(1)\ \chi_{2i}(P_n) = \begin{cases} 1 & n = 2 \\ 2 & n = 3, 4 \\ 3 & n \geq 5 \end{cases}.$$

And for a positive integer k,

$$(2)\ \chi_{2i}(C_n) = \begin{cases} 3 & n = 3k \\ 2 & n = 4 \\ 4 & n \in \{6k+1, 6k+2, 6k+4, 6k+5\},\ n \neq 10 \\ 5 & n \in \{5, 10\} \end{cases}.$$

We discuss the 2-distance injective coloring of a family of Cartesian products of two paths (ladder and grid graphs) and determine their 2-distance injective chromatic number.

Let $G_{m,n}$ represent the grid graph $P_m \square P_n$ $(m, n > 2)$, with mn vertices and $2mn - m - n$ edges.

A ladder graph L_n is a connected special grid graph $P_2 \square P_n$ with $2n$ vertices and $3n-2$ edges, where $V(L_n) = \{v_{ij} : i = 1, 2 \ \textit{and}\ 1 \leq j \leq n\}$.

Proposition 4.2. *Let $G \cong L_n$ be a ladder graph. Then*

$$\chi_{2i}(L_n) = \begin{cases} 3 = \Delta^{(2)}(L_3) + 1 & \text{if } n = 3 \\ 4 = \Delta^{(2)}(L_4) + 1 & \text{if } n = 4 \\ 5 = \Delta^{(2)}(L_n) + 1 & \text{if } n \geq 5 \end{cases}.$$

Theorem 4.3. *Let $G_{m,n}$ be a grid graph with $m, n \geq 2$. Then*

$$\chi_{2i}(G_{m,n}) = \begin{cases} 9 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m, n \geq 5 \\ 8 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 4, n \geq 5 \\ 7 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n \geq 5 \\ 7 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 4, n = 4 \\ 6 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n = 4 \\ 5 = \Delta^{(2)}(G_{m,n}) + 1 & \text{if } m = 3, n = 3 \end{cases}.$$

5. Conclusion

The introduced concept of 2-distance injective coloring provides a unified framework that bridges injective and 2-distance colorings, offering new insights into their structural and combinatorial relationships.

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Some more properties of the graph $SC^*(M)$

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Abstract. In this paper, we investigate some more results of the comaximal colon ideal graph $\mathrm{C}^*(M)$. Moreover, we introduced and studied a supergraph of the comaximal colon ideal graph for a commutative ring R, denoted by $\mathrm{SC}^*(R)$ and for an R-module M, $\mathrm{SC}^*(M)$. In addition, we study the interplay between the graph-theoretic properties and the corresponding algebraic structures.

Keywords: Colon ideal, Comaximal ideals, Connected graph

AMS Mathematics Subject Classification [2020]: 13A15, 13F05, 05C75

1. Introduction

In this document, all rings are assumed to be commutative with a non-zero identity, and all modules are assumed to be unitary. A *local module* is a module with exactly one maximal submodule. Also, a *local ring* is a ring (not necessarily Noetherian) with exactly one maximal ideal. A *semi-local module* is a module with a finite number of maximal submodules. A ring R is a *semi-local ring* whenever it is a semi-local module as an R-module. An R-module M is named *prime* if for all non-zero submodules X of M, $\mathrm{ann}_R(X) = \mathrm{ann}_R(M)$.

Lately, there has been significant focus on linking graphs with algebraic structures in the research. There exist numerous methods to link a graph to a given ring R, we direct the reader to [2, 3, 6, 10, 11, 13, 15].

Also, in ([11], 2022), the authors introduced and studied the comaximal colon ideal graph $\mathrm{C}^*(R)$ for a ring R, which is a simple graph with vertex set $\mathbb{I}^*(R)$ and two distinct vertices I, J of $\mathbb{I}^*(R)$ are adjacent if and only if $\mathrm{C}(I, J) = (I : J) + (J : I) = R$. Clearly, $\mathrm{C}^*(R)$ is a supergraph of $\mathrm{C}(R)$.

In this paper, we present and research the supergraph $\mathrm{SC}^*(R)$ of graph $\mathrm{C}^*(R)$ which is a simple graph with vertex set $\mathbb{I}^*(R)$ and two distinct vertices I, J of $\mathbb{I}^*(R)$ are adjacent if and only if $\mathrm{C}(I, J)$ is an essential ideal of R.

For a graph G, by $\mathrm{V}(G)$ and $\mathrm{E}(G)$, we mean the set of all vertices and edges of G, respectively. We recall that a graph is *connected* if there exists a path connecting any two of it's distinct vertices. A graph G is named *empty* or *totally disconnected* if $\mathrm{E}(G) = \emptyset$.

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The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of a shortest path connecting them (if such a path does not exist, then $\mathrm{d}(a,b)=\infty$; we also define $\mathrm{d}(a,a)=0$). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is equal to $\sup\{\mathrm{d}(a,b) : a,b \in \mathrm{V}(G)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. The *girth* of a graph G, represented by $\mathrm{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise; $\mathrm{gr}(G)=\infty$. We denote the complete graph on n vertices by K_n and the complete bipartite graph on m and n vertices by $K_{m,n}$.

2. Main results

Consider R as a commutative ring with a non-zero identity element. We present a supergraph of the comaximal colon ideal graph $\mathrm{C}^*(R)$ as discussed in [11]. Let I and J be two elements of $\mathbb{I}^*(R)$ and represent $\mathrm{C}(I,J)$ as the sum of two colon ideals $(I:J)$ and $(J:I)$.

Definition 2.1. The comaximal colon ideal graph $\mathrm{C}^*(R)$ which is a simple graph with vertex set $\mathbb{I}^*(R)$ and two vertices I,J of $\mathbb{I}^*(R)$ are adjacent if and only if $\mathrm{C}(I,J)=R$.

Definition 2.2. The graph $\mathrm{SC}^*(R)$ is a simple undirected graph with vertex set $\mathbb{I}^*(R)$ and two vertices I,J of $\mathbb{I}^*(R)$ are adjacent if $C(I,J) \trianglelefteq^e R$.

Note that in general case, $\mathrm{V}(\mathrm{C}(R)) \subsetneq \mathrm{V}(\mathrm{C}^*(R)) = \mathrm{V}(\mathrm{SC}^*(R))$. Also, $\mathrm{E}(\mathrm{C}(R)) \subsetneq \mathrm{E}(\mathrm{C}^*(R)) \subseteq \mathrm{E}(\mathrm{SC}^*(R))$.

Example 2.3. Consider $R=\mathbb{Z}_{18}$, then $\mathrm{J}(R)=\langle\bar{6}\rangle$. Clearly, $\mathrm{V}(\mathrm{C}(R))=\{\langle\bar{2}\rangle,\langle\bar{3}\rangle,\langle\bar{9}\rangle\}$, $\mathrm{V}(\mathrm{C}^*(R))=\mathrm{V}(\mathrm{SC}^*(R))=\{\langle\bar{2}\rangle,\langle\bar{3}\rangle,\langle\bar{6}\rangle,\langle\bar{9}\rangle\}$. One can check that, $\mathrm{C}(R)$ is the line graph $\langle 9\rangle-\langle 2\rangle-\langle 3\rangle$, and also $\mathrm{C}^*(R)=\mathrm{SC}^*(R)$ is the complete graph K_4.

Remmeber that the *singular submodule* of M represented by $\mathrm{Z}(M)=\{m\in M \mid \mathrm{ann}_R(m)\subseteq^e R\}$. Then M is named *singular* if $\mathrm{Z}(M)=M$ and *non-singular* if $\mathrm{Z}(M)=0$.

Theorem 2.4. *Let R be a commutative ring. Then the following assertions are true.*

(i) *If $I\in\mathbb{I}^*(R)$ is an essential ideal, then I is a universal vertex of $\mathrm{SC}^*(R)$. Furthermore, in this case, $\mathrm{diam}(\mathrm{SC}^*(R))\le 2$.*
(ii) *Suppose that $(R,\mathfrak{m})$ is a local ring. Then $\mathfrak{m}$ is a universal vertex of $\mathrm{SC}^*(R)$ and $\mathrm{diam}(\mathrm{SC}^*(R))\le 2$. Moreover, if $|\mathbb{I}^*(R)|=3$, then $\mathrm{gr}(\mathrm{SC}^*(R))=3$.*
(iii) *For every non-zero element $m\in\mathrm{Z}(M)$, $\mathrm{ann}_R(m)$ is a universal vertex of $\mathrm{SC}^*(R)$.*

Proof. (i) Assume that I is an essential ideal of R and J is any vertex in $\mathrm{SC}^*(R)$ distinct from I. Since $I\subseteq(I:J)\subseteq\mathrm{C}(I,J)\subseteq R$, hence $\mathrm{C}(I,J)\trianglelefteq^e R$. Thus, $I-J$ is an edge of $\mathrm{SC}^*(R)$, as needed. In this case, $\deg(I)=\mathrm{card}(\mathbb{I}^*(R))-1$. Therefore for arbitrary ideals J,K of $\mathbb{I}^*(R)$ distinct from I, $J-I-K$ is a path in $\mathrm{SC}^*(R)$, so $\mathrm{diam}(\mathrm{SC}^*(R))\le 2$.
(ii) Clearly, $\mathrm{J}(R)=\mathfrak{m}$ is an essential ideal of R and the proof results by (i). Suppose that $|\mathbb{I}^*(R)|=3$. Take, $\mathbb{I}^*(R)=\{I_1,I_2,\mathfrak{m}\}$. Two following cases may be occure:
(a). If $I_1+I_2=\mathfrak{m}\subsetneq^e R$, then $\mathrm{C}(I_1,I_2)\subseteq^e R$ since $I_1+I_2\subseteq\mathrm{C}(I_1,I_2)$, so $I_1-\mathfrak{m}-I_2-I_1$.
(b). If $I_1+I_2\neq\mathfrak{m}$, then either $I_1\subset I_2$ or $I_2\subset I_1$. Again, $I_1-\mathfrak{m}-I_2-I_1$. Hence, $\mathrm{gr}(\mathrm{SC}^*(R))=3$.
(iii) Suppose that $0\neq m\in\mathrm{Z}(M)$. Clearly, $\mathrm{ann}_R(m)$ is a non-zero proper essential ideal of R and so it is a vertex of $\mathrm{SC}^*(R)$. The proof results by (i). □

Corollary 2.5. *Every proper ideal $I+\mathrm{ann}(I)$ is a universal vertex of $\mathrm{SC}^*(R)$.*

Proof. Clearly, for every arbitrary ideal I of R, $I+\mathrm{ann}(I)$ is a non-zero ideal. By [4, Observation 1.3], $I+\mathrm{ann}(I)$ is an essential ideal of R. The proof results by Theorem 2.4 (i). □

Corollary 2.6. *If* $\mathrm{SC}^*(R)$ *is an empty graph with at least two vertices, then* M *is a non-singular module.*

Proposition 2.7. *The following assertions are true.*

(i) *If* R *is a uniform ring, then* $\mathrm{SC}^*(R)$ *is a complete graph.*
(ii) *Assume that* R *is a finite commutative ring but not a field which is contain* $I \in \mathbb{I}^*(R)$ *such that* $I^2 \neq 0$. *Then* $\mathrm{gr}(\mathrm{SC}^*(R)) = 3$.

Proof. (i) It results by Theorem 2.4 (i).
(ii) Since R is a finite commutative ring but not a field, hence the non-zero proper ideal I of R is an annihilating ideal of R, i.e., $\mathrm{ann}(I) \neq 0$. By assumption, $\mathrm{ann}(I) \neq I$. By Corollary 2.5, it results that $I - \mathrm{ann}(I) - I + \mathrm{ann}(I) - I$ is a 3-cycle in $\mathrm{SC}^*(R)$, as we claimed. □

Definition 2.8. Let M be an R-module. The graph $\mathrm{SC}^*(M)$ is a simple undirected graph with vertex set of $\mathrm{L}^*(M)$ and two distinct vertices N and K are adjacent if and only if $\mathrm{C}(N,K) = (N:K) + (K:N)$ is an essential ideal of R.

Clearly, $\mathrm{SC}^*(M)$ *contains the inclusion graph (two distinct submodules* N *and* K *are adjacent if and only if* N *and* K *are comparable) and it is a supergraph of* $\mathrm{C}^*(M)$, *see [11, Definition 3.1]. Recall that an* R*-module* M *is named a* Bezout module *if for all* $m, n \in M$, $Rm + Rn$ *is cyclic, equivalently every finitely generated submodule of* M *is cyclic. An* R*-module* M *is named a* valuation module *if for all* $m, n \in M$, *either* $Rm \subseteq Rn$ *or* $Rn \subseteq Rm$. *Equivalently, for all submodules* N, K *of* M, *either* $N \subseteq K$ *or* $K \subseteq N$.

Theorem 2.9. *Let* M *be an* R*-module and* $N \in \mathrm{L}^*(M)$. *If* $\mathrm{ann}(N) \subseteq^e R$, *then* N *is a universal vertex in* $\mathrm{SC}^*(M)$.

Proof. Assume that $K \in \mathrm{L}^*(M) \setminus \{N\}$. Since $\mathrm{ann}(N) \subseteq (K:N) \subseteq \mathrm{C}(N,K) \subseteq R$, hence by assumption, $\mathrm{C}(N,K) \subseteq^e R$. This implies that N is adjacent to K in $\mathrm{SC}^*(M)$. □

Remmeber that a submodule N *of* M *is named an* annihilator essential submodule, *represented by* $N \leq^{ae} M$, *if* $N \cap X = 0$ *results that* $\mathrm{ann}(X) \subseteq^e R$ *for each submodule* X *of* M, *see [12, Definition 3.1].*

Corollary 2.10. *Let* M *be an* R*-module and* $N \leq^{ae} M$. *Then every submodule* $X \in \mathrm{L}^*(M)$ *with* $N \cap X = 0$ *is a universal vertex in* $\mathrm{SC}^*(M)$. *Moreover, the subgraph of* $\mathrm{SC}^*(M)$ *generated by minimal annihilator essential submodules of* M *is a complete graph.*

Corollary 2.11. *Let* $X = \{Rm \mid m \in \mathrm{Z}(M)\}$. *Then the subgraph of* $\mathrm{SC}^*(M)$ *generated by* X *is a complete graph.*

Proposition 2.12. *Let* R *be an integral domain and* M *a faithful multiplication* R*-module. If either* M *is a valuation module or* M *is a Bezout local module, then* $\mathrm{SC}^*(M)$ *is the complte graph* K_s *where* $s = |\mathrm{L}^*(M)|$.

Proof. The proof is straightforward by [1, Corollary 2.4]. □

Remmber that an R*-module* M *is named* sum-irreducible *whenever it is non-zero and cannot be expressed as the sum of two proper submodules of itself.*

THEOREM 2.13. *Let M be a sum-irreducible R-module. Then* $\mathrm{SC}^*(M)$ *is a connected graph and* $\mathrm{diam}(\mathrm{SC}^*(M)) \leq 2$.

PROOF. Suppose that N, K are distinct vertices of $\mathrm{SC}^*(M)$. If either $N \subset K$ or $K \subset N$, then $\mathrm{C}(N, K) = R$ is an essential ideal of R and so $N - K$ is an edge in $\mathrm{SC}^*(M)$. Otherwise, if $N \nsubseteq K$ and $K \nsubseteq N$, then by assumption, $N + K \neq M$ and so $N - N + K - K$ is a path in $\mathrm{SC}^*(M)$. Hence, $\mathrm{diam}(\mathrm{SC}^*(M)) \leq 2$, as needed. □

THEOREM 2.14. *Let M be a sum-irreducible R-module. Then* $\mathrm{gr}(\mathrm{SC}^*(M)) = 3$ *or* ∞.

PROOF. Assume that N and K are two arbitrary distinct vertices of $\mathrm{SC}^*(M)$. Two situations are possible as follows:

a) N and K are comparable. Then $\mathrm{C}(N, K) = R$, so $N - K$ is an edge of $\mathrm{SC}^*(M)$ and $\mathrm{d}(N, K) = 1$. If all nonzero proper submodules of M are comparable, then for $|\mathrm{L}^*(M)| \leq 2$, there exists no cycle and so $\mathrm{gr}(\mathrm{SC}^*(M)) = \infty$. Also, in this case, if $|\mathrm{L}^*(M)| \geq 3$, then $\mathrm{gr}(\mathrm{SC}^*(M)) = 3$.

b) Suppose that N and K are non-comparable. This results that there exist $x \in N \setminus K$ and $y \in K \setminus N$.

case 1. Suppose that $\mathrm{C}(Rx, Ry)$ is essential in R. Then one of the following situation may occur:
(i) If $N = Rx$ and $K = Ry$, then $N - K - N + K - N$ is a 3-cycle in $\mathrm{SC}^*(M)$.
(ii) If $N \neq Rx$ and $K = Ry$, then $K - Rx - N + K - K$ is a 3-cycle.
(iii) If $N = Rx$ and $K \neq Ry$, then $N - Ry - N + K - N$ is a 3-cycle.
(iv) If $N \neq Rx$ and $K \neq Ry$, then $N - Rx - N + K - N$ is a 3-cycle.

case 2. Suppose that $\mathrm{C}(Rx, Ry)$ is not essential in R. Then one of the following situation may occur:
(i) If $N = Rx$ and $K = Ry$, then since $M \neq N + K$ hence there exists $z \in M \setminus (N + K)$. Then $N - Rx + Ry - Rx + Ry + Rz - N$ is a 3-cycle in $\mathrm{SC}^*(M)$.
(ii) If $N = Rx$ and $K \neq Ry$, then $N - Rx + Ry - N + K - N$ is a 3-cycle in $\mathrm{SC}^*(M)$.
(iii) If $N \neq Rx$ and $K = Ry$, then $N - Rx - N + K - N$ is a 3-cycle in $\mathrm{SC}^*(M)$.
(iv) If $N \neq Rx$ and $K \neq Ry$, then $N - Rx - N + K - N$ is a 3-cycle in $\mathrm{SC}^*(M)$. □

3. *Conclusion*

We investigated some more results of the graph $\mathrm{SC}^*(M)$.

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A Generalization of Lifting Properties for Directed Graphs

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Abstract. This exposition generalizes the fundamental lifting properties of A-homotopy theory for graphs to a broader framework suitable for directed graphs (digraphs). By leveraging the concept of a Directed Semicovering—a weakening of the classical covering map condition—we introduce the μ-fundamental monoid, denoted $A_1^{\mu}(D, v_0)$. This algebraic invariant serves as an analogue to the generalized fundamental group introduced by de Oliveira and da Silva. We systematically reconstruct the Path Lifting, Homotopy Lifting, and Lifting Criterion theorems within this new μ-homotopy theory, demonstrating that while uniqueness of lifts is forfeited, the existence properties remain, there by providing a powerful new tool for the classification and analysis of directed graph structures.

Keywords: fundamental group, directed graphs, lifting properties, Homotopy Lifting

AMS Mathematics Subject Classification [2020]: Primary: 57M10, 57M12, Secondary: 57M05

1. Preliminaries: The μ-Homotopy Framework

Recall that a continuous map $p : \tilde{X} \longrightarrow X$ is called a covering of X, if for every $x \in X$ there is an open subset U of X with $x \in U$ such that U is evenly covered by p i.e. $p^{-1}(U)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p.

Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f.

The map p has *path lifting property* if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has *unique path lifting property* if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f (see [6].)

Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has *continuous lifting of paths* if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$

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defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism, for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0,1] \to Y | \alpha(0) = y\}$. Also, a map $p : Y \to X$ *has continuous lifting of homotopies* if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism, for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. He also simplified the definition of semicovering maps by showing that having continuous lifting of paths implies having continuous lifting of homotopies (see [1].) The notion of a semicovering map, originally introduced by Brazas and later simplified by Kowkabi et al. [3] to a local homeomorphism with unique path lifting properties, provides the foundational motivation for our work. While these definitions effectively clarify the concept in the context of topological spaces, their reliance on the continuity of path spaces is not directly applicable to the discrete nature of graphs. We therefore formulate a combinatorial analogue that preserves the spirit of guaranteed path lifting, replacing the analytical condition of a homeomorphism on path spaces with a local structural property suitable for digraphs [5]. Before proceeding to the main theorems, we must formally establish the foundational definitions upon which our generalized framework rests. These definitions directly adapt the concepts from semicovering theory to the context of directed graphs and the non-invertible nature of path concatenation, naturally leading to a monoid structure for more references, see [2], [4].

Definition 1.1 (Directed Semicovering). Let D' and D be two digraphs. A digraph homomorphism $p : D' \to D$ is a *Directed Semicovering* if, for every vertex $v' \in V(D')$, the restriction of p to the open out-neighborhood of v', denoted $p|_{N^+(v')}$, is a surjective map onto the open out-neighborhood of its image, i.e., $p|_{N^+(v')} : N^+(v') \to N^+(p(v'))$ is surjective.

Intuition: This condition guarantees that any directed path segment in the base digraph D can be lifted, but it does not guarantee a unique lift, as the fiber $p^{-1}(w)$ for a neighbor w of $p(v')$ may contain multiple vertices adjacent from v'.

Definition 1.2 (μ-Trivial Loop). A directed loop $f : P_m \to D$ based at v_0 is called *μ-trivial* if there exists a directed semicovering $p : D' \to D$ and a path $\tilde{f} : P_m \to D'$ such that $p \circ \tilde{f} = f$ and $\tilde{f}$ is itself a loop in D', i.e., $\tilde{f}(0) = \tilde{f}(m)$.

Definition 1.3 (The μ-Fundamental Monoid, A_1^μ). The *μ-fundamental monoid* $A_1^\mu(D, v_0)$ is the quotient of the standard directed A-fundamental monoid $A_1(D, v_0)$ by the smallest congruence relation R_μ that identifies every μ-trivial loop with the identity element (the trivial loop at v_0). Formally:

$$A_1^\mu(D, v_0) := A_1(D, v_0)/R_\mu$$

where $[f]R_\mu[g]$ if g can be obtained from f by a finite sequence of insertions or deletions of μ-trivial loops.

2. Generalization of Lifting Properties for Directed Semicoverings

We now reconstruct the central theorems of classical A-homotopy lifting theory as presented in [2] within our μ-homotopy framework. The primary theme is the transition from "unique existence" to "guaranteed existence".

Theorem 2.1 (Existence of Path Lifts). *Let $p : D' \to D$ be a directed semicovering. For any directed path $f : P_m \to D$ and any vertex $v'_0 \in p^{-1}(f(0))$, there exists at least one directed path $\tilde{f} : P_m \to D'$ such that $\tilde{f}(0) = v'_0$ and $p \circ \tilde{f} = f$.*

Proof Sketch. The proof proceeds by induction on the length m of the path.

- **Base Case (m=0):** The path f is a single vertex $f(0)$. The lift $\tilde{f}$ is the path of length 0 at v'_0, which trivially satisfies the conditions.
- **Inductive Step:** Assume the theorem holds for all paths of length $k < m$. Let $f : P_m \to D$ be a path. By the inductive hypothesis, there exists a lift $\tilde{f}_k : P_k \to D'$ for the prefix of f of length $k = m - 1$. Let $v'_k = \tilde{f}_k(k)$. The next segment of the path in D is the edge $(f(k), f(k+1))$. Since $p(v'_k) = f(k)$ and p is a directed semicovering, the map $p|_{N^+(v'_k)}$ is surjective onto $N^+(f(k))$. Therefore, as $f(k+1) \in N^+(f(k))$, there must exist at least one vertex $v'_{k+1} \in N^+(v'_k)$ such that $p(v'_{k+1}) = f(k+1)$. We define $\tilde{f}(k+1) = v'_{k+1}$ and extend the lift.

Since the choice of v'_{k+1} may not be unique, the lift $\tilde{f}$ is not guaranteed to be unique. □

□

Theorem 2.2 (Existence of Homotopy Lifts). *Let $p : D' \to D$ be a directed semicovering and $H : X \times P_n \to D$ be a directed A-homotopy. For any given lift $\tilde{h}_0 : X \to D'$ of the initial map $h_0(x) = H(x, 0)$, there exists a directed A-homotopy $\tilde{H} : X \times P_n \to D'$ such that $\tilde{H}(x, 0) = \tilde{h}_0(x)$ for all $x \in V(X)$ and $p \circ \tilde{H} = H$.*

Proof Sketch. The construction of the lifted homotopy $\tilde{H}$ is performed vertex by vertex and time-step by time-step. For each vertex $x \in V(X)$, the map $H(x, \cdot) : P_n \to D$ is a directed path in D starting at $H(x, 0) = h_0(x)$. We have a starting point for its lift, namely $\tilde{h}_0(x)$. By Theorem 2.1, we can lift this "vertical" path to a path $\tilde{H}(x, \cdot) : P_n \to D'$.

The core of the proof lies in showing that the collection of these lifted paths constitutes a valid digraph homomorphism $\tilde{H}$ from the product graph $X \times P_n$. This requires verifying that for any edge $(x_1, x_2) \in E(X)$ and any $j \in \{0, \ldots, n\}$, the adjacency $(\tilde{H}(x_1, j), \tilde{H}(x_2, j))$ is preserved, and for any $x \in V(X)$ and $j \in \{0, \ldots, n-1\}$, the adjacency $(\tilde{H}(x, j), \tilde{H}(x, j+1))$ is preserved. This follows from the homomorphism property of H and the fact that p is a homomorphism that preserves adjacency locally. The potential non-uniqueness of path lifts at each step means the resulting lifted homotopy $\tilde{H}$ is not unique. □ □

Theorem 2.3 (The μ-Lifting Criterion). *Let $p : D' \to D$ be a directed semicovering. A digraph homomorphism $f : X \to D$ has a lift $\tilde{f} : X \to D'$ if and only if every loop γ in X (based at some x_0) maps to a loop $f \circ \gamma$ in D that is μ-trivial.*

Proof Sketch. **(⇒) Necessity:** Assume a lift $\tilde{f} : X \to D'$ exists. Let γ be any loop in X. The composition $\tilde{f} \circ \gamma$ is a path in D'. Since γ is a loop, $(\tilde{f} \circ \gamma)(0) = \tilde{f}(\gamma(0)) = \tilde{f}(\gamma(m)) = (\tilde{f} \circ \gamma)(m)$, so $\tilde{f} \circ \gamma$ is a loop in D'. We also have $p \circ (\tilde{f} \circ \gamma) = (p \circ \tilde{f}) \circ \gamma = f \circ \gamma$. By Definition 1.2, since $f \circ \gamma$ is the projection of a loop in the semicovering space D', $f \circ \gamma$ is by definition a μ-trivial loop.

(⇐) Sufficiency: Assume that for every loop γ in X, $f \circ \gamma$ is μ-trivial. We must construct a lift $\tilde{f}$. We choose a basepoint $x_0 \in V(X)$ and a lift $x'_0 \in p^{-1}(f(x_0))$. For any other vertex $x \in V(X)$, we choose a path α from x_0 to x. The composition $f \circ \alpha$ is a path in D starting at $f(x_0)$. By Theorem 2.1, we can lift this to a path $\widetilde{(f \circ \alpha)}$ in D' starting at x'_0. We define $\tilde{f}(x)$ to be the endpoint of this lifted path.

The crucial step is to show that this definition is independent of the choice of path α. If β is another path from x_0 to x, then $\alpha * \beta^{-1}$ (conceptually, in the undirected sense) forms a loop γ based at x_0. The condition that $f \circ \gamma$ is μ-trivial

ensures that the lifts of $f \circ \alpha$ and $f \circ \beta$ must end at the same point in D'. This consistency allows for a well-defined mapping $\tilde{f}$. Finally, one verifies that $\tilde{f}$ is a digraph homomorphism.

□

3. Conclusion and Implications

The reconstruction of these fundamental theorems demonstrates that the theory of A-homotopy can be successfully and coherently generalized to the μ-homotopy framework. The substitution of "uniqueness" with "existence" is not a flaw, but rather the central feature of this broader theory. It allows for the analysis of a much larger class of digraph morphisms that are prevalent in areas such as computer science (e.g., system simulations and abstractions) and complex systems, where local isomorphism is too restrictive. The μ-fundamental monoid A_1^μ emerges as a robust algebraic invariant capable of classifying digraphs based on these more flexible covering relations, opening a promising avenue for future research in combinatorial and computational topology.

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Some properties of Scalar Graphs Product Graphs on Modules

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Abstract: If we have a commutative ring R and it is identified as $1 \neq 0$ there is an R-module M, $G_R(M)$ will denote the Scalar product graph of M. Vertices of $G_R(M)$ are elements of M, and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. In this paper, we observe some properties of scalar Graphs product graphs on modules.
Keywords: Graph Join, Topological Index, Module.

1. Introduction

Everywhere in this paper, All graphs are simple and connected. Let G be a graph with sets V (G) and $E(G)$ as vertex and edge set. For two arbitrary vertices, u and v of G, the distance between u and v, denoted by $d(u;\ v)$, is the length of the shortest path between them. For a graph G, the degree of a vertex v, denoted by $deg(v)$, is the number of edges incident to v. We will suppose that there are two separate graphs as G and H. One of operation between two graphs is $G + H$. This is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ as vertex set and edge set, respectively. This operation named the join of G and H. An appropriate assigning k colors to the $V(G)$ such that no two adjacent vertices have the same color, named k-coloring. The chromatic number of G, denoted by $\chi(G)$, is the smallest number of k with this property. A complete subgraph of G is a clique of G. The clique number of G is cardinality of largest clique of G and denoted by $\omega(G)$. If $\chi(G) = \omega(G)$, then G is called weakly perfect.
Nouri-Jouybari et al.[4], defined a graph over R −module, named Scalar product graph, denoted by $G_R(M)$. Vertices of this graph is M, and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. Nouri-Jouybari et al. [2], [3] proved scalar product graphs on $\mathbb{Z}_{2p}$; $\mathbb{Z}_{3p}$; $5p$ as $\mathbb{Z}$-modules are weakly perfect. We state some properties of these graphs.

1.1. Primary Definition

In this section, first, we review some primary properties of scalar product graphs on some modules.

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Definition 2.1 ([4])**.** For a commutative ring R with identity $1 \neq 0$ and an R-module M, let a, b be two arbitrary elements in M. A graph with vertices in M, named Scalar product graph, denoted by $G_R(M)$, if $a = rb$ or $b = ra$ for some $r \in R$, then a, b are adjacent. For example, See Fig1.

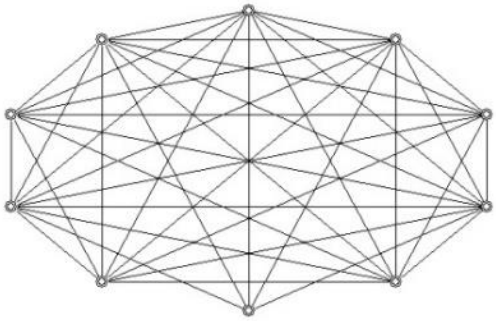

Figure 1: Scalar-product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{10}$

Remark 2.2. If M be an R-module, then for $G_R(M)$, We have, $a, b \in M$ are two adjacent vertices if and only if $Ra \subseteq Rb$ or $Rb \subseteq Ra$.
Remark 2.3. $G_R(M)$ is join of two graphs G_1 and G_2, that is, vertices of G_1 are unit elements of M and formed a complete graph. Also, vertices of G_2 are nonzero non-unit elements of M and formed the complement of the cozero-divisor graph of M.

The Wiener index of G is half of the sum of the distance between two distinct vertices. For example, we have $W(K_n) = \frac{1}{2}n(n-1)$ and $W(K_{1,n-1}) = (n-1)^2$.
we compute Wiener indices of some Scalar product graph of $\mathbb{Z}$-module.

Definition 2.4. The Wiener index of a graph G are defined as:

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u,v)$$

Theorem 2.5. For any two graphs G_1 and G_2, we have:

$$W(G_1 + G_2) = |V(G_1)|^2 - |V(G_1)| + |V(G_2)|^2 - |V(G_2)| + |V(G_1)||V(G_2)| - |E(G_1)| - |E(G_2)|.$$

Proposition 2.6. Suppose that p is a prime number. Then, we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 2p^2 - 1$.
Proposition 2.7. Suppose that p is a prime number. Then, we have $W\left(G_{\mathbb{Z}}(\mathbb{Z}_{3p})\right) = \frac{9}{2}p^2 + \frac{1}{2}p - 1$.
Proposition 2.8. Suppose that p is a prime number. Then, we can have $W\left(G_{\mathbb{Z}}(\mathbb{Z}_{5p})\right) = \frac{25}{2}p^2 + \frac{3}{2}p - 4$.

3. Scalar product graph of module $\mathbb{Z}_{pq}$

Let $p < q$ be two prime number. In this section, we want to calculate the number of edges of the scalar product graph of the $\mathbb{Z}$-modules $\mathbb{Z}_{pq}$. First, the structure of this graph is discussed.

Lemma 3.1. Suppose that $p < q$ is two prime number. Then, we have:

$$G_{\mathbb{Z}}(\mathbb{Z}_{pq}) = K_{(p-1)q-p+2} + \overline{K_{p-1,q-1}}$$

Proof: $G_{\mathbb{Z}}(\mathbb{Z}_{pq})$ is join of two graphs: vertices of first graph are unit elements of $\mathbb{Z}_{pq}$ and formed a complete graph; vertices of second graph are nonzero non-unit elements of $\mathbb{Z}_{pq}$ and formed the complement of the cozero-divisor graph of $\mathbb{Z}_{pq}$.

Theorem 3.2. [1] Let G_1, G_2 be two graphs. Then:

$$|E(G_1 + G_2)| = |E(G_1)| + |E(G_2)| + |V(G_1)||V(G_2)|$$

Theorem 3.3. Suppose that $p < q$ is two prime number and G is Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{pq}$. Then $|E(G)| = \frac{p^2q^2}{2} - \frac{3p-2}{2}q + p - 1$.

Proof. By theorem 3.2 and lemma 3.1, we have:

$$\begin{aligned}
E(G) &= E\left(K_{(p-1)q-p+2} + \overline{K_{p-1,q-1}}\right) \\
&= \left|E\left(K_{(p-1)q-p+2}\right)\right| + \left|E\left(\overline{K_{p-1,q-1}}\right)\right| + \left|V\left(K_{(p-1)q-p+2}\right)\right|\left|V\left(\overline{K_{p-1,q-1}}\right)\right| \\
&= \frac{[(p-1)q-p+2][(p-1)q-p+2]}{2} + \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} \\
&\quad +[(p-1)q-p+2][(p-1)+(q-1)] \\
&= \frac{[pq-q-p+2][pq-q-p+2]}{2} + \frac{p^2-3p+2}{2} + \frac{q^2-3q+2}{2} \\
&\quad +[pq-q-p+2][p+q-2] \\
&= \frac{p^2q^2}{2} + \frac{5}{2}pq - 4pq + q + p - 1 \\
&= \frac{p^2q^2}{2} - \frac{3}{2}pq + q + p - 1 \\
&= \frac{p^2q^2}{2} - \frac{3p-2}{2}q + p - 1
\end{aligned}$$

Example 3.4. Let G be a scalar product graph of $\mathbb{Z}$ -modules $\mathbb{Z}_{\mathbf{10}}$. Then, $E(G) = \frac{2^2 5^2}{2} - \frac{3(2)-2}{2}5 + 2 - 1 = 41$. See Fig.1. Let G be a scalar product graph of $\mathbb{Z}$ -modules $\mathbb{Z}_{\mathbf{21}}$. Then, $(G) = \frac{3^2 7^2}{2} - \frac{3(3)-2}{2}7 + 3 - 1 = 198$. See Fig.2.

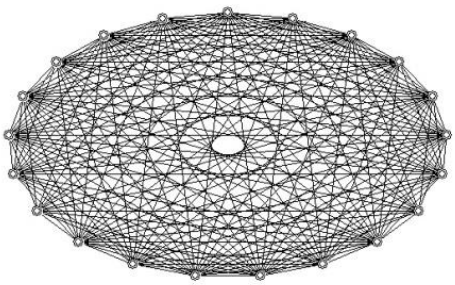

Figure 2: Scalar-product graph of $\mathbb{Z}$ -modules $\mathbb{Z}_{\mathbf{21}}$

4. Conclusion

Considering the theorems obtained from the calculation of topological indices in scalar product graphs on simple modules, we conclude that the structure of scalar product graphs, which is the sum of two graphs, can be used to calculate the rest of the topological indices. It is suggested that the calculation of the topological indices of scalar product graphs from other modules, such as direct product modules, etc. should be studied in future researches.

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Automorphism Groups of Scalar Product graph of Modules

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Abstract: Let R be a commutative ring with condition $1 \neq 0$, and M be a R-module. The Scalar product graph$G_R(M)$ of M is a graph with vertex set M and two vertices $a, b \in M$ are adjacent if and only if $a = rb$ or $b = ra$ for some $r \in R$. In this paper we investigate automorphism groups of scalar product graph on modules.
Keywords: Automorphism Group, Module, Scalar Product Graph, Ring.

1. Introduction

Throughout this paper all graphs mentioned are assumed to be finite simple graph. Let $G = (V, E)$ be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, $E \subseteq P_2(V)$ and $|E| = m$. The automorphism group of a graph G is denoted by $Aut(G)$. We will suppose that there are two separate graphs as G and H. One of operation between two graphs is $G + H$. This is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ as vertex set and edge set, respectively. This operation named the join of G and H.
Nouri-Jouybari et al.[4], defined a graph over R −module, named Scalar product graph, denoted by $G_R(M)$. Vertices of this graph is M, and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. Nouri-Jouybari et al. [2], [3] presented some properties of scalar product graphs on $\mathbb{Z}_{2p}$; $\mathbb{Z}_{3p}$; $5p$ as $\mathbb{Z}$-modules.

An automorphism of a graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge–vertex connectivity. Formally, an automorphism of a graph $G = (V, E)$ is a permutation σ of the vertex set V, such that the pair of vertices (u, v) form an edge if and only if the pair $(\sigma(u), \sigma(v))$ also form an edge. That is, it is a graph isomorphism from G to itself.
In this paper we compute automorphism group of scalar product graphs on $\mathbb{Z}$-modules..

1.1. Primary Definition

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In this section, first, we review some primary properties of scalar product graphs on some modules.

Definition 2.1 ([4])**.** For a commutative ring R with identity $1 \neq 0$ and an R-module M, let a, b be two arbitrary elements in M. A graph with vertices in M, named Scalar product graph, denoted by $G_R(M)$, if $a = rb$ or $b = ra$ for some $r \in R$, then a, b are adjacent. For example, See Fig1.

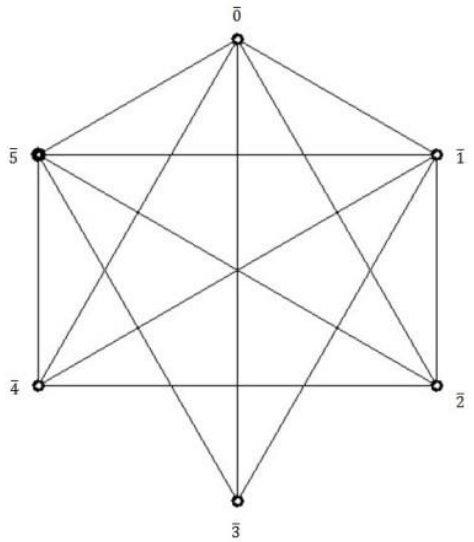

Figure 1: Scalar-product graph of $\mathbb{Z}$-modules $\mathbb{Z}_6$

Remark 2.2. If M be an R-module, then for $G_R(M)$, We have, $a, b \in M$ are two adjacent vertices if and only if $Ra \subseteq Rb$ or $Rb \subseteq Ra$.

Remark 2.3. $G_R(M)$ is join of two graphs G_1 and G_2, that is, vertices of G_1 are unit elements of M and formed a complete graph. Also, vertices of G_2 are nonzero non-unit elements of M and formed the complement of the cozero-divisor graph of M.

Proposition 2.4. Let $\mathbb{Z}_n$ be $\mathbb{Z}$-module. If p, m are prime and positive integer number, then for $n = 1, p, p^m$, Scalar product graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is complete.

Proposition 2.5. Let M be an R-module and N submodule of M. Then $G_R(N)$ is an induced subgraph of $G_R(M)$.

Theorem 2.6. Let M be a R-module. Then scalar product graph $G_R(M)$ is complete if and only if the cyclic submodules of M are linearly ordered by inclusion relation.

3. Automorphism group of $G_{\mathbb{Z}}(\mathbb{Z}_n)$ for some n

In this section we compute order of automorphism group of scalar product graphs of $\mathbb{Z}_n$-module as $\mathbb{Z}$- module.

Theorem 3.1. [1] Suppose G_1 and G_2 are two graphs. If $H_1 \leq Aut(G_1)$ and $H_2 \leq Aut(G_2)$ then $H_1 \times H_2 \leq Aut(G_1 \times G_2)$. Also, if $|d(x_i) - d(y_j)| \neq |n_1 - n_2|$, $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$ then $Aut(G_1 + G_2) \cong Aut(G_1) \times Aut(G_2)$.

Lemma 3.2. Suppose that p is a prime number. Then the scalar product graph of $\mathbb{Z}_{2p}$ is the join of K_p and $\overline{K_{1,p-1}}$.
Proof. By remark 2. We have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = G_1 + G_2$ where G_1 is a complete graph with $|W_{\mathbb{Z}}(\mathbb{Z}_{2p})^*| = p$ vertices and G_2 is the complement of the the cozero-divisor graph of $\mathbb{Z}_{2p}$ that is $\overline{K_{1,p-1}}$.

According to the lemma 3.2 and theorem 3.1 we have :

Theorem 3.3. Suppose that $p > 2$ is a prime number. Then we have:

$$Aut\left(G_{\mathbb{Z}}(\mathbb{Z}_{2p})\right) \cong Aut(K_p) \times Aut(\overline{K_{1,p-1}}) = S_{\phi(2p)+1} \times \prod_{1,n\neq d|n} S_{\phi(d)}$$

Example 3.4. According to the last theorem we compute order of automorphism group of scalar product graph of $\mathbb{Z}_6$ showed in fig. 1.

$$G_{\mathbb{Z}}(\mathbb{Z}_6) = K_3 + \overline{K_{1,2}}$$

$$Aut(G_{\mathbb{Z}}(\mathbb{Z}_6)) \cong Aut(K_3) \times Aut(\overline{K_{1,2}}) = S_3 \times \prod_{1,6\neq d|6} S_{\phi(d)}$$

Order of this group is 12.

4. Conclusion

Considering the theorems obtained from the calculation of automorphism groups in scalar product graphs on simple modules, we conclude that the structure of scalar product graphs, which is the sum of two graphs, can be used to calculate the rest of the automorphism groups. It is suggested that the calculation of the automorphism groups of scalar product graphs from other modules, such as direct product modules, etc. should be studied in future researches.

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Mathematical Analysis

Inequalities for sector matrices and matrix monotone functions

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Abstract. Several inequalities are presented for sector matrices and matrix monotone functions. As applications of these inequalities, similar results are provided for singular values and norms. Finally, some unitarily invariant norm inequalities are derived for sector matrices.

Keywords: matrix mean, matrix monotone function, sector matrices.

AMS Mathematics Subject Classification [2020]: 15A45, 15A42, 47A30.

Denote the set of all $n \times n$ complex matrices as $\mathbb{M}_n$. In the special case a Hermitian matrix is said to be positive semidefinite, if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$ and we write $A \geq 0$. It is called positive definite if it is positive semidefinite and invertible, we will write $A > 0$, the class of positive definite matrices is denoted by $\mathbb{M}_n^+$. A matrix $A \in \mathbb{M}_n$ is said to be accretive if, in the Cartesian or Toeplitz decomposition, can be expressed as $A = Re(A) + iIm(A)$, where $Re(A)$ is positive definite $Re(A) > 0$, with $Re(A) = \frac{A+A^*}{2}$, and $Im(A) = \frac{A-A^*}{2i}$. The numerical range of $A \in \mathbb{M}_n$ is defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

The sector region $\mathcal{S}_\theta$ is defined as follows:

$$\mathcal{S}_\theta = \{z \in \mathbb{C} : Re(z) > 0,\ |Im(z)| \leq Re(z)\tan\theta\}.$$

The expression $W(A) \subset \mathcal{S}_\theta$, where $0 \leq \theta < \frac{\pi}{2}$, means that if the numerical range of matrix A is contained within the sector region $\mathcal{S}_\theta$ in the complex plane, then A is referred to as a sectorial matrix and is represented as $A \in \mathcal{S}_\theta$. Furthermore, since $0 \notin \mathcal{S}_\theta$, then every element in $\mathcal{S}_\theta$ is invertible. Additionally, it is mentioned that sectorial matrix is necessarily accretive.

A matrix mean σ on $\mathbb{M}_n^+$, is a binary operation defined by $A\sigma B$ that satisfies the following conditions:

- $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$; for any $A, B, C, D \in \mathbb{M}_n^+$.
- $C^*(A\sigma B)C = (C^*AC)\sigma(C^*BC)$; for any $A, B \in \mathbb{M}_n^+$ and any invertible $C \in \mathbb{M}_n$.

*Speaker.

- $A_n \downarrow A$ and $B_n \downarrow B$ imply that $A_n \sigma B_n \downarrow A\sigma B$; for any $A_n, B_n, A, B \in \mathbb{M}_n^+$
- $I\sigma I$.

The matrix mean σ in the sence of Kubo-Ando is determined by a matrix monotone function $f : (0,\infty) \to (0,\infty)$ with $f(1) = 1$ (we will denote this as $f \in \mathbf{m}$). It can be expressed as follows:

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \tag{1}$$

where A and B are positive invertible matrices. The function f that satisfies the conditions for the matrix mean σ referred to as the "representing function" of σ. Below are some important matrix means:

$$t - Weighted\ arithmetic\ mean : A\nabla_t B = (1-t)A + tB. \quad (0 < t < 1)$$

$$t - Weighted\ harmonic\ mean : A!_t B = [(1-t)A^{-1} + tB^{-1}]^{-1}. \quad (0 < t < 1)$$

$$t - Weighted\ geometric\ mean : A\sharp_t B = A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}. \quad (0 < t < 1)$$

Definition 0.1. ([5]), Let σ be an matrix mean with the representing function f. The dual of σ, denoted by $\sigma^\perp$, is the mean with a representing function $\dfrac{t}{f(t)}$. Formula (1) provides an explicit form for the adjoint

$$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1} \qquad for\ invertible\ A\ and\ B.$$

If $f : (0,\infty) \to (0,\infty)$ corresponds to the mean of σ, then

$$f^*(x) = \frac{1}{f(1/x)}, \quad x > 0.$$

The function $f^*(x)$ corresponds to the adjoint of the mean σ. Recall that, if $f \in \boldsymbol{m}$, then $f^* \in \boldsymbol{m}$. Therefore, if $A \in \mathcal{S}_\theta$, then $f^*(A) \in \mathcal{S}_\theta$.
For further details, we refer the reader to [5].
Additional results related to this topic can be found in [1, 2, 4]

Lemma 0.2. *([3]), Let A and B be two definite matrices and let σ_t be a mean between ∇_t and $!_t$, where $0 \le t \le 1$. If $f : (0,\infty) \to (0,\infty)$ is an matrix monotone decreasing function, then*

$$f(A)\sigma_t f(B) \le f(A!_t B).$$

1. Main results

In this section, the main objective is to prove inequalities for sectorial matrices and matrix monotone functions. Subsequently, we present their applications in norms, eigenvalues, and other related concepts.

Theorem 1.1. *Let σ and τ be binary operations for invertible accretive matrices and $f, g, h \in \boldsymbol{m}$ be continuous functions. If*

$$\sec^4\theta\ Re\,(g(A)\tau h(B)) \le Re(f\,(A\sigma B))\ \ for\ all\ A, B \in \mathcal{S}_\theta, \tag{2}$$

then

$$Re\left(f^*(B\sigma^\perp A)\right) \le \sec^4\theta\ Re\left(h^*(B)\ \tau^\perp g^*(A)\right).$$

Based on the initial part of the proof for Theorem 1.1, we obtain the following useful result.

Corollary 1.2. *Let σ and τ be binary operations for invertible accretive matrices and $f, g, h \in \boldsymbol{m}$ be continuous functions. If*

$$g(Re(A))\tau h(Re(B)) \leq f\left(Re(A)\sigma Re(B)\right) \ \ for\ all\ A, B \in \mathcal{S}_\theta,$$

then

$$Re\left(f^*(B\sigma^\perp A)\right) \leq \sec^4\theta\ Re\left(h^*(B)\ \tau^\perp g^*(A)\right).$$

Proposition 1.3. *Let $A, B \in \mathcal{S}_\theta$ and let σ_t be a mean between ∇_t and $!_t$, where $0 \leq t \leq 1$. If $f : (0, \infty) \to (0, \infty)$ is an matrix monotone decreasing function, then*

$$Re\left(f((1-t)A + tB)\right) \leq \sec^4\theta\ Re\left(f(B)\ \sigma_t^\perp f(A)\ \right). \tag{3}$$

Inequality (3) can be seen as an extension of the established fact in [3, Lemma 2.4].

We can use the previous theorems to derive matrix inequalities involving matrix means. The following lemma assists us in presenting the norm version of Proposition 1.3.

Lemma 1.4. *([1]), If $A \in \mathcal{S}_\theta$, then for any unitarily invariant norm $\|.\|$ on $\mathbb{M}_n$,*

$$\|Re(A)\| \leq \|A\| \leq sec\theta\ \|Re(A)\|.$$

Theorem 1.5. *Let $A, B \in \mathcal{S}_\theta$ and σ_t be a arbitrary mean between ∇_t and $!_t$, where $0 \leq t \leq 1$. If $f : (0, \infty) \to (0, \infty)$ is a matrix monotone decreasing function, then*

$$\|f((1-t)A + tB)\| \leq \sec^5\theta\ \|f(B)\ \sigma_t^\perp\ f(A)\|,$$

for any unitarily invariant norm $\|.\|$ on $\mathbb{M}_n$.

Using the well-known inequalities mentioned below, we can obtain a useful inequality for the determinant version of the inequality in Proposition 1.3.

Lemma 1.6. *([1]), If $A \in \mathcal{S}_\theta$, then*

$$det(Re(A)) \leq |det(A)| \leq sec^n\theta\ det(Re(A)).$$

Theorem 1.7. *Let $A, B \in \mathcal{S}_\theta$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f \in \boldsymbol{m}$ and decreasing, then*

$$|det\left(f((1-t)A + tB)\right)| \leq \sec^{5n}\theta\ \left|det\left(f(B)\ \sigma_t^\perp\ f(A)\right)\right|.$$

In the next lemma, a useful relation between eigenvalues and singular values of a sector matrix is mentioned. By using this relation, we were able to obtain a useful inequality for the eigenvalues of sector matrices.

Lemma 1.8. *([1]), Let $A \in \mathcal{S}_\theta$. Then*

$$\lambda_j(Re(A)) \leq s_j(A) \leq sec^2\theta\ \lambda_j(Re(A)), \qquad j = 1, \ldots, n,$$

where s_j and λ_j denote the jth largest singular value and eigenvalue of a matrix.

Theorem 1.9. *Let $A, B \in \mathcal{S}_\theta$, $0 \leq t \leq 1$ and σ_t be an arbitrary mean between ∇_t and $!_t$. If $f \in \boldsymbol{m}$ and decreasing , then*

$$s_j\left(f((1-t)A + tB)\right) \leq \sec^6(\theta)\ s_j\left(f(B)\ \sigma_t^\perp\ f(A)\right).$$

The numerical radius, denoted by $\omega(A)$, of an element $A \in \mathbb{M}_n$ is defined as the supremum of the inner product $\langle Ay, y\rangle$, where y is an element of $\mathbb{C}^n$ with a norm of 1. In other words:

$$\omega(A) = \sup\{\langle Ay, y\rangle : y \in \mathbb{C}^n, \|y\| = 1\}.$$

When $A \in \mathcal{S}_0$, we have $\omega(A) = \|A\|$, and thus $\omega(Re(A)) = \|Re(A)\|$. Bedrani et al. in [1] demonstrated that if $A \in \mathcal{S}_\theta$, the following inequality holds.

$$\omega(Re(A)) \leq \omega(A) \leq \sec\theta\ \omega(Re(A)). \tag{4}$$

In the last theorem, we obtain a relation for the numerical range of sectorial matrices, which includes the arithmetic mean.

THEOREM 1.10. *Let $A, B \in \mathcal{S}_\theta$ and σ_t be a mean between ∇_t and $!_t$, where $0 \leq t \leq 1$. If $f \in \boldsymbol{m}$ is decreasing, then*

$$\omega\left(f((1-t)A + tB)\right) \leq \sec^5(\theta)\ \omega\left(f(B)\ \sigma_t^{\perp}\ f(A)\right).$$

2. Conclusion

Some well-known inequalities that hold for positive matrices and matrix monotone functions also hold in the sectorial case with specific coefficients. These inequalities apply to the determinant, norm, eigenvalues, and numerical range of these matrices.

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Inequalities involving interpolation paths for sectorial matrices

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Abstract. In this paper, we examine interpolation paths for symmetric matrix means. We define the interpolation paths for both matrix means and power means, followed by an analysis of their properties. Additionally, we present new inequalities for these paths in the context of sectorial matrices.

Keywords: Interpolation paths, Matrix inequalities, matrix means, Sectorial matrices.

AMS Mathematics Subject Classification [2020]: 15A60, 47A64, 47B65.

1. Introduction

Let $\mathbb{M}_n$ denote the set of all $n \times n$ complex matrices. A Hermitian matrix $A \in \mathbb{M}_n$ is said to be positive semidefinite, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. It is called positive definite if it is positive semidefinite and invertible, we will write $A > 0$, the class of positive definite matrices is denoted by $\mathbb{M}_n^+$. A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian or Toeplitz decomposition, $A = \mathcal{R}A + i\mathcal{I}A$, $\mathcal{R}A$ is positive definite ($\mathcal{R}A > 0$), where $\mathcal{R}A = \frac{A+A^*}{2}$, $\mathcal{I}A = \frac{A-A^*}{2i}$. The numerical range of $A \in \mathbb{M}_n$ is defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

The sector region $\mathcal{S}_\alpha$ is defined as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0,\ |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\}.$$

The given statement $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, says that if the numerical range of a matrix A is a subset of a sector region $\mathcal{S}_\alpha$ in the complex plane, then A is called a sectorial matrix and is denoted by $A \in \mathcal{S}_\alpha$. Furthermore, since $0 \notin \mathcal{S}_\alpha$, then each member of $\mathcal{S}_\alpha$ is invertible. It is also stated that a sectorial matrix is necessarily accretive.
A matrix mean σ in the sense of Kubo-Ando [5] is defined by a matrix monotone function $f : (0, \infty) \to (0, \infty)$ with $f(1) = 1$ as

$$A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}, \tag{1}$$

*Speaker.

for positive invertible matrices A and B. The function f that satisfies the conditions for the matrix mean σ is called the "representing function" of σ. For more information about the means and sectors, readers can refer to [1–4]

In this article, we study an matrix version of Uhlmann's interpolation. We recall that an matrix mean σ is symmetric if $A\sigma B = B\sigma A$ for all $A, B \in B(H)$.

We define interpolational path for sector matrices:

Definition 1.1. For a symmetric matrix mean σ, a parameterized matrix mean σ_t is called an interpolational path for σ, if for $A, B \in \mathcal{S}_\alpha$, it satisfies

(*i*) $A\sigma_0 B = A$, $A\sigma_{\frac{1}{2}} B = A\sigma B$ and $A\sigma_1 B = B$

(*ii*) $(A\sigma_p B)\sigma(A\sigma_q B) = A\sigma_{\frac{p+q}{2}} B$ for all $p, q \in [0, 1]$

(*iii*) the map $t \in [0, 1] \mapsto A\sigma_t B$ is norm continuous for each A and B.

Typical interpolational means are so-called power means:

$$Am_r B = A^{\frac{1}{2}} \left(\frac{1 + (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r}{2} \right)^{\frac{1}{r}} A^{\frac{1}{2}} \text{ for } r \in [-1, 1]$$

and their interpolational paths are

$$Am_{r,t} B = A^{\frac{1}{2}} \left(1 - t + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}} \text{ for } t \in [0, 1].$$

For each $r \in [-1, 1]$ and $t \in [0, 1]$, $Am_{r,t} B$ is a path from A to B. Specifically,

$$Am_{1,t} B = A\nabla_t B = (1 - t)A + tB,$$
$$Am_{0,t} B = A\sharp_t B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}},$$
$$Am_{-1,t} B = A!_t B = \left((1 - t)A^{-1} + tB^{-1} \right)^{-1}.$$

They are called the arithmetic, geometric and harmonic interpolations respectively.

In [5], generally, for $A, B \in \mathcal{S}_\alpha$ on $\mathbb{M}_n$, Pečarić et al. defined a norm continuous path of positive invertible matrices $Am_{r,t} B$ by

$$Am_{r,t} B = A^{\frac{1}{2}} \left(1 - t + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

for all real numbers $r \in \mathbb{R}$ and t with $0 \leq t \leq 1$ as an extension of $m_{r,t}$. We also define the representing function $F_{r,t}$ for $m_{r,t}$ by

$$F_{r,t}(x) = 1 m_{r,t} x = (1 - t + tx^r)^{\frac{1}{r}} \text{ for all } x > 0.$$

Then we have

$$Am_{r,t} B = A^{\frac{1}{2}} F_{r,t} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

First, in the following lemmas, we present certain properties that will be used to prove the main results.

Lemma 1.2. *[5, lemma 5.19] Every function $F_{r,t}(x)$ is strictly increasing and strictly convex (resp. strictly concave) for $r > 1$ (resp. $r < 1$).*

Lemma 1.3. *[5, Theorem 5.20] The inequality $|r| \leq 1$ is the equivalent condition that $m_{r,t}$ is matrix mean, or equivalently $F_{r,t}$ is matrix monotone.*

Lemma 1.4. *[5, Theorem 5.21] For each $t \in (0,1)$ a path $Am_{r,t}B$ is nondecreasing and norm continuous for $r \in \mathbb{R}$: For $r \leq s$*

$$Am_{r,t}B \leq Am_{s,t}B$$

Lemma 1.5. *[5, Theorem 5.24] A path $Am_{r,t}B$ is interpolational that*

$$(Am_{r,p}B)m_{r,t}(Am_{r,q}B) = Am_{r,(1-t)p+tq}B$$

for all real numbers $r \in \mathbb{R}$ and $0 \leq p, q, t \leq 1$. In particular, the transposition formula holds:

$$Bm_{r,t}A = Am_{r,1-t}B.$$

Lemma 1.6. *[1, proposition 4.1 and 4.2] Let $A, B \in \mathcal{S}_\alpha$. Then $A\sigma B \in \mathcal{S}_\alpha$ and*

$$\mathcal{R}A\sigma\mathcal{R}B \leq \mathcal{R}(A\sigma B) \leq \sec^2(\alpha)\ (\mathcal{R}A\sigma\mathcal{R}B).$$

2. Main results

Using the definition of the interpolation path and Lemma 1.6, we obtain the following important proposition:

Proposition 2.1. *Let $A, B \in \mathcal{S}_\alpha$ and matrix mean σ_t be the interpolational path for σ. Then the following inequalities hold*

(i) $\mathcal{R}A \leq \mathcal{R}(A\sigma_0 B) \leq \sec^2(\alpha)\mathcal{R}B$ *and* $\mathcal{R}B \leq \mathcal{R}(A\sigma_{\frac{1}{2}}B) \leq \sec^2(\alpha)\mathcal{R}B$,

(ii) $\cos^2(\alpha)\mathcal{R}(A\sigma B) \leq \mathcal{R}(A\sigma_1 B) \leq \sec^2(\alpha)\mathcal{R}(A\sigma B)$,

(iii) $\cos^2(\alpha)\mathcal{R}(A\sigma_{\frac{p+q}{2}}B) \leq \mathcal{R}\left((A\sigma_p B)\sigma\mathcal{R}(A\sigma_q B)\right) \leq \sec^4(\alpha)\mathcal{R}(A\sigma_{\frac{p+q}{2}}B)$;

where $p, q \in [-1, 1]$.

Using Lemma 1.6, we obtain the following inequality, which is an extension of [5, Theorem 5.21]:

Theorem 2.2. *Let $A, B \in \mathcal{S}_\alpha$. For each $t \in (0,1)$, the path $Am_{r,t}B$ is nondecreasing and norm continuous with respect to $r \in [-1, 1]$. in other words, for $r \leq s$, we have*

$$\mathcal{R}(Am_{r,t}B) \leq \sec^2(\alpha)\mathcal{R}(Am_{s,t}B).$$

By applying Lemma 1.6, we have stated and proved an extension of [5, Theorem 5.21] in the following theorem.

Theorem 2.3. *Let $A, B \in \mathcal{S}_\alpha$, for each $t \in (0,1)$ a path $Am_{r,t}B$ is interpolational that*

$$\begin{aligned}\cos^2(\alpha)\mathcal{R}(Am_{r,(1-t)p+tq}B) &\leq \mathcal{R}\left((Am_{r,p}B)m_{r,t}(Am_{r,q}B)\right)\\ &\leq \sec^4(\alpha)\mathcal{R}(Am_{r,(1-t)p+tq}B),\end{aligned}$$

for all real numbers $r \in [-1, 1]$ and $0 \leq p, q, t \leq 1$.

Using the previous theorem, we obtain the following useful result for sectorial matrices.

Corollary 2.4. *Let $A, B \in \mathcal{S}_\alpha$. Then*

$$\begin{aligned}\cos^2(\alpha)\mathcal{R}(Am_{r,1-t}B) &\leq \mathcal{R}\left(Bm_{r,t}A)\right)\\ &\leq \sec^4(\alpha)\mathcal{R}(Am_{r,1-t}B).\end{aligned}$$

3. Conclusion

We investigated how equalities related to interpolation paths and means for positive matrices transform into inequalities in the sectorial case. Additionally, we found that inequalities involving interpolation paths and means for positive matrices similarly transform into corresponding inequalities in the sectorial context.

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A Theoretical Study of Multipliers in Semisimple Banach Algebras

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Abstract. This paper is devoted to a theoretical investigation of multipliers in semisimple Banach algebras. We examine the structure and properties of multiplier operators, emphasizing their characterization via the Gelfand transform. In particular, we show that for commutative semisimple Banach algebras, the space of multipliers can be identified with a function algebra on the maximal ideal space. The results highlight the algebraic and topological aspects of multiplier theory and provide a simplified framework compared to the classical setting.

Keywords: multipliers, Banach algebras, Gelfand transform, semisimple structure

AMS Mathematics Subject Classification [2020]: 46J20, 46H10, 47B48

1. Introduction

The study of multipliers in Banach algebras has a long history, beginning with the pioneering work of Helgason [8] in the context of commutative semisimple Banach algebras. Since then, this theory has been extended and developed by various authors, including Wang [10] and Birtal [6], who examined multipliers in more general and faithful Banach algebras. The notion of multipliers provides a powerful tool to analyze the algebraic and topological structures of Banach algebras and has motivated many developments in functional analysis.

In essence, a multiplier can be viewed either as a bounded linear operator preserving the product structure of a Banach algebra, or equivalently, as a bounded continuous function acting on its maximal ideal space through the Gelfand transform. Larson [9] established that these approaches coincide in the semisimple case, which makes the theory particularly elegant and applicable.

Beyond their abstract significance, multiplier-type operators appear in various branches of mathematics and applied sciences. For example, extensions of multiplier theory, such as quasi-multipliers, naturally arise in the duals of Banach algebras and in several classes of topological algebras [1–5]. Moreover, similar structures occur in applied areas under different names (such as filters), with applications in engineering and signal analysis.

2. Main results

Definition 2.1. Suppose that A is a *commutative Banach algebra with identity* over $\mathbb{C}$. The *maximal ideal space* of A, denoted by $\Delta(A)$, consists of all nonzero homomorphisms from A into $\mathbb{C}$. This space carries the weak-* topology induced from the dual space A^*. Via the kernel map, each element of $\Delta(A)$ corresponds uniquely to a maximal ideal of A.

For each $a \in A$, the *Gelfand transform* associated with a is defined as the mapping

$$\hat{a} : \Delta(A) \to \mathbb{C}, \qquad \hat{a}(\phi) = \phi(a)$$

for all $\phi \in \Delta(A)$.

The correspondence $a \mapsto \hat{a}$ defines an algebra homomorphism from A into $C(\Delta(A))$, the algebra of continuous complex-valued functions on $\Delta(A)$. If A is semisimple (i.e., the Jacobson radical of A is zero), then this map is injective. If the Banach algebra A is *not unital*, then the maximal ideal space $\Delta(A)$ is locally compact but not compact, and the Gelfand transform maps A into $C_0(\Delta(A))$, the algebra of continuous functions vanishing at infinity on $\Delta(A)$. Thus, the unitality of A determines whether the Gelfand transform takes values in $C(\Delta(A))$ or in $C_0(\Delta(A))$.

Remark 2.2. The following theorem is a classical result in the theory of multipliers on commutative Banach algebras; see for example [7, Theorem 4.1.5]. In the classical setting, one usually assumes that the algebra A is *faithful* (or semisimple), which ensures the *uniqueness* of the multiplier function.

However, in our applications to multiplier operators on signals (Section 4), we do not require uniqueness of the multiplier function φ. Therefore, we can state and prove a version of the theorem under the *weaker assumption* that A is commutative, without assuming faithfulness. This allows a simpler and more direct proof than the classical one.

Theorem 2.3. *Let A be a commutative Banach algebra, and let T be a multiplier operator on A. Then there exists a bounded continuous function φ defined on the maximal ideal space $\Delta(A)$ such that for every $a \in A$, the Gelfand transform of $T(a)$ satisfies*

$$\widehat{T(a)} = \varphi \cdot \hat{a}.$$

Proof. For each $m \in \Delta(A)$, choose any $a \in A$ with $\hat{a}(m) \neq 0$ (if none exists, then $\hat{b}(m) = 0$ for all b, and the formula holds trivially). Define

$$\varphi(m) := \frac{\widehat{T(a)}(m)}{\hat{a}(m)}.$$

The multiplier property $T(ab) = aT(b) = bT(a)$ ensures that this definition is independent of the choice of a at points where $\hat{a}(m) \neq 0$. Continuity of φ follows from continuity of the Gelfand transforms $\widehat{T(a)}$ and $\hat{a}$, since locally one can always choose a with $\hat{a}(m) \neq 0$. Finally, for each $a \in A$ and $m \in \Delta(A)$, the definition ensures

$$\widehat{T(a)}(m) = \varphi(m)\hat{a}(m),$$

so

$$\widehat{T(a)} = \varphi \cdot \hat{a}.$$

□

Definition 2.4. A Banach algebra A is called *semisimple* if its radical ideal is trivial, i.e.,

$$\mathrm{rad}(A) = \{0\}.$$

Equivalently, this means that the *Gelfand transform*

$$a \mapsto \hat{a}$$

acts an injective algebra homomorphism from A into $C_0(\Delta(A))$, the algebra of bounded continuous functions on the maximal ideal space $\Delta(A)$. Consequently, the algebraic and topological structure of A is faithfully represented by the function algebra $\hat{A}$.

Note that every commutative semisimple Banach algebra is faithful. Indeed, semisimplicity implies that the Jacobson radical is zero, i.e., $\mathrm{rad}(A) = \{0\}$, which ensures that the Gelfand transform

$$a \mapsto \hat{a}$$

is injective. As a result, if $a \in A$ satisfies $aA = \{0\}$ or $Aa = \{0\}$, then $\hat{a} \cdot \hat{b} = 0$ for all $b \in A$, implying $\hat{a} = 0$, and hence $a = 0$. Therefore, no nonzero element of A can annihilate the whole algebra, and A is faithful.

Corollary 2.5. *Let A be a semisimple commutative Banach algebra. Then the equation $\widehat{T(a)} = \varphi\ \hat{a},\ a \in A$, defines a continuous isomorphism of $M(A)$ onto $C_0(\Delta(A))$.*

Proof. Let A be a commutative semisimple Banach algebra. By definition, semisimplicity means that the Jacobson radical of A is trivial, i.e.,

$$\mathrm{rad}(A) = \{0\},$$

which implies that the Gelfand transform

$$a \mapsto \hat{a}$$

is an injective algebra homomorphism from A into $C_0(\Delta(A))$. By Theorem 2.3, for every multiplier $T \in M(A)$, there corresponds a unique function $\varphi \in C_0(\Delta(A))$, which is bounded and continuous, such that for all $a \in A$,

$$\widehat{T(a)} = \varphi \cdot \hat{a},$$

and

$$\|\varphi\|_\infty \leq \|T\|.$$

Since A is faithful (as every semisimple Banach algebra is faithful), the Gelfand transform is injective and separates points of A. Therefore, the map

$$\Phi : M(A) \to C_0(\Delta(A)), \quad \Phi(T) = \varphi,$$

is injective.

To prove surjectivity, let $\varphi \in C_0(\Delta(A))$ be arbitrary. Define the operator $T_\varphi : A \to A$ by

$$T_\varphi(a) = \mathcal{G}^{-1}(\varphi \hat{a}),$$

where $\mathcal{G}^{-1}$ denotes the inverse of the Gelfand transform restricted to its image. Since multiplication by φ preserves $\hat{A} \subseteq C_0(\Delta(A))$, it follows that T_φ is a multiplier on A.

Moreover, the norm satisfies

$$\|T_\varphi\| = \|\varphi\|_\infty,$$

showing that Φ is an isometry.

Hence, Φ is a continuous algebra isomorphism of $M(A)$ onto $C_0(\Delta(A))$, as claimed. $\square$

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Lie mappings on some matrix rings

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Abstract. We study Lie derivations on *trivial generalized matrix algebras* (TG-algebras), i.e. block algebras of the form $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $MN = NM = 0$, without assuming the existence of a unit. Under natural hypotheses on the center and on faithfulness of the bimodules M, N, we show that every Lie derivation can be decomposed into a sum of an associative derivation, a possible extremal Lie part coming from the off-diagonal blocks, and a center-valued linear map annihilating all commutators. We state precise sufficient conditions for the absence of the extremal part and give illustrative examples.

Keywords: Lie derivation, generalized matrix algebra, trivial extension, central map

AMS Mathematics Subject Classification [2020]: 15A78, 16W10

1. Introduction

Derivations and Lie-derivations encode infinitesimal symmetries of associative algebras and play a central role in ring theory and operator algebras [1–3]. Most classical results treat unital algebras; here we focus on *trivial generalized matrix algebras* without assuming identities in the corner algebras. Such TG-algebras include triangular algebras and trivial extensions, and their Lie-type mappings exhibit phenomena not present in the unital case. Our aim is to describe the general structure of Lie derivations in this setting and to single out algebraic conditions guaranteeing the "standard" decomposition.

2. Preliminaries

Let A and B be associative algebras over a commutative base ring R, and let M be an (A, B)-bimodule and N a (B, A)-bimodule. Assume $MN = 0$ and $NM = 0$. The *trivial generalized matrix algebra* is

$$\mathcal{G} = \begin{pmatrix} A & M \\ N & B \end{pmatrix},$$

with entrywise addition and the natural block multiplication (which vanishes on $M \cdot N$ and $N \cdot M$). For $x, y \in \mathcal{G}$ write $[x, y] = xy - yx$.

A linear map $L : \mathcal{G} \to \mathcal{G}$ is a *Lie derivation* if

$$L([x, y]) = [L(x), y] + [x, L(y)] \qquad (\forall x, y \in \mathcal{G}).$$

*Speaker.

We denote by $Z(\mathcal{G})$ the center. A linear map $\tau : \mathcal{G} \to Z(\mathcal{G})$ is called central-valued and trivial on commutators if $\tau([x, y]) = 0$ for all x, y.

We say M (resp. N) is *faithful* if $aM = 0 \Rightarrow a = 0$ (resp. $Mb = 0 \Rightarrow b = 0$) as usual. We also use π_A, π_B for natural projections of diagonal entries.

3. Main results

The following theorem summarizes the main structural description.

THEOREM 3.1. *Let* $\mathcal{G} = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a trivial generalized matrix algebra with* $MN = NM = 0$. *Assume*

(1) $\pi_A(Z(\mathcal{G})) = Z(A)$ *and* $\pi_B(Z(\mathcal{G})) = Z(B)$,
(2) M *is faithful as an* (A, B)*-bimodule and* N *is faithful as a* (B, A)*-bimodule,*
(3) *for any* $m \in M$ *we have* $Am = 0 \iff mB = 0$ *(and the dual condition for* N*).*

Then every Lie derivation $L : \mathcal{G} \to \mathcal{G}$ *can be written as*

$$L = \mathcal{D} + \Delta + \tau,$$

where $\mathcal{D}$ *is an associative derivation of* $\mathcal{G}$, Δ *is an 'extreme' Lie derivation arising only from the off-diagonal structure (and which vanishes under extra compatibility hypotheses), and* $\tau : \mathcal{G} \to Z_1(\mathcal{G})$ *is linear with* $\tau([x, y]) = 0$ *for all* $x, y \in \mathcal{G}$. *Moreover, if additionally* $A \circ A = A$ *and* $B \circ B = B$ *(i.e. each diagonal algebra is generated by its commutators), then the extreme part* Δ *vanishes and* $L = \mathcal{D} + \tau$.

REMARK 3.2. The theorem generalizes known decompositions for triangular algebras to the non-unital trivial generalized matrix algebras and isolates the precise module-theoretic conditions required in the absence of units.

4. Sketch of proof

The proof proceeds in three steps (details appear in the full paper):

(I) Block form. Writing an arbitrary linear map in 2×2 block form, standard computations using the Lie-derivation identity on diagonal and off-diagonal elementary matrices force the image of each block to have a restricted form. This yields a decomposition of L into ten linear component maps acting on A, M, N, B respectively.

(II) Faithfulness and central constraints. Using faithfulness of M, N and the projection of the center onto A, B, one shows that many component maps must take values in centers $Z(A)$ or $Z(B)$, and certain off-diagonal components are scalar-multiples of the corresponding entries. The implication $Am_0 = 0 \iff m_0B = 0$ is used to eliminate ambiguous components.

(III) Identification of derivation and central part. Isolating the associative part $\mathcal{D}$ is achieved by checking the Leibniz rule on diagonal blocks; the remainder is central-valued and vanishes on commutators. Under the extra commutator-generation hypothesis the extreme off-diagonal Lie part must be zero, completing the proof of Theorem 3.1.

5. Examples and remarks

- If $A = M_n(F)$ and $B = M_m(F)$ and M and N are the natural bimodules with $MN = NM = 0$ (choose M and N as complementary rectangular blocks), then the hypotheses of Theorem 3.1 are satisfied and every Lie derivation has the standard decomposition. This recovers and extends classical matrix results.

- If A or B lacks enough commutators (fails $A \circ A = A$) then nontrivial extreme Lie parts can occur; we give a small explicit 2-by-2 example in the full paper.

6. Conclusion

We provide a comprehensive description of Lie derivations on TG-algebras under natural center and faithfulness conditions. The work clarifies which module-theoretic obstructions arise in the non-unital case and gives criteria ensuring the 'proper' (derivation + central) form. Future work includes applications to operator-algebraic contexts and classification of extreme Lie parts.

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Linear mappings through commutative zero products

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Abstract. This paper studies the structure of linear mappings on generalized matrix algebras that behave like Lie centralizers through commutative zero products. We characterize such mappings and provide necessary and sufficient conditions for their representation in terms of centralizers. Applications to commutant preservers and double commutant preservers are also investigated. Our results extend and unify several known characterizations in the literature.

Keywords: Generalized matrix algebra, Lie centralizer, commutant preserver, Double commutant preserver, Zero product preserving map.

AMS Mathematics Subject Classification [2020]: 47L35, 17B40, 16W25

1. Introduction

Lie centralizers and their various generalizations have been extensively studied in the theory of algebras and operator algebras [1, 2, 4, 5]. A linear mapping Φ on an algebra $\mathcal{A}$ is called a *Lie centralizer* if $\Phi([x, y]) = [\Phi(x), y]$ for all $x, y \in \mathcal{A}$, where $[x, y] = xy - yx$ denotes the Lie product. While every centralizer is a Lie centralizer, the converse does not hold in general.

Note: The main results of this paper are based on our journal paper [3] which has been published in the *Journal of algebra and it's applications.*

In this paper, we investigate linear mappings on generalized matrix algebras that satisfy a weaker condition related to commutative zero products. Specifically, we study mappings $\Phi : \mathcal{G} \to \mathcal{G}$ on a generalized matrix algebra $\mathcal{G}$ that satisfy:

$$X, Y \in \mathcal{G},\ XY = YX = 0 \Longrightarrow [\Phi(X), Y] = 0. \quad (\star)$$

Generalized matrix algebras provide a unified framework that includes many important classes of algebras such as full matrix algebras, triangular algebras, nest algebras, and various operator algebras. The study of linear preservers and mappings with local properties on such algebras has been an active research area.

Our main contributions are:

- Characterizing the structure of linear mappings satisfying condition $(\star)$

*Speaker.

- Establishing necessary and sufficient conditions for such mappings to be representable in terms of centralizers
- Applying these results to characterize commutant preservers and double commutant preservers

2. Preliminaries

Let $\mathcal{R}$ be a commutative unital ring. A *Morita context* consists of two algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules $\mathcal{M}$ (an $(\mathcal{A},\mathcal{B})$-bimodule) and $\mathcal{N}$ (a $(\mathcal{B},\mathcal{A})$-bimodule), and two bimodule homomorphisms satisfying certain compatibility conditions. The associated *generalized matrix algebra* is:

$$\mathcal{G} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

with matrix operations defined in the natural way.

Throughout this paper, we assume $\mathcal{G}$ is a 2-torsion free unital generalized matrix algebra with unity $I = \begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 1_{\mathcal{B}} \end{bmatrix}$. A bimodule $\mathcal{M}$ is called *faithful* if $a\mathcal{M} = 0$ implies $a = 0$ and $\mathcal{M}b = 0$ implies $b = 0$.

The center of $\mathcal{G}$ is characterized by:

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in Z(\mathcal{A}), b \in Z(\mathcal{B}), am = mb, na = bn \ \forall m \in \mathcal{M}, n \in \mathcal{N} \right\}.$$

When $\mathcal{M}$ or $\mathcal{N}$ are faithful, there exists a unique algebra isomorphism $\eta : \pi_{\mathcal{A}}(Z(\mathcal{G})) \to \pi_{\mathcal{B}}(Z(\mathcal{G}))$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $a \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$, $m \in \mathcal{M}$, $n \in \mathcal{N}$.

3. Main Results

We begin by characterizing the general form of linear mappings satisfying condition $(\star)$.

THEOREM 3.1. *Let $\mathcal{G} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ be a 2-torsion free unital generalized matrix algebra and $\Phi : \mathcal{G} \to \mathcal{G}$ a linear mapping satisfying $(\star)$. Then Φ has the form:*

$$\Phi\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ nf_1(1_{\mathcal{A}}) - f_4(1_{\mathcal{A}})n & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

where the component mappings satisfy specific compatibility conditions.

SKETCH OF PROOF. The proof proceeds by analyzing the condition $(\star)$ for various choices of elements with commutative zero products. By considering elements of the form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, we determine the structure of the off-diagonal components. The diagonal components are then analyzed using more complicated products that yield zero. □

Our main characterization theorem provides necessary and sufficient conditions for such mappings to be representable in terms of centralizers.

THEOREM 3.2. *Suppose $\mathcal{M}$ or $\mathcal{N}$ are faithful bimodules in $\mathcal{G}$. Let $\Phi : \mathcal{G} \to \mathcal{G}$ be a linear mapping satisfying $(\star)$ with the form given in Theorem 3.1. Then the following are equivalent:*

(1) $\Phi(X) = \lambda X + \mu(X)$ *for all* $X \in \mathcal{G}$, *where* $\lambda \in Z(\mathcal{G})$ *and* $\mu : \mathcal{G} \to Z(\mathcal{G})$ *is a linear mapping with* $\mu([X, Y]) = 0$ *for all* $X, Y \in \mathcal{G}$ *with* $XY = YX = 0$.
(2) $k_1(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{G}))$ *and* $f_4(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{G}))$.
(3) $k_1(1_{\mathcal{B}}) \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$ *and* $f_4(1_{\mathcal{A}}) \in \pi_{\mathcal{B}}(Z(\mathcal{G}))$.

As an immediate corollary, we obtain:

Corollary 3.3. *Let* $\mathcal{M}$ *or* $\mathcal{N}$ *be faithful bimodules in* $\mathcal{G}$, *and assume* $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ *and* $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$. *Then a linear mapping* $\Phi : \mathcal{G} \to \mathcal{G}$ *satisfies* $(\star)$ *if and only if* $\Phi(X) = \lambda X + \mu(X)$ *for all* $X \in \mathcal{G}$, *where* $\lambda \in Z(\mathcal{G})$ *and* $\mu : \mathcal{G} \to Z(\mathcal{G})$ *is a linear mapping with* $\mu([X, Y]) = 0$ *for all* $X, Y \in \mathcal{G}$ *with* $XY = YX = 0$.

These conditions are satisfied by many important classes of algebras, including full matrix algebras $M_n(\mathcal{A})$ $(n \geq 2)$, upper triangular matrix algebras $T_n(\mathcal{A})$, standard operator algebras on Banach spaces, factor von Neumann algebras, and non-trivial nest algebras.

4. Applications to Commutant Preservers

Let $\mathcal{A}$ be an algebra. For $a \in \mathcal{A}$, the *commutant* of a is $\{a\}' = \{x \in \mathcal{A} : ax = xa\}$, and the *double commutant* is $\{a\}'' = \{x \in \mathcal{A} : xy = yx \ \forall y \in \{a\}'\}$. A linear mapping $\Psi : \mathcal{A} \to \mathcal{A}$ is called a *commutant preserver* if $\{a\}' = \{\Psi(a)\}'$ for all $a \in \mathcal{A}$, and a *double commutant preserver* if $\{\Psi(a)\}'' = \{a\}''$ for all $a \in \mathcal{A}$.

Theorem 4.1. *Let* $\mathcal{M}$ *or* $\mathcal{N}$ *be faithful bimodules in* $\mathcal{G}$, *and assume* $\pi_{\mathcal{A}}(Z(\mathcal{G})) = Z(\mathcal{A})$ *and* $\pi_{\mathcal{B}}(Z(\mathcal{G})) = Z(\mathcal{B})$. *If* $\Psi : \mathcal{G} \to \mathcal{G}$ *is a commutant preserver, then* $\Psi(X) = \lambda X + \mu(X)$ *for all* $X \in \mathcal{G}$, *where* $0 \neq \lambda \in Z(\mathcal{G})$ *and* $\mu : \mathcal{G} \to Z(\mathcal{G})$ *is a linear mapping with* $\mu([X, Y]) = 0$ *for all* $X, Y \in \mathcal{G}$ *with* $XY = YX = 0$.

Proof. If $XY = YX = 0$, then $Y \in \{X\}'$. Since Ψ is a commutant preserver, $Y \in \{\Psi(X)\}'$, so $[\Psi(X), Y] = 0$. Thus Ψ satisfies $(\star)$, and the result follows from Corollary 3.3. The condition $\lambda \neq 0$ follows from the non-commutativity of $\mathcal{G}$. □

Since a linear mapping is a commutant preserver if and only if it is a double commutant preserver, we immediately obtain:

Corollary 4.2. *Under the same hypotheses as Theorem 4.1, if* $\Psi : \mathcal{G} \to \mathcal{G}$ *is a double commutant preserver, then* $\Psi(X) = \lambda X + \mu(X)$ *for all* $X \in \mathcal{G}$, *where* $0 \neq \lambda \in Z(\mathcal{G})$ *and* $\mu : \mathcal{G} \to Z(\mathcal{G})$ *is a linear mapping with* $\mu([X, Y]) = 0$ *for all* $X, Y \in \mathcal{G}$ *with* $XY = YX = 0$.

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Topological Structures on the Space of Module Actions in Banach Algebras

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Abstract. In this paper, we investigate the space of bilinear and separately continuous module actions over the dual of a Banach algebra. We introduce the notions of ultra*-approximate identities and study their role in the structure of module multipliers. Several topologies, including the norm, strict, and quasi-strict topologies, are defined and compared on this space. We prove that these topologies induce the same bounded sets and establish completeness results under suitable conditions. Our approach provides new insights into the interaction between multipliers, approximate identities, and topological structures in Banach algebras.

Keywords: Banach algebras; multipliers; module actions; strict topology; approximate identity

AMS Mathematics Subject Classification [2020]: 46H05; 46H20; 46J10

1. Introduction

Multipliers and module actions play an essential role in the structure theory of Banach algebras [1–3]. They provide tools to study continuity, approximate identities, and dual module structures. In particular, the investigation of strict and quasi-strict topologies on spaces of module multipliers has attracted attention in recent years [6, 7].

In this paper, we introduce the notion of ultra*-approximate identities and study their influence on bilinear and separately continuous module actions over the dual of a Banach algebra. We compare several natural topologies on this space and prove results concerning boundedness and completeness, which extend earlier work on quasi-multipliers and strict topologies [4, 5].

2. Main results

Definition 2.1. Let $m : A^* \times A \to A^*$ be a bilinear map. We say that m defines a *right module action* over A^* if it satisfies the following conditions:

$$m(a \cdot \zeta, b) = a \cdot m(\zeta, b) \quad \text{and} \quad m(\zeta, ba) = m(\zeta, b) \cdot a, \tag{1}$$

for all $\zeta \in A^*$ and $a, b \in A$.

Similarly, a bilinear map $m' : A \times A^* \to A^*$ is called a *left module action* over A^* if the following equalities hold:

$$m'(ab, \zeta) = a \cdot m'(b, \zeta) \quad \text{and} \quad m'(b, \zeta \cdot a) = m'(b, \zeta) \cdot a,$$

for every $\zeta \in A^*$ and $a, b \in A$.

Let $\mathcal{B}_{\text{mod}}(A^*)$ be the set of all bilinear and separately continuous right module actions over A^*. It is clear that $\mathcal{B}_{\text{mod}}(A^*)$ forms a linear space. Moreover, it becomes a Banach space when equipped with the norm

$$\|m\| = \sup \{\|m(\zeta, a)\| : \zeta \in A^*,\ a \in A,\ \|\zeta\| \leq 1,\ \|a\| \leq 1\}.$$

Let A be a Banach algebra. A mapping $T\colon A^* \to A^*$ is called a *right module action* over A^* if

$$T(a \cdot \zeta) = a \cdot T(\zeta), \quad \text{for all } \zeta \in A^*,\ a \in A.$$

We denote by $\mathcal{M}_{\text{mod}}(A^*)$ the space of all bounded linear right module actions over A^*. It is clear that for each $a \in A$, the right multiplication operator R_a defined by

$$R_a(\zeta) = \zeta \cdot a$$

is a right module action over A^*.

Definition 2.2. A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra A is called an *ultra***-approximate identity* if, for every $m \in \mathcal{B}_{\text{mod}}(A^*)$ and every $\zeta \in A^*$, the net $\{m(\zeta, e_\lambda) : \lambda \in I\}$ is Cauchy in A^*.

Theorem 2.3. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity $\{e_\lambda\}$. Then the mapping*

$$\rho\colon \mathcal{M}_{\text{mod}}(A^*) \to \mathcal{B}_{\text{mod}}(A^*), \quad \rho_T(\zeta, a) = (T\zeta) \cdot a$$

for all $T \in \mathcal{M}_{\text{mod}}(A^*)$, $\zeta \in A^*$, and $a \in A$, is bijective with $\|\rho\| \leq 1$. Furthermore, if $\{e_\lambda\}$ is an approximate identity of norm one, then ρ is an isometry.

Note that $\mathcal{B}_{\text{mod}}(A^*)$ can be endowed with an A-bimodule structure as follows. For $m \in \mathcal{B}_{\text{mod}}(A^*)$ and $a \in A$, define the left and right actions $a * m$ and $m * a$ as functions from $A^* \times A$ to A^*, given by

$$\begin{aligned} (a * m)(\zeta, b) &= m(\zeta \cdot a, b), \\ (m * a)(\zeta, b) &= m(\zeta, ab), \end{aligned} \qquad (\zeta \in A^*,\ b \in A).$$

It is straightforward to verify that $a*m$ and $m*a$ both belong to $\mathcal{B}_{\text{mod}}(A^*)$, thus confirming that $\mathcal{B}_{\text{mod}}(A^*)$ indeed forms an A-bimodule.

Let τ denote the norm topology on $\mathcal{B}_{\text{mod}}(A^*)$.

Definition 2.4. The *strict topology* β on $\mathcal{B}_{\text{mod}}(A^*)$ is defined by the family of seminorms

$$m \mapsto \|m * a\|, \qquad (a \in A).$$

Definition 2.5. The *quasi-strict topology* γ on $\mathcal{B}_{\text{mod}}(A^*)$ is defined by the family of seminorms

$$m \mapsto \|m(\zeta, a)\|, \qquad (\zeta \in A^*,\ a \in A).$$

Lemma 2.6. *If A is a factorable Banach algebra, then $\gamma \subseteq \beta \subseteq \tau$.*

Theorem 2.7. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity. Then the topological vector spaces $(\mathcal{B}_{\text{mod}}(A^*), \gamma)$, $(\mathcal{B}_{\text{mod}}(A^*), \tau)$, and $(\mathcal{B}_{\text{mod}}(A^*), \beta)$ have identical collections of bounded subsets.*

Proof. Since $\gamma \subseteq \tau$, every τ-bounded set is necessarily γ-bounded.

Conversely, suppose that $K \subseteq \mathcal{B}_{\mathrm{mod}}(A^*)$ is γ-bounded. Then for each $a \in A$ and $\zeta \in A^*$, there exists a constant $r = r(a, \zeta) > 0$ such that

$$\|m(a, \zeta)\| \leq r, \quad \text{for all } m \in K. \tag{2}$$

For each $m \in K$ and $\zeta \in A^*$, define the operator $M_\zeta : A \to A^*$ by

$$M_\zeta(a) := m(a, \zeta), \quad (a \in A).$$

Let $\mathcal{K} := \{M_\zeta : m \in K\}$. By inequality (2), we have

$$\|M_\zeta(b)\| = \|m(b, \zeta)\| \leq r(b, \zeta), \quad \text{for all } m \in K,\, b \in A.$$

Thus, $\mathcal{K}$ is pointwise bounded. By the principle of uniform boundedness, there exists a constant $c = c(\zeta) > 0$ such that

$$\|M_\zeta\| \leq c, \quad \text{for all } m \in K. \tag{3}$$

Now, for each $m \in K$, define a seminorm λ_m on A^* by

$$\lambda_m(\zeta) := \|M_\zeta\| = \sup_{\|a\| \leq 1} \|m(a, \zeta)\|, \quad (\zeta \in A^*).$$

To verify continuity of λ_m, let $\{\zeta_n\} \subseteq A^*$ converge to $\zeta_0 \in A^*$. Then

$$|\lambda_m(\zeta_n) - \lambda_m(\zeta_0)| \leq \lambda_m(\zeta_n - \zeta_0) = \sup_{\|a\| \leq 1} \|m(a, \zeta_n - \zeta_0)\| \to 0,$$

so λ_m is continuous.

Since the family $\Lambda := \{\lambda_m : m \in K\}$ is pointwise bounded and each λ_m is continuous, the BanachSteinhaus theorem (see [**?**, p. 142]) implies that there exist a ball

$$B(\zeta_0, r) := \{\zeta \in A^* : \|\zeta - \zeta_0\| \leq r\}$$

and a constant $\alpha_0 > 0$ such that

$$\lambda_m(\zeta) \leq \alpha_0, \quad \text{for all } \zeta \in B(\zeta_0, r),\, m \in K.$$

Now, for any $\zeta \in A^*$ with $\|\zeta\| \leq 1$, we estimate

$$\lambda_m(\zeta) = \frac{\lambda_m(r\zeta + \zeta_0 - \zeta_0)}{r} \leq \frac{1}{r}\left(\lambda_m(r\zeta + \zeta_0) + \lambda_m(\zeta_0)\right) \leq \frac{2\alpha_0}{r}.$$

Hence,

$$\|m\| = \sup_{\|a\| \leq 1, \|\zeta\| \leq 1} \|m(a, \zeta)\| = \sup_{\|\zeta\| \leq 1} \lambda_m(\zeta) \leq \frac{2\alpha_0}{r},$$

which shows that K is τ-bounded. This establishes that the bounded sets in $(\mathcal{B}_{\mathrm{mod}}(A^*), \gamma)$ and $(\mathcal{B}_{\mathrm{mod}}(A^*), \tau)$ coincide.

Finally, by Lemma 2.6, we conclude that the bounded sets in all three topologies γ, τ, and β on $\mathcal{B}_{\mathrm{mod}}(A^*)$ coincide. □

Theorem 2.8. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity.*

(i) *The space $(\mathcal{B}_{mod}(A^*), \gamma)$ is complete.*
(ii) *If A admits a norm-one approximate identity, then the space $(\mathcal{B}_{mod}(A^*), \beta)$ is also complete.*

Proof. (i) Let $\{m_\alpha\}_{\alpha\in I}$ be a γ-Cauchy net in $\mathcal{B}_{mod}(A^*)$. By the definition of the γ-topology, for any $\zeta \in A^*$ and $a \in A$, the net $\{m_\alpha(\zeta, a)\}_{\alpha\in I}$ is Cauchy in the norm topology of A^*. Define

$$m(\zeta, a) = \lim_\alpha m_\alpha(\zeta, a).$$

Clearly, m is a bilinear mapping on $A^* \times A$ satisfying condition (1). Moreover, by the Uniform Boundedness Principle [**?**, p. 172], m is separately continuous, and thus $m \in \mathcal{B}_{mod}(A^*)$.

(ii) Suppose now that $\{m_\alpha\}_{\alpha\in I}$ is a β-Cauchy net in $\mathcal{B}_{mod}(A^*)$. For each $a \in A$, define the operator

$$T_a^\alpha : A^* \to A^*, \quad T_a^\alpha(\zeta) = m_\alpha(\zeta, a),$$

which belongs to $\mathcal{M}_{mod}(A^*)$. It follows that $\rho_{T_a^\alpha} = m_\alpha * a$. By the definition of the β-topology, the net $\{\rho_{T_a^\alpha}\}_{\alpha\in I}$ is Cauchy in the norm of $\mathcal{B}_{mod}(A^*)$.

Using Theorem 2.3, we know that the map ρ is an isometry, so the net $\{T_a^\alpha\}$ is Cauchy in the norm of $\mathcal{M}_{mod}(A^*)$. Since $\mathcal{M}_{mod}(A^*)$ is a Banach space, there exists $T_a \in \mathcal{M}_{mod}(A^*)$ such that $\|T_a^\alpha - T_a\| \to 0$.

By Lemma 2.6, the net $\{m_\alpha\}$ is γ-Cauchy. From part (i), we conclude that $\mathcal{B}_{mod}(A^*)$ is γ-complete. Hence, there exists $m \in \mathcal{B}_{mod}(A^*)$ such that

$$\lim_\alpha m_\alpha(\zeta, a) = m(\zeta, a) \quad \text{for all } \zeta \in A^*,\ a \in A.$$

Now, for each $b \in A$, we compute:

$$\begin{aligned}\rho_{T_a}(\zeta, b) &= \lim_\alpha \rho_{T_a^\alpha}(\zeta, b) = \lim_\alpha (m_\alpha * a)(\zeta, b)\\ &= \lim_\alpha m_\alpha(\zeta, ab) = m(\zeta, ab) = (m * a)(\zeta, b).\end{aligned}$$

Therefore,

$$\|m_\alpha * a - m * a\| = \|\rho_{T_a^\alpha} - \rho_{T_a}\| = \|T_a^\alpha - T_a\| \to 0,$$

which implies that m is the β-limit of the net $\{m_\alpha\}_{\alpha\in I}$. Hence, $\mathcal{B}_{mod}(A^*)$ is complete with respect to the β-topology. □

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Investigation Hyers-Ulam stability of hyper 3-homomorphisms and hyper 3-derivations in complex ternary algebras

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Abstract. In this work, we study the notions of *hyper 3-homomorphisms* and *hyper 3-derivations* in the framework of complex ternary algebras. We establish the Hyers–Ulam stability results for these mappings corresponding to the following 3-additive functional equation:

$$f(x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2) = \sum_{i,j,k=1}^{2} f(x_i,\, y_j,\, z_k). \tag{1}$$

In addition, we examine the structure of isomorphisms between complex ternary algebras that are related to the above 3-additive functional equation.

Keywords: 3-additive functional equation, ternary algebra,Hyers-Ulam stability, hyper 3-homomorphism, hyper 3-derivation

AMS Mathematics Subject Classification [2020]: 11E20, 39B52, 39B82

1. Introduction

The first stability problem was posed by Ulam [1], and a first affirmative solution for Banach spaces was provided by Hyers [2]. Let $f : E \to E'$ be a mapping between Banach spaces satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \delta, \quad x, y \in E,\ \delta > 0.$$

Then there exists a unique additive mapping $l : E \to E'$ such that

$$\|f(x) - l(x)\| \le \delta, \quad x \in E.$$

This property is called the *Hyers-Ulam stability* of the additive functional equation $g(x+y) = g(x)+g(y)$. Rassias generalized this result by allowing unbounded Cauchy differences, and Găvruta [3] further extended it using general control functions.

Ternary and n-ary structures have important applications in physics [4]. A *complex ternary algebra* $(A, [\cdot,\cdot,\cdot])$ is a complex vector space equipped with a mapping $[\cdot,\cdot,\cdot] : A^3 \to A$ that is complex-linear in the first and third variables, conjugate-linear in the

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second variable, and satisfies the associativity condition $[[x,y,z],u,v] = [x,[y,z,u],v] = [x,y,[z,u,v]], \quad x,y,z,u,v \in A$. If there exists an element $e \in A$ such that $[e,e,a] = [e,a,e] = [a,e,e] = a$ for all $a \in A$, then A is said to have a *unit*.

A mapping $H : A \to A'$ is a *ternary algebra homomorphism* if

$$H([x,y,z]) = [H(x), H(y), H(z)], \quad x,y,z \in A,$$

and it is an *isomorphism* if bijective. A *ternary derivation* $\delta : A \to A$ satisfies

$$\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)], \quad x,y,z \in A.$$

A mapping $f : X^3 \to X$ is *3-additive* if

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k), \quad x_1,x_2,y_1,y_2,z_1,z_2 \in X,$$

and it is *3-linear* if it is 3-additive and complex-linear in each variable.

Throughout this paper, X is assumed to be a complex ternary algebra, Y a complex ternary Banach algebra, and t a fixed nonzero real number with $|t| < 1$.

2. Main Results

In this section, we prove the Hyers-Ulam stability of hyper 3-homomorphisms in complex ternary algebras and we investigate ternary algebra isomorphisms between complex ternary algebras, associated with the 3-additive functional equation (**??**).

Definition 2.1. Let X and Y be complex ternary algebras. A 3-linear mapping $h : X^3 \to Y$ is called a *hyper 3-additive* mapping if h satisfies

$$8h(x_1,y_1,z_1) = \sum_{i,j,k=1}^{2} h(x_i + (-1)^i x_2,\ y_j + (-1)^j y_2,\ z_1 + (-1)^k z_2) \tag{2}$$

for all $x_1,y_1,z_1,x_2,y_2,z_2 \in X$.

Definition 2.2. Let X and Y be complex ternary algebras. A 3-linear mapping $h : X^3 \to Y$ is called a *hyper 3-homomorphism* if h satisfies

$$h([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) = [h(x_1,x_2,x_3),\ h(y_1,y_2,y_3),\ h(z_1,z_2,z_3)] \tag{3}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in X$.

Theorem 2.3. *Let X and Y be complex ternary algebras and t be a real number satisfying $|t| < 1$. Assume that a mapping $h : X^3 \to Y$ satisfies $h(0,a,b) = h(a,0,b) = h(a,b,0) = 0$ and*

$$\begin{aligned} &\|8h(x_1,y_1,z_1) - \sum_{i,j,k=1}^{2} h(x_1 + (-1)^i x_2,, y_1 + (-1)^j y_2,, z_1 + (-1)^k z_2)\| \\ &\le \|t\left(8\sum_{i,j,k=1}^{2} h\left(\frac{x_1+(-1)^i x_2}{2},, \frac{y_1+(-1)^j y_2}{2},, \frac{z_1+(-1)^k z_2}{2}\right) - 8h(x_1,y_1,z_1)\right)\| \end{aligned} \tag{4}$$

for all $a,b \in X$ and all $(x_1,y_1,z_1),(x_2,y_2,z_2) \in X^3$. Then h is hyper 3-additive.

THEOREM 2.4. *Let X be a complex ternary algebra, Y be a complex ternary Banach algebra, and t be a real number satisfying $|t| < 1$. Let $\varphi : X^6 \to [0, \infty)$ and $\psi : X^9 \to [0, \infty)$ be functions such that*

$$\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty,$$

$$\sum_{j=1}^{\infty} 8^{3j} \psi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in X$. Assume that a mapping $h : X^3 \to Y$ satisfies $h(0, a, b) = h(a, 0, b) = h(a, b, 0) = 0$ and

(5)

$$\begin{aligned}
&\|8\mu h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h(\mu(x_1 + (-1)^i x_2), \mu(y_1 + (-1)^j y_2), \mu(z_1 + (-1)^k z_2))\| \\
&\leq \|t\left(8\mu h(x_1, y_1, z_1) - \sum_{i,j,k=1}^{2} h\left(\mu\frac{x_1 + (-1)^i x_2}{2}, \mu\frac{y_1 + (-1)^j y_2}{2}, \mu\frac{z_1 + (-1)^k z_2}{2}\right)\right)\| \\
&\quad + \varphi(x_1, y_1, z_1, x_2, y_2, z_2)
\end{aligned}$$

for all $a, b \in X$, all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, and all $\mu \in T_1$. Let $h : X^3 \to X$ satisfy

$$\begin{aligned}
&\big\|h([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\
&\quad - [h(x_1, x_2, x_3), h(y_1, y_2, y_3), h(z_1, z_2, x_3)]\big\| \\
&\leq \psi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)
\end{aligned} \tag{6}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Then there exists a unique hyper 3-homomorphism $H : X^3 \to Y$ such that

$$\|h(x, y, z) - H(x, y, z)\| \leq \sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \tag{7}$$

for all $x, y, z \in X$.

THEOREM 2.5. *Let X be a complex ternary algebra, Y be a complex ternary Banach algebra, and t be a real number satisfying $|t| < 1$. Let $h : X^3 \to Y$ be a bijective mapping satisfying (5) such that*

$$h([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) = [h(x_1, x_2, x_3), h(y_1, y_2, y_3), h(z_1, z_2, z_3)] \tag{8}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$.

If $h(\alpha x_0, \beta y_0, \gamma z_0)$ is continuous in $\alpha, \beta, \gamma \in \mathbb{R}$ for each fixed $(x_0, y_0, z_0) \in X^3$ and $\lim_{n\to+\infty} 8^n h\left(\frac{e}{2^n}, \frac{e}{2^n}, \frac{e}{2^n}\right) = e'$, then the mapping $h : X^3 \to Y$ is a hyper 3-isomorphism.

In this following, we prove the Hyers-Ulam stability of hyper 3-derivations in complex ternary algebras.

DEFINITION 2.6. Let X be a ternary algebra. A 3-linear mapping $f : X^3 \to X$ is called a *hyper 3-derivation* if f satisfies

$$\begin{aligned}
&f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\
&\quad = [f(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]]
\end{aligned}$$

$$+ [[x_1, x_2, x_3], f(y_1, y_2, y_3), [z_1, z_2, z_3]]$$
$$+ [[x_1, x_2, x_3], [y_1, y_2, y_3], f(z_1, z_2, z_3)]$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$.

THEOREM 2.7. *Let X be a ternary Banach algebra and t be a real number satisfying $|t| < 1$. Let $\varphi : X^6 \to [0, \infty)$ and $\psi : X^9 \to [0, \infty)$ be functions such that*

$$\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty$$

and

$$\sum_{j=1}^{\infty} 8^{3j} \psi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}, \frac{z}{2^j}\right) < \infty$$

for all $x, y, z \in X$. Let $f : X^3 \to X$ be a mapping satisfying

$$\|f\big(\mu(x_1 + x_2), \mu(y_1 + y_2), \mu(z_1 + z_2)\big) - \mu \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k)\|$$
$$\leq \|t\left(8f\left(\mu\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)) - \mu \sum_{i,j,k=1}^{2} f(x_i, y_j, z_k)\right)\|$$
$$+ \varphi(x_1, y_1, z_1, x_2, y_2, z_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$ and all $\mu \in T_1$, and

$$\big\|f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) - [f(x_1, x_2, x_3), [y_1, y_2, y_3], [z_1, z_2, z_3]]$$
$$- [[x_1, x_2, x_3], f(y_1, y_2, y_3), [z_1, z_2, z_3]] - [[x_1, x_2, x_3], [y_1, y_2, y_3], f(z_1, z_2, z_3)]\big\|$$
$$\leq \psi(x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in X$. Then there exists a unique hyper 3-derivation $D : X^3 \to X$ such that

$$\|f(x, y, z) - D(x, y, z)\| \leq \sum_{j=0}^{\infty} 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}, \frac{z}{2^{j+1}}\right) \tag{9}$$

for all $x, y, z \in X$.

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Generalized Hyers–Ulam Stability of an Additive Quadratic Functional Equation in 2-Banach Space

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Abstract. In this paper, the Hyers–Ulam stability of an additive quadratic-type functional equation is investigated, and a generalized form of the Hyers–Ulam stability for the equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y)$$

is obtained for any integer $a \neq -1, 0, 1$ in a 2-Banach space.

Keywords: Quadratic function, Additive function, Hyers–Ulam stability, 2-Banach space.
AMS Mathematics Subject Classification [2020]: 46K05, 46H05.

1. Introduction

The stability of a functional equation for a function from a normed space to a Banach space was first studied by Hyers [3]. He proved that for a function $f : X \to Y$, where X is a normed space and Y is a Banach space, if

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for each $x, y \in X$ and some $\delta > 0$, then there exists a unique additive function $T : X \to Y$ such that

$$\|f(x)-T(x)\| \leq \delta$$

for all $x \in X$. This provides a positive answer to a problem raised by Ulam [5] for a functional equation on a metric group. The stability of functional equations from normed spaces to Banach spaces has since been widely studied by many authors. The terminology *generalized Hyers–Ulam stability* originates from these historical foundations.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is called a *quadratic functional equation*, and every solution of the quadratic functional equation (1) is said to be a quadratic function. The stability problem for the quadratic functional equation (1) has been solved by many authors [1].

*Speaker.

In this paper, we investigate the Hyers–Ulam stability of the following functional equations:

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y); \tag{2}$$

$$f(x+\alpha y)+\alpha f(x-y)=f(x-\alpha y)+\alpha f(x+y), \tag{3}$$

for any fixed integer $\alpha \neq -1, 0, 1$, introduced by Jun and Kim [4], for a function from a 2-normed space to a 2-Banach space. It is shown in [4] that these two functional equations (2) and (3) are equivalent, and such f satisfies

$$f(x)=B(x,x)+A(x)+f(0),$$

for each $x \in E_1$, where $B : E_1 \times E_1 \to E_2$ is a symmetric biadditive mapping, and $A : E_1 \to E_2$ is additive. In the 1960s, Gähler [2] introduced the concept of 2-normed spaces.

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$, and let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ be a function satisfying:

(1) $\|x,y\|=0$ if and only if x and y are linearly dependent,
(2) $\|x,y\|=\|y,x\|$,
(3) $\|ax,y\|=|a|\,\|x,y\|$,
(4) $\|x,y+z\| \leq \|x,y\|+\|x,z\|$,

for each $x,y,z \in X$ and $a \in \mathbb{R}$. Then $\|\cdot,\cdot\|$ is called a *2-norm* on X, and $(X,\|\cdot,\cdot\|)$ is called a *2-normed space*.

We introduce a basic property of 2-normed spaces as follows. Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space, $x \in X$, and suppose $\|x,y\|=0$ for each $y \in X$. If $x \neq 0$, then since $\dim X > 1$, there exists $y \in X$ such that $\{x,y\}$ is linearly independent, hence $\|x,y\| \neq 0$, a contradiction. Therefore, we have the following lemma.

Lemma 1.2. *Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. If $x \in X$ and $\|x,y\|=0$ for each $y \in X$, then $x=0$.*

Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. For $x,z \in X$, let $p_z(x)=\|x,z\|$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) \geq 0$, $p_z(\alpha x)=|\alpha|p_z(x)$, and

$$p_z(x+y)=\|x+y,z\|=\|z,x+y\| \leq \|z,x\|+\|z,y\|=p_z(x)+p_z(y),$$

for all $\alpha \in \mathbb{R}$ and $x,y \in X$. Thus, p_z is a seminorm for each $z \in X$. By Lemma 1.2, if $p_z(x)=0$ for all $z \in X$, then $x=0$. Hence, the family $\{p_z : z \in X\}$ is a separating family of seminorms, inducing a locally convex topology on X, which makes X a locally convex topological vector space.

Definition 1.3. A sequence $\{x_n\}$ in a 2-normed space X is called a *2-Cauchy sequence* if

$$\lim_{m,n\to\infty} \|x_n-x_m,x\|=0$$

for each $x \in X$.

Definition 1.4. A sequence $\{x_n\}$ in a 2-normed space X is called a *2-convergent sequence* if there exists $x \in X$ such that

$$\lim_{n\to\infty} \|x_n-x,y\|=0$$

for each $y \in X$. If $\{x_n\}$ converges to x, we write $\lim_{n\to\infty} x_n = x$.

Definition 1.5. We say that a 2-normed space $(X, \|\cdot,\cdot\|)$ is a *2-Banach space* if every 2-Cauchy sequence in X is 2-convergent in X.

By using properties (2) and (4) of Definition 1.1, one can see that $\|\cdot,\cdot\|$ is continuous in each component. More precisely, for a convergent sequence $\{x_n\}$ in a 2-normed space X,

$$\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$$

for each $y \in X$.

2. Main Results

Throughout this section, let X be a real normed linear space, equipped with a 2-norm that makes $(X, \|\cdot,\cdot\|)$ a 2-Banach space. For a mapping $f : (X, \|\cdot,\cdot\|) \to (X, \|\cdot,\cdot\|)$, define

$$\Delta_f(x; y) = f(x + ay) + af(x - y) - f(x - ay) - af(x + y),$$

for each $x, y \in X$ and $a \neq -1, 0, 1$.

Theorem 2.1. *Let $\varepsilon \geq 0$, $0 < p < 2$, $r > 0$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x; y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to X$ such that

$$\left\| \frac{f(x) + f(-x)}{2} - Q(x) - f(0), z \right\| \leq \varepsilon \|x\|^p \|z\|^r \frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)}$$

for each $x, z \in X$.

Theorem 2.2. *Let $\varepsilon > 0$, $r > 0$, $p > 2$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x; y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to X$ such that

$$\left\| \frac{f(x) + f(-x)}{2} - Q(x) - f(0), z \right\| \leq \varepsilon \|x\|^p \|z\|^r \frac{|a|^p + 2|a| + 1}{2^p(|a|^p - |a|^2)}$$

for each $x, z \in X$.

Theorem 2.3. *Let $\varepsilon > 0$, $r > 0$, $0 < p < 1$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x; y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to X$ such that

$$\left\| \frac{f(x) - f(-x)}{2} - A(x), z \right\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{2(|a| - |a|^p)}$$

for each $x, z \in X$.

Theorem 2.4. *Let $\varepsilon > 0$, $r > 0$, $p > 1$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x; y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to X$ such that

$$\left\| \frac{f(x) - f(-x)}{2} - A(x), z \right\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{2(|a|^p - |a|)}$$

for each $x, z \in X$.

Corollary 2.5. *Let $\varepsilon > 0$, $r > 0$, $0 < p < 1$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x;y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to X$ and a unique quadratic mapping $Q : X \to X$ such that

$$\|f(x) - Q(x) - A(x) - f(0), z\| \leq \varepsilon \|x\|^p \|z\|^r \left[\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} + \frac{1}{2(|a| - |a|^p)}\right]$$

for all $x, z \in X$.

Corollary 2.6. *Let $\varepsilon > 0$, $r > 0$, $p > 2$, and $f : X \to X$ satisfy*

$$\|\Delta_f(x;y), z\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right) \|z\|^r$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to X$ and a unique quadratic mapping $Q : X \to X$ such that

$$\|f(x) - Q(x) - A(x) - f(0), z\| \leq \varepsilon \|x\|^p \|z\|^r \left[\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} + \frac{1}{2(|a| - |a|^p)}\right]$$

for all $x, z \in X$.

3. Conclusion

We have proved the generalized Hyers–Ulam stability of the additive quadratic-type functional equation

$$f(x + ay) + af(x - y) = f(x - ay) + af(x + y)$$

for any integer $a \neq -1, 0, 1$ in a 2-Banach space. This result provides an extension and improvement in the field of functional equations.

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Solvability of an infinite system of the fractional order non-linear fractional Hadamard functional integral equations in the Banach algebra $(C(\mathfrak{J}))^{\infty}$

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Abstract. In the present paper, we investigate the existence of solutions of an infinite system of the fractional order non-linear Hadamard functional integral equations. The problem is discussed in the sequence Banach algebra $(C(\mathfrak{J}))^{\infty}$ ($\mathfrak{J}$ is a bounded interval), using the theory of measure of non-compactness together with the fixed point theory in Banach algebras. An example is provided to support our main result.

Keywords: Hadamard functional integral equation, infinite system, measure of non-compactness

AMS Mathematics Subject Classification [2020]: 45G10, 26A33

1. Introduction

The theory of fractional functional integral equations is an extraordinary and prestigious branch of nonlinear analysis and seeks various invocations in demonstrating numerous real-life scenarios. It also plays a significant role in some fields such as biology, traffic theory, the theory of optimal control, economics, etc. Extensive studies on the existence solutions of these equations have been done by employing the technique of measure of non-compactness (for short MNC) together with the fixed point theory (FPT) (see [1]). The degree of noncompactness of a bounded set is described by means of functions called measures of noncompactness. They were first introduced by Kuratowski. They have been effectively used to study the solvability of various kinds of boundary value problems for differential equations and their systems in sequence spaces. In such studies, the existence of solutions is proved with the help of the fixed point theory. Verifying the behavioral properties of solutions is also extensively studied. In recent times, the fixed point theory is applicable in different scientific fields suggested by Stephen Banach.

Now, by using the technique of measure of noncompactness, we study the solvability of the infinite system of functional non-linear integral equation with the Hadamard fractional operator

$$x_i(\tau) = F\Big(\tau, x_i(\tau), \frac{1}{\Gamma(\alpha)}\int_1^{\tau} \frac{u(\tau, s, x(\varphi_1(s)))}{s}\Big(\log\frac{\tau}{s}\Big)^{\alpha-1} ds, x_i(\Psi_1(s))\Big)\times \tag{1}$$

*Speaker.

$$G\Big(\tau, x_i(\tau), \int_1^\tau v(\tau, s, x(\varphi_2(s)))ds, x_i(\Psi_2(\tau))\Big),\ \tau \in \mathfrak{J} = [1, T], \quad i \in \mathbb{N},$$

where Γ is the gamma function, $u, v : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R}^\infty \to \mathbb{R}$ are Lebesgue integral functions and $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ are two functions. The principal construction of this study is as follows: Section 2 explores some notations, definitions, and preliminary results. In Section 3, we adopt the technique established by the applications of MNC and FPT in Banach algebra to find the existence result. Also, we present an example to illustrate our existence result.

2. Preliminaries

For a real Banach space $(\Omega, \|.\|)$, let us denote

- $D(z, \iota) = \{\varsigma \in \Omega :\ \|\varsigma - z\| \leq \iota\}$.
- If $\emptyset \neq \Lambda \subset \Omega$, then $\overline{\Lambda}$ and $\mathrm{Conv}\Lambda$ are the closure and closed convex hull of Λ, respectively.
- $\mathfrak{M}_\Omega = \{\Lambda \subset \Omega :\ \emptyset \neq \Lambda$ and it is bounded$\}$.
- $\mathfrak{N}_\Omega = \{\Lambda \in \mathfrak{M}_\Omega :\ \Lambda$ is relatively compact$\}$.

Definition 2.1. [1] The function $\tilde{\mu} : \mathfrak{M}_\Omega \to [0, +\infty)$ is a measure of noncompactness (MNC) in Ω if for any $\mathcal{A}, \mathcal{B} \in \mathfrak{M}_\Omega$ it satisfies the properties:

(i) $\emptyset \neq \ker \tilde{\mu} = \{\mathcal{A} \in \mathfrak{M}_\Omega : \tilde{\mu}(\mathcal{A}) = 0\} \subseteq \mathfrak{N}_\Omega$.
(ii) If $\mathcal{A} \subset \mathcal{B}$, then $\tilde{\mu}(\mathcal{A}) \leq \tilde{\mu}(\mathcal{B})$.
(iii) $\tilde{\mu}(\overline{\mathcal{A}}) = \tilde{\mu}(\mathrm{Conv}\mathcal{A}) = \tilde{\mu}(\mathcal{A})$.
(iv) For each $\kappa \in [0, 1]$, $\tilde{\mu}(\kappa\mathcal{A} + (1 - \kappa)\mathcal{B}) \leq \kappa\tilde{\mu}(\mathcal{A}) + (1 - \kappa)\tilde{\mu}(\mathcal{B})$.
(v) If for each $\alpha \in \mathbb{N}$, $\mathcal{A}_\alpha = \overline{\mathcal{A}_\alpha} \in \mathfrak{M}_\Omega$, $\mathcal{A}_{\alpha+1} \subset \mathcal{A}_\alpha$ and $\lim_{\alpha\to\infty} \tilde{\mu}(\mathcal{A}_\alpha) = 0$, then $\mathcal{A}_\infty = \bigcap_{\alpha=1}^{\infty} \mathcal{A}_\alpha \neq \emptyset$.

Definition 2.2. [1] Let $(\mathcal{W}, d)$ be a metric space, $\mathfrak{M}_\mathcal{W}$ be the family of all bounded subsets of $\mathcal{W}$, and let $\aleph \in \mathfrak{M}_\mathcal{W}$. The Kuratowski measure of noncompactness of $\aleph$ is

$$\alpha(\aleph) = \inf\Big\{\xi > 0 : \aleph \subset \bigcup_{i=1}^{n} \ell_i,\ \ell_i \subset \mathcal{W},\ \mathrm{diam}(\ell_i) < \xi,\ n \in \mathbb{N}\Big\},$$

where $\mathrm{diam}(\ell_i) = \sup\{d(w, u) : w, u \in \ell_i\}$ and ball measure of noncompactness (Hausdorff MNC) is $\chi(\aleph) = \inf\Big\{\xi > 0 : \aleph \subset \bigcup_{i=1}^{n} B(w_i, \zeta_i),\ w_i \in \mathcal{W},\ \zeta_i < \xi,\ n \in \mathbb{N}\Big\}$.

Definition 2.3. Suppose that $\mathfrak{J} = [1, T]$. The Banach algebra $C(\mathfrak{J})$ is equipped with the usual norm $\|f\|_{C(\mathfrak{J})} = \sup\{|f(t)| : t \in \mathfrak{J}\}$, for $f \in C(\mathfrak{J})$. Let us suppose $\mathfrak{W}$ is a nonempty, bounded, fixed subset of $C(\mathfrak{J})$, $\varrho > 0$, and $x \in \mathfrak{W}$. we define $\omega(x, \varrho)$, the modulus of continuity of x as

$$\omega(x, \varrho) = \sup\{|x(\tau_1) - x(\tau_2)| : \tau_1, \tau_2 \in \mathfrak{J}, |\tau_1 - \tau_2| \leq \varrho\}.$$

Further, we define

$$\omega(\mathfrak{W}, \varrho) = \sup\{\omega(x, \varrho) : x \in \mathfrak{W}\},$$
$$\omega_0(\mathfrak{W}) = \lim_{\varrho\to 0} \omega(\mathfrak{W}, \varrho),$$

where $\omega_0(\mathfrak{W})$ is an MNC in the space $C(\mathfrak{J})$. Furthermore, $\eta(\mathfrak{W}) = \frac{1}{2}\omega_0(\mathfrak{W})$ is a ball MNC in the space $C(\mathfrak{J})$.

Definition 2.4. [4] The Hadamard integral of fractional order $\alpha \in \mathbb{R}^+$ for a map $f \in L^1([1, T], \Omega)$ is defined by

$$I^\alpha f(\tau) = \frac{1}{\Gamma(\alpha)} \int_1^\alpha \frac{f(s)}{s} \Big(\log \frac{\tau}{s}\Big)^{\alpha-1} ds,$$

where the gamma function, Γ is defined as $\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau$.

THEOREM 2.5. *[2] Consider $\mathfrak{M}$ to be a nonempty, closed, convex, and bounded subset of $C[1,T]$, together with the operators F and G , which transform continuously on the set $\mathfrak{M}$ into $C[1,T]$ in such a manner that $F(\mathfrak{M})$ and $G(\mathfrak{M})$ are bounded. In addition, suppose that the operator $S = F \times G$ transforms $\mathfrak{M}$ into itself. Further, each of the operators F and G fulfill the Darbo condition on the set $\mathfrak{M}$, together with the constants κ_1 and κ_2, correspondingly. Then, the operator S fulfills the Darbo condition on $\mathfrak{M}$ along with the constant $\|F(\mathfrak{M})\|\kappa_2 + \|G(\mathfrak{M})\|\kappa_1$, provided that*

$$S(\mathfrak{M}) \leq \|F(\mathfrak{M})\|\kappa_2 + \|G(\mathfrak{M})\|\kappa_1.$$

Moreover, if $\|F(\mathfrak{M})\|\kappa_2 + \|G(\mathfrak{M})\|\kappa_1 < 1$, then S possesses at least one fixed point in the set $\mathfrak{M}$.

3. Main results

In this section, we first introduce an MNC in the sequence Banach algebra $(C(\mathfrak{J}))^\infty$.

PROPOSITION 3.1. *Suppose that $\mathfrak{J} = [1,T]$. The Banach algebra structure of the sequence space $(C(\mathfrak{J}))^\infty$ is equipped with the multiplication componentwise and the norm*

$$\|x\|_{(C(\mathfrak{J}))^\infty} = \sup \|\pi_i(x)\|_{C(\mathfrak{J})}$$

for $x = (x_i) \in (C(\mathfrak{J}))^\infty$.

PROPOSITION 3.2. *Let $\mathfrak{W}$ be a nonempty and bounded subset of $(C(\mathfrak{J}))^\infty$. Then*

$$\varpi_0(\mathfrak{W}) = \sup_{i\in\mathbb{N}} \omega_0(\pi_i(\mathfrak{W}))$$

is an MNC in $(C(\mathfrak{J}))^\infty$, where ω_0 is defined as Definition 2.3. Furthermore, $\bar{\eta}(\mathfrak{W}) = \frac{1}{2}\varpi_0(\mathfrak{W})$ is a ball MNC in the Banach algebra $(C(\mathfrak{J}))^\infty$.

Here, we discuss the existence of solution of the infinite system of functional nonlinear integral equation with the Hadamard fractional operator (1) in the Banach algebra $(C(\mathfrak{J}))^\infty$.

Consider the following hypotheses:

(I) The maps $u, v : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R}^\infty \to \mathbb{R}$ are continuous. Moreover, there are increasing maps $\Phi, \Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\tau, s \in \mathfrak{J}$ and $x \in (C(\mathfrak{J}))^\infty$ we have

$$|u(\tau, s, x(\varphi_1(s)))| \leq \Phi(\|x\|_{(C(\mathfrak{J}))^\infty}),$$

$$|v(\tau, s, x(\varphi_2(s)))| \leq \Psi(\|x\|_{(C(\mathfrak{J}))^\infty}).$$

(II) There are continuous maps $B_i : \mathfrak{J} \to \mathfrak{J}$, for $i = 1,2,3,4,5,6$ in such a way

$$|F(\tau, y_1, y_2, y_3) - F(\tau, \acute{y}_1, \acute{y}_2, \acute{y}_3)| \leq B_1(\tau)|y_1 - \acute{y}_1| + B_2(\tau)|y_2 - \acute{y}_2| + B_3(\tau)|y_3 - \acute{y}_3|,$$

$$|G(\tau, y_1, y_2, y_3) - G(\tau, \acute{y}_1, \acute{y}_2, \acute{y}_3)| \leq B_4(\tau)|y_1 - \acute{y}_1| + B_5(\tau)|y_2 - \acute{y}_2| + B_6(\tau)|y_3 - \acute{y}_3|,$$

for all $\tau \in \mathfrak{J}$ and $y_1, y_2, y_3, \acute{y}_1, \acute{y}_2, \acute{y}_3 \in \mathbb{R}$. Furthermore, there is a non-negative constant Ξ so that

$$\Xi = \max_{1\leq i\leq 6}\{B_i(\tau) : \tau \in \mathfrak{J}\}.$$

(III) There is a nonnegative constant P_1 so that

$$|F(\tau, 0, 0, 0) \leq P_1,$$

$$|G(\tau, 0, 0, 0) \leq P_1,$$

for all $\tau \in \mathfrak{J}$.

(IV) There is a solution $r_0 > 0$ of the following inequality

$$(2\Xi r_0 + P_1 + \frac{\Phi(r_0)}{\Gamma(\alpha+1)}(\log T)^\alpha) \times (2\Xi r_0 + P_1 + \Psi(r_0)(T-1)) \leq r_0.$$

Moreover,

$$(2\Xi r_0 + P_1 + \Phi(r_0)(\log T)^\alpha)2\Xi + (2\Xi r_0 + P_1 + \Psi(r_0)(T-1))(2\Xi + \frac{\Phi(r_0)}{\Gamma(\alpha+1)}(\log T)^\alpha) < 1.$$

The new result of the our study is as follows.

THEOREM 3.3. *Under the hypotheses* (I)*-*(IV)*, the infinite system* (1) *has at least one solution in* $(C(\mathfrak{J}))^\infty$.

We examine the following example to validate our result.

EXAMPLE 3.4. Let us suppose the following infinite system of non-linear Hadamard fractional functional integral equation

(2)

$$x_i(\tau) = \Big(\frac{1}{9\Gamma(\alpha)}\int_1^\tau \frac{\frac{\tau+s}{7}\sin(\|x(\sqrt{s})\|)}{s}\Big(\log\frac{\tau}{s}\Big)^{\alpha-1}ds\Big) + \frac{1}{14}\int_1^2 \frac{\cos(\|x(1-s)\|)}{2} + 3\tau^2\arctan(\frac{\|x(1-s)\|}{1+\|x(1-s)\|})ds,$$

where $i \in \mathbb{N}$. The infinite system (2) is a special case of the infinite system (1) in the Banach algebra $(C([1,2])^\infty$. Indeed, $F(\tau,x,y,z) = \frac{1}{9}y$, $G(\tau,x,y,z) = \frac{1}{14}y$. Thus, we have $B_1 = B_3 = B_4 = B_6 = 0$, $B_2(\tau) = \frac{1}{9}$, $B_5(\tau) = \frac{1}{14}$, and $P_1 = 0$. In this case, we have

$$\Xi = \max\{\frac{1}{14}, \frac{1}{9}\} = \frac{1}{9}.$$

Furthermore, the maps $u, v : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R}^\infty \to \mathbb{R}$ are continuous, where $\mathfrak{J} = [1,2]$. The first and second inequalities, arising in Hypothesis (IV) are fulfilled for $r_0 = 1$ and for each $\alpha > 0$. Hence, $r_0 = 1$ is a solution of the inequalities which have occurred in Hypothesis (IV).

Now, all the hypotheses from (I) to (IV) are fulfilled and therefore in accordance with Theorem 3.3, the infinite system 2 has at least one solution in $(C([1,2])^\infty$.

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Existence Results of a Nonlinear Fractional Integral Equation in the Space of Regulated Functions

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Abstract. In the present work, we introduce a measure of noncompactness on the space of regulated functions defined on Davison spaces. By using the technique of the measure of noncompactness and a fixed point theorem of Darbo type, we study the existence of solutions of a nonlinear fractional integral equation of two variables in the space of regulated functions. Also, we add an example in verification of our existence results.

Keywords: Davison space, fractional integral equation, measure of noncompactness, regulated function

AMS Mathematics Subject Classification [2020]: 47H10, 47H08, 26A33

1. Introduction

The fractional calculus is a significant topic for studying natural phenomena in many fields including biology, biophysics and dynamical systems coupled with lots of mathematical modelings formulated. A new approach for solving fractional differential and integral equations consists in applying new measures of noncompactness (for short MNC) [2] for some Banach spaces. In this way, extensions of some fixed point theorems such as extension of Darbo's fixed point theorem via measure of noncompactness can be very useful. The notion of a regulated function was presented in the middle of the twentieth century. Subsequently, some scholars introduced this notion from different points of view and obtained some of its applications. Particularly the approach offered in [4] seems to be appropriate and transparent. In 2019, Olszowy [5] constructed arithmetically convenient measures of noncompactness in the spaces of regulated functions. Davison generalized regulated functions on a topological space [3] and proved some remarkable facts concerning these new types of functions. However, up to now, it has not been defined an MNC in the space of regulated functions $S(Y, Z)$ for a Davison space (Y, τ, σ). In this work, we are going to construct such MNC. Fractional integral equations appear in various real-life problems. Some mathematicians focus on such type of equations with the help of fixed point theory. Rabbani et al. [6] studied the existence of solution of a fractional integral equation in two variables, by using the technique of measure of noncompactness. In the

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present manuscript, we are motivated to discuss the solvability of the following nonlinear fractional integral equation of two variables

$$x(t,s) = G(t,s) + F\Big(t, s, \int_0^t \int_0^s \frac{u(t,s,v,w,x(v,w))}{(s-v)^{1-\alpha}(t-w)^{1-\beta}} dvdw, x(t,s)\Big) \tag{1}$$

in the space of regulated functions defined on a Davison space, where $\alpha, \beta \in (0,1)$ and $t, s \in [0,T]$. This type of integral equations may be occurred in the mathematical physics modeling, engineering, and so on. We organize this paper as follows. In Section 2, we collect some auxiliary facts. In Section 3, we aim to characterize relatively compact sets in the space of regulated functions defined on a Davison space, and then we construct an MNC on this space. In Section 4, we review the existence of solutions of the fractional integral equation in two variables (1) in a regulated space. We also provide an example to show applicability of our existence theorem.

2. Preliminaries

In [3] Davison introduced the concepts of Davison space and regulated functions.

Definition 2.1. A triple (Y, τ, σ) is called a Davison space if (Y, τ) is a topological space and σ is an algebra of sets on Y, so that $\tau \bigcap \sigma$ is a base for τ.

Example 2.2. Assume that $\Omega \subseteq \mathbb{R}^2$ is a compact set. Also, assume that τ is the usual topology on $\mathbb{R}^2$ and ω is the algebra of sets with finite perimeter. Then this is a Davison space.

Definition 2.3. [3] Let (Y, τ, σ) be a Davison space and $(Z, \|.\|_Z)$ be a normed space. Then $h : Y \to Z$ is called σ-regulated at $y \in Y$ if for each $\varepsilon > 0$, there exist $B_1, B_2, \ldots, B_n \in \sigma$ such that $y \in \text{int}(\bigcup_{i=1}^{n} B_i)$ and $\|h(y_1) - h(y_2)\|_Z < \varepsilon$ for each $y_1, y_2 \in B_i$ $(i = 1, 2, \ldots, n)$.
If h is σ-regulated (regulated for short) at each point of Y, then h is σ-regulated on Y.

Continuous functions are examples of regulated functions. Now, suppose that (Y, τ, σ) is a Davison space and $(Z, \|.\|_Z)$ is a normed space. Also, suppose that $B(Y, Z)$ is the set of all bounded functions, i.e., the set of functions $f : Y \to Z$ in which $\|f\|_{sup} = \sup\{\|f(y)\|_Z : y \in Y\} \leq M$ for some $M > 0$. The space of regulated functions from Y to Z is denoted by $S(Y, Z)$. It is known that if Y is compact, then $S(Y, Z) = \overline{S(Y, Z)} \subseteq B(Y, Z)$ [3].
We end this section by recalling the well-known Darbo type fixed point theorem.

Theorem 2.4. *[1] Suppose that Ω is a nonempty, bounded, closed and convex subset of a Banach space Υ and $H : \Omega \to \Omega$ is a continuous mapping. Assume that there exists a constant $k < 1$ such that $\eta(HX) \leq k\eta(X)$ for any nonempty subset X of Ω, where η is an MNC defined in Υ. Then H has a fixed point in the set Ω.*

3. On the Space of Regulated Functions

In this part, we aim to characterize relative compact sets in the regulated function space $S(Y, Z)$ (Y is a Davison space). Then we introduce an MNC on this space.

Definition 3.1. Suppose that $\mathfrak{S} \subseteq S(Y, Z)$. We say that $\mathfrak{S}$ is equiregulated if for each $\varepsilon > 0$ and $y \in Y$ there exist $B_1, B_2, \ldots, B_m \in \sigma$ such that $y \in \text{int}(\bigcup_{i=1}^{m} B_i)$ and for all $s_1, s_2 \in B_i$, $(i = 1, 2, \ldots, m)$, and for all $u \in \mathfrak{S}$, we have $\|u(s_1) - u(s_2)\|_Z < \varepsilon$.

LEMMA 3.2. *Suppose that $\mathfrak{S} \subseteq S(Y,Z)$ is relatively compact. Then $\mathfrak{S}$ is equiregulated.*

Under additional conditions, the reverse of the above lemma is also true.

LEMMA 3.3. *Assume that Y is compact and $\mathfrak{S} \subseteq S(Y,Z)$ is equiregulated. Also, assume that $\varepsilon > 0$ is given. Then there exist $Y_1, Y_2, \ldots, Y_m \in \sigma$ such that $Y = Y_1 \bigcup Y_2 \bigcup \ldots \bigcup Y_m$ and if $y, y' \in Y_i$ $(i = 1, 2, \ldots, m)$, then $\|u(y) - u(y')\|_Z \leq \varepsilon$ for each $u \in \mathfrak{S}$.*

LEMMA 3.4. *Assume that Y is a compact Davison space, Z is a normed space with the Heine-Borel property, and $\mathfrak{S} \subseteq S(Y,Z)$ is equiregulated. Also, assume that $\varepsilon > 0$ is given. If $Y = Y_1 \bigcup Y_2 \bigcup \ldots Y_m$ is as Lemma 3.3 and for each $1 \leq i \leq m$ there exists $a_i \in Y_i$ in which $\{\|u(a_i)\|_Z : \ u \in \mathfrak{S}\} \leq k$ for some $k > 0$, then $\mathfrak{S}$ is relatively compact.*

From now on, we assume that Y is a compact Davison space and Z is a normed space with the Heine-Borel property. Also, assume that $\mathfrak{M}_{S(Y,Z)}$ is the collection of bounded sets of regulated functions from Y into Z and $\mathfrak{N}_{S(Y,Z)}$ is its subfamily consisting of nonempty relatively compact subsets of $S(Y,Z)$. For $\emptyset \neq \Omega \subseteq S(Y,Z)$, the symbols $\overline{\Omega}$ and $\mathrm{Conv}(\Omega)$ will denote the closure and closed convex hull of Ω, respectively.
As a consequences of Lemma 3.4 we have the following result.

PROPOSITION 3.5. *Assume that $\mathfrak{S} \in \mathfrak{M}_{S(Y,Z)}$ is equiregulated. Then $\mathfrak{S}$ is relatively compact.*

DEFINITION 3.6. A mapping $\eta : \mathfrak{M}_{S(Y,Z)} \to \mathbb{R}_+$ is called an MNC in $S(Y,Z)$ if for any $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2 \in \mathfrak{M}_{S(Y,Z)}$ it fulfills the following conditions:
(1) $\emptyset \neq \ker(\eta) = \{\mathcal{W} \in \mathfrak{M}_{S(Y,Z)} : \eta(\mathcal{W}) = 0\} \subseteq \mathfrak{N}_{S(Y,Z)}$.
(2) $\mathcal{W}_1 \subseteq \mathcal{W}_2 \Rightarrow \eta(\mathcal{W}_1) \leq \eta(\mathcal{W}_2)$.
(3) $\eta(\overline{\mathcal{W}}) = \eta(\mathcal{W}) = \eta(\mathrm{Conv}(\mathcal{W}))$.
(4) $\eta(\gamma\mathcal{W}_1 + (1-\gamma)\mathcal{W}_2) \leq \gamma\eta(\mathcal{W}_1) + (1-\gamma)\eta(\mathcal{W}_2)$ $\forall \gamma \in [0,1]$.
(5) If $\{\mathcal{W}_j\}$ is a sequence of closed chains in $\mathfrak{M}_{S(Y,Z)}$ such that $\mathcal{W}_{j+1} \subset \mathcal{W}_j$ for $j = 1, 2, \ldots$ and $\lim_{j\to\infty} \eta(\mathcal{W}_j) = 0$, then $\mathcal{W}_\infty = \bigcap_{j=1}^{\infty} \mathcal{W}_j \neq \emptyset$.

Now, for $\mathcal{B} \in \mathfrak{M}_{S(Y,Z)}$, $u \in \mathcal{B}$, $x \in Y$ and $\varepsilon > 0$, let us put

$$\eta(u,x,\varepsilon) = \sup\left\{\|u(t) - u(s)\|_Z : \ \exists K_1, \ldots, K_n \in \sigma; x \in \mathrm{int}(\bigcup_{i=1}^{n} K_i), \ \mathrm{diam}(K_i) \leq \varepsilon, \ s,t \in K_i\right\},$$

where $\mathrm{diam}(K_i) = \sup\{d(\varsigma,\nu) : \varsigma, \nu \in K_i\}$. Then we define

$$\eta(B,x,\varepsilon) = \sup\{\eta(u,x,\varepsilon), \ u \in B\}, \ \eta(B,\varepsilon) = \sup\{\eta(B,x,\varepsilon), \ x \in Y\},$$

$$\eta(B) = \lim_{\varepsilon\to 0} \eta(B,\varepsilon). \tag{2}$$

THEOREM 3.7. *The function $\eta : \mathfrak{M}_{S(Y,Z)} \to \mathbb{R}_+$ given by (2), defines an MNC in $S(Y,Z)$.*

4. Solvability Results

In this section, we investigate the solvability of the nonlinear fractional integral equation (1) in a space of regulated functions. We also provide an example to show the applicability of our result. Consider the following hypotheses.
(1) $G : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function.
(2) $F : [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function and there exist $k_1 > 0$ and $0 < k_2 < 1$ such that

$$|F(t,s,p_1,q) - F(t,s,p_2,q)| \leq k_1|p_1 - p_2|,$$
$$|F(t,s,p,q_1) - F(t,s,p,q_2)| \leq k_2|q_1 - q_2|$$

for each $t, s \in [0,T]$ and $p, q, p_1, p_2, q_1, q_2 \in \mathbb{R}$.
(3) $|F(t,s,0,0)| \leq \varphi(t,s)$, where $\varphi : [0,T] \times [0,T] \to \mathbb{R}$ is a bounded function.
(4) The function $u : [0,T] \times [0,T] \times [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Furthermore, there exists a continuous mapping $\theta : [0,T] \times [0,T] \to \mathbb{R}$ such that

$$|u(t,s,v,w,x) - u(t,s,v,w,x')| \leq \theta(t,s)|x - x'|$$

for all $t, s, v, w \in [0,T]$ such that $v \leq t$, $w \leq s$ and for all $x, x' \in \mathbb{R}$.
(5) If $u_0(t,s) = \sup\{|u(t,s,v,w,0)| : \ 0 \leq v \leq t, \ 0 \leq w \leq s\}$, $\eta_1(t,s) = \theta(t,s)s^\alpha t^\beta$, $\eta_2(t,s) = u_0(t,s)s^\alpha t^\beta$, $\overline{\eta_1} = \sup\{\eta_1(t,s) : \ t, s \in [0,T]\}$, and $\overline{\eta_2} = \sup\{\eta_2(t,s) : \ t, s \in [0,T]\}$, then there exists a positive solution r_0 for the inequality

$$\|G\|_{sup} + \|\varphi\|_{sup} + k_2 r_0 + \frac{k_1}{\alpha\beta}(\overline{\eta_1} r_0 + \overline{\eta_2}) \leq r_0.$$

(6) For each x in $\mathbb{R}$, $F(.,.,.,x)$ is continuous on $[0,T] \times [0,T] \times [-r_0, r_0]$.

THEOREM 4.1. *Under the assumptions (1)-(6), the nonlinear fractional integral equation (1) has at least one solution in the space* $S(([0,T] \times [0,T], \tau, \sigma), \mathbb{R})$, *where* $([0,T] \times [0,T], \tau, \sigma)$ *is the Davison space as Example 2.2* (ii).

EXAMPLE 4.2. Consider the following equation

$$x(t,s) = \frac{1}{2}e^{-t^3 s^3} + \frac{1}{14}e^{-(t^2+s^2)} \cos\Big(\int_0^t \int_0^s \frac{4^{-\frac{ts}{3}}|x(v,w)|}{(s-v)^{\frac{1}{2}}(t-w)^{\frac{1}{2}}} dv dw + \frac{x(t,s)}{8}\Big) \tag{3}$$

for $t, s \in [0, 1.1]$. Put $\alpha = \beta = \frac{1}{2}$, $G(t,s) = \frac{1}{2}e^{-t^3 s^3}$, and $F(t,s,p,q) = \frac{1}{2}e^{-t^3 s^3} + \frac{1}{14}e^{-(t^2+s^2)}\cos(p + \frac{q}{8})$. Clearly, F is continuous, and condition (2) of Theorem 4.1 holds. Also, $F(t,s,0,0) = \frac{1}{2}$, $\varphi(t,s) = \frac{1}{2}$ is bounded with $\|\varphi\|_{sup} = \frac{1}{2}$, $u(t,s,v,w,x) = 4^{-\frac{ts}{3}}|x(v,w)|$, and $|u(t,s,v,w,x) - u(t,s,v,w,x')| \leq 4^{-\frac{ts}{3}}|x(v,w) - x'(v,w)|$. Hence $\theta(t,s) = 4^{-\frac{ts}{3}}$ and so it is continuous. Furthermore, $u_0(t,s) = \sup\{|u(t,s,v,w,0)| : \ 0 \leq v \leq t, \ 0 \leq w \leq s\} = 0$, $\eta_1(t,s) = 4^{-\frac{ts}{3}} s^{\frac{1}{2}} t^{\frac{1}{2}}$, and $\eta_2(t,s) = 0$. Also, $\overline{\eta_1} = 0.23211$ and $\overline{\eta_2} = 0$. Finally, the inequality in condition (5) of Theorem 4.1 has positive solution $r_0 = 2$. Thus all assumptions of Theorem 4.1 hold and therefore Eq. (3) has a solution in the regulated space $S(([0,1.1] \times [0,1.1], \tau, \sigma), \mathbb{R})$.

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Solvability of an Infinite System of Differential Equations of Order n in the space bv_q

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Abstract. In this paper, we study the existence of solutions of an infinite system of differential equations of order n with initial conditions. The problem is discussed in the sequence spaces bv_q $(1 \leq q < \infty)$, with the help of the technique of measure of noncompactness and Meir–Keeler condensing mappings. We also give an example to support our result.

Keywords: Measure of noncompactness, Meir-Keeler condensing map, Sequence space.

AMS Mathematics Subject Classification [2020]: 47H09, 47H10, 34A12

1. Introduction

The degree of noncompactness of a bounded set is described by means of functions called measures of noncompactness (MNC). They were first introduced by Kuratowski. There are different types of MNC in metric and linear topological spaces. They have been effectively used to study the solvability of various types of boundary value problems for differential equations and their systems in sequence spaces (see [6]). In relation to this topic, Alotaibi et al. discussed the solvability of infinite system of second-order differential equations in the sequence space $n(\phi)$. Recently, a lot of scholars have been devoted to the study of the solvability of infinite systems of fractional differential equations in weighted and tempered sequence spaces.

On the other hand, Aghajani et al. [2] proved some coupled fixed point theorems for Meir–Keeler condensing mappings with the help of measure of noncompactness. Now, by using the technique of measure of noncompactness, we study the solvability of the infinite system of differential equations of order n with boundary conditions

$$\begin{cases} \eth_i^{(n)}(\xi) + f_i(\xi, \eth(\xi)) = 0, \quad i = 1, 2, \ldots, \\ \eth_i(\sigma) = \eth_i'(\sigma) = \ldots = \eth_i^{(n-2)}(\sigma) = 0 \text{ and } \eth_i(\varrho) = 0, \end{cases} \tag{1}$$

in the new sequence space bv_q $(1 \leq q < \infty)$, where $\eth(\xi) = (\eth_i(\xi))$ $(\xi \in [\sigma, \varrho])$.

Differential equations are used to describe heat transfer and flow fields, the vibrations of systems and the propagation of waves in physics. In this work, we consider the solvability

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of an infinite system of differential equations by using Meir–Keeler contractive mapping associated with Hausdorff measure of noncompactness. The paper is organized as follows. In the next section, we present some auxiliary facts that we will need further on. In Section 3, we explain about the construction of the Banach sequence space bv_q $(1 \leq q < \infty)$, and we define an MNC in this space. In Section 4, we investigate solvability of the infinite system (1) in the sequence spaces bv_q and give an example.

2. Preliminaries

In this section, we give some relevant facts that we will need further on. Throughout the manuscript, unless otherwise specified, we consider Λ is a real Banach space. We also consider

- $B(\vartheta, \zeta)$ is a ball with center ϑ and radius ζ.
- If $\emptyset \neq \mathfrak{L} \subseteq \Lambda$, then Conv$\mathfrak{L}$ is the closure of convex hull of $\mathfrak{L}$.
- $\mathfrak{M}_\Lambda = \{\mathcal{W} : \emptyset \neq \mathcal{W} \subseteq \Lambda$ and it is bounded$\}$.
- $\mathfrak{N}_\Lambda = \{\mathcal{W} : \mathcal{W} \in \mathfrak{M}_\Lambda$ and it is relatively compact$\}$.

Definition 2.1. [1] A measure of noncompactness (MNC) in Λ is a map $\mu : \mathfrak{M}_\Lambda \to \mathbb{R}_+$ so that for any $\mathcal{W}, \mathcal{Q} \in \mathfrak{M}_\Lambda$ we have:

(1) $\mathfrak{N}_\Lambda \supseteq \ker \mu = \{\mathcal{W} \in \mathfrak{M}_\Lambda : \mu(\mathcal{W}) = 0\} \neq \emptyset$.
(2) $\mathcal{W} \subset \mathcal{Q} \Rightarrow \mu(\mathcal{W}) \leq \mu(\mathcal{Q})$.
(3) $\mu(\mathcal{W}) = \mu(\overline{\mathcal{W}}) = \mu(\text{Conv}\mathcal{W})$.
(4) $\mu(\lambda\mathcal{W} + (1-\lambda)\mathcal{Q}) \leq \lambda\mu(\mathcal{W}) + (1-\lambda)\mu(\mathcal{Q})$ for each $\lambda \in [0,1]$.
(5) Suppose that for each $n \in \mathbb{N}$, $\overline{\mathcal{W}_n} = \mathcal{W}_n \subseteq \mathfrak{M}_\Lambda$, and $\mathcal{W}_{n+1} \subset \mathcal{W}_n$. Then

$$\lim_{n\to\infty} \mu(\mathcal{W}_n) = 0 \Rightarrow \emptyset \neq \mathcal{W}_\infty = \bigcap_{n=1}^{\infty} \mathcal{W}_n.$$

Definition 2.2. [3] Let $(\mathcal{W}, d)$ be a metric space, $\mathfrak{M}_\mathcal{W}$ be the family of all bounded subsets of $\mathcal{W}$, and let $\aleph \in \mathfrak{M}_\mathcal{W}$. Then the Kuratowski measure of noncompactness of $\aleph$ is

$$\alpha(\aleph) = \inf\Big\{\xi > 0 : \aleph \subset \bigcup_{i=1}^{n} \ell_i,\ \ell_i \subset \mathcal{W},\ \text{diam}(\ell_i) < \xi,\ n \in \mathbb{N}\Big\},$$

where $\text{diam}(\ell_i) = \sup\{d(w,u) : w, u \in \ell_i\}$ and ball measure of noncompactness is

$$\chi(\aleph) = \inf\Big\{\xi > 0 : \aleph \subset \bigcup_{i=1}^{n} B(w_i, \zeta_i),\ w_i \in \mathcal{W},\ \zeta_i < \xi,\ n \in \mathbb{N}\Big\}.$$

Definition 2.3. [2] Let $\emptyset \neq C \subseteq \Lambda$ and μ be an MNC in a metric space Λ. A mapping $K : C \to C$ is Meir–Keeler condensing if for any $\varepsilon > 0$, there is $\delta > 0$ in which

$$\varepsilon \leq \mu(\mathcal{W}) < \varepsilon + \delta \ \Rightarrow\ \mu(K(\mathcal{W})) < \varepsilon$$

for any bounded subset $\mathcal{W}$ of C.

Theorem 2.4. *[2] Let $\emptyset \neq C \subseteq \Lambda$ be bounded, convex, and closed and let μ be an MNC in Λ. If $K : C \to C$ is a continuous Meir–Keeler condensing mapping, then K has a fixed point and the set of all fixed points of K is compact.*

Remark 2.5. [3] If Γ is equicontinuous (i.e., for any $\varepsilon > 0$, there is $\delta > 0$ in which if for each $\varsigma, \varsigma' \in I$, $|\varsigma - \varsigma'| < \delta$, then $\|f(\varsigma) - f(\varsigma')\| < \varepsilon$ for each $f \in \Gamma$) and bounded subset

of $C(I,\Lambda)$, then $\mu(\Gamma(.))$ is continuous on I and

$$\mu(\Gamma)=\sup_{\xi\in I}\mu(\Gamma(\xi)),\quad \mu\Big(\int_0^{\xi}\Gamma(\wp)d\wp\Big)\leq\int_0^{\xi}\mu(\Gamma(\wp))d\wp.$$

3. The space bv_q $(1\leq q<\infty)$.

In this part, we recall the Banach sequence space bv_q $(1\leq q<\infty)$, and we define an MNC in this space. By $\hbar$ we mean the space of all complex sequences. For sequence spaces $\mathcal{W}$ and $\mathcal{U}$, the class of all infinite matrices A that map $\mathcal{W}$ into $\mathcal{U}$ is denoted by $(\mathcal{W},\mathcal{U})$. Now if $N=(b_{\imath k})_{\imath,k=0}^{\infty}$ is a complex matrix and $N_\imath$ is its $\imath$-th row, then

$$N_\imath(w)=\sum_{k=0}^{\infty}b_{\imath k}w_k \text{ and } N(w)=(N_\imath(w))_{\imath=0}^{\infty}.$$

Also, $N\in(\mathcal{W},\mathcal{U})$ if and only if $N_\imath(w)$ converges for each $w\in\mathcal{W}$ and $\imath$ and $N(w)\in\mathcal{U}$. The matrix domain of N in $\mathcal{W}$, $\mathcal{W}_N$, [5] is defined by

$$\mathcal{W}_N=\{w\in\hbar : N(w)\in\mathcal{W}\}. \tag{2}$$

The difference space of the sequence space l_q $(1\leq q<\infty)$ [4] is defined as follow:

$$bv_q=\big\{w=(w_\jmath)\in\hbar:\ \sum_{\jmath=0}^{\infty}|w_\jmath-w_{\jmath-1}|^q<\infty\big\}.$$

Lemma 3.1. *[6] Let $\Upsilon\in\mathfrak{M}_{\mathcal{W}}$, where $\mathcal{W}=l_q$ or c_0. Also, let $\Re_m:\mathcal{W}\to\mathcal{W}$ be the map given by $\Re_m((z_k))=(z_0,z_1,\ldots,z_m,0,0,\ldots)$, then*

$$\chi(\Upsilon)=\lim_{m\to\infty}\big\{\sup_{z\in\Upsilon}\|(I-\Re_m)z\|\big\}.$$

Theorem 3.2. *[5] For normed sequence space $\mathcal{W}$, let χ_K and χ be the ball MNCs on $\mathfrak{M}_{\mathcal{W}_K}$ and $\mathfrak{M}_{\mathcal{W}}$, respectively. Then $\chi_K(\Upsilon)=\chi(K(\Upsilon))$ for any $\Upsilon\in\mathfrak{M}_{\mathcal{W}_K}$.*

Now, we are in position to compute the ball MNC in the space bv_q.

Corollary 3.3. *[5] Let $K=\triangle$ and $\mathcal{W}=l_q$ for $1\leq q<\infty$. Then*

$$\chi_K(\Upsilon)=\chi(K(\Upsilon))=\lim_{m\to\infty}\sup_{w\in K(\Upsilon)}\|(I-\Re_m)(w)\|=\lim_{m\to\infty}\sup_{z\in\Upsilon}\|(I-\Re_m)(K(z))\|$$

for all $\Upsilon\in\mathfrak{M}_{\mathcal{W}_K}$, where $\|(I-\Re_m)(K(z))\|=\big(\sum_{k\geq m}|z_k-z_{k-1}|^p\big)^{\frac{1}{q}}$.

4. Solvability Results

In this section, we investigate solvability of the system of differential equations of order $n\geq 2$ (1) in the space bv_q.

Put $I=[\sigma,\varrho]$. Assume that $C^n(I,\mathbb{R})$ is the set of all real-valued n-times continuously differentiable functions on I. Then $\eth\in C^n(I,\mathbb{R})$ is a solution of (1) if and only if $\eth\in C(I,\mathbb{R})$ and it is a solution of the infinite system of integral equations:

$$\eth_i(\xi)=\int_{\sigma}^{\varrho}G(\xi,\wp)f_i(\wp,\eth(\wp))d\wp,\quad i=1,2,\ldots,$$

where $f_i(\xi, \eth(\xi)) \in C(I, \mathbb{R})$ $(\xi \in I)$ and the associated Green's function is defined by

$$G(\xi, \wp) = \begin{cases} \frac{(\varrho-\wp)^{n-1}(\xi-\sigma)^{n-1}-(\varrho-\sigma)^{n-1}(\xi-\wp)^{n-1}}{(\varrho-\sigma)^{n-1}(n-1)!}, & \sigma \leq \wp \leq \xi \leq \varrho, \\ \frac{(\varrho-\wp)^{n-1}(\xi-\sigma)^{n-1}}{(\varrho-\sigma)^{n-1}(n-1)!}, & \sigma \leq \xi \leq \wp \leq \varrho. \end{cases}$$

Here, using Green's function we transform the system (1) into a system of integral equations and focus on the solvability of this new one. Consider the subsequent hypotheses.
(i) Suppose that for any $i \in \mathbb{N}$, $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ is continuous and $f : I \times bv_q \to bv_q$ is given by $(\xi, \eth) \to (f\eth)(\xi) = (f_1(\xi, \eth(\xi)), f_2(\xi, \eth(\xi)), \ldots)$ so that the family $((f\eth)(\xi))_{\xi \in I}$ is equicontinuous at any point of bv_q.
(ii) For any $\wp \in I$, $k \in \mathbb{N}$ and $\eth \in bv_q$ we have

$$|f_k(\wp, \eth(\wp)) - f_{k-1}(\wp, \eth(\wp))|^q \leq p_k(\wp) + q_k(\wp)|\eth_k(\wp) - \eth_{k-1}(\wp)|^q,$$

when $q_k, p_k \in C([\sigma, \varrho], \mathbb{R}_+)$. Also, $(q_k(\wp))$ is equibounded on interval I, $\sum_{k=1}^{\infty} p_k(\wp)$ uniformly converges on I, and $\limsup_{k\to\infty} q_k(\wp)$ is integrable on I. Put $P = \sup_{\wp\in[\sigma,\varrho]} \sum_{k=1}^{\infty} p_k(\wp) < \infty$, and $\aleph = \sup_{k\in\mathbb{N}} \sup_{\wp\in I} q_k(\wp)$.

Theorem 4.1. *If the conditions $(i), (ii)$ hold and $(\frac{n!}{2})^q \frac{1}{(\varrho-\sigma)^{nq}} > \aleph$, then the system (1) has a solution $\eth = (\eth_i) \in C(I, bv_q)$.*

Example 4.2. Consider the system of third-order differential equations

$$\eth_i^{(3)}(\xi) + \frac{\xi}{5}\sum_{k=1}^{i} \frac{k+2}{k+1} + \frac{\sin(\eth_k(\xi))\cos(2\xi+3)\tanh(\eth_k(\xi)+1)}{e^{\xi}} = 0, \ i = 1, 2, \ldots \tag{3}$$

with the boundary conditions given by $\eth_i(1) = \eth_i'(1) = 0$ and $\eth_i(1.7) = 0$, also $\eth_i^{(3)}(1) + \frac{1}{5}\sum_{k=1}^{i} \frac{k+2}{k+1} + \frac{\sin(\eth_k(1))\cos(5)\tanh(\eth_k(1)+1)}{e^1} = 0$ and

$$v_i^{(3)}(1.7) + \frac{1.7}{5}\sum_{k=1}^{i} \frac{k+2}{k+1} + \frac{\sin(\eth_k(1.7))\cos(6.4)\tanh(\eth_k(1.7)+1)}{e^{1.7}} = 0$$

where $\xi \in [1, 1.7]$, and $i = 1, 2, \ldots$. Equation (3) is clearly a special case of equation (1) when

$$f_i(\xi, \eth(\xi)) = \frac{\xi}{5}\sum_{k=1}^{i} \frac{k+2}{k+1} + \frac{\sin(\eth_k(\xi))\cos(2\xi+3)\tanh(\eth_k(\xi)+1)}{e^{\xi}}.$$

Taking into account Theorem 4.1 the infinite system of differential equations of order 3 (3) has at least a solution in $C(I, bv_q)$ for all $q \geq 1$.

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Continuity Results of the Boundary Spectrum in p-Banach Algebras

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Abstract. In this paper, we derive some results concerning the continuity and upper semi-continuity of the boundary spectrum function in p-Banach algebras. Also, we support our discussions by two examples.

Keywords: p-Banach algebras, Boundary spectrum, Upper semi-continuity.

AMS Mathematics Subject Classification [2020]: 46H05, 46H20

1. Introduction

Throughout this paper, all p-Banach algebras will be assumed to be unital and complex. The identity element of a p-Banach algebra is denoted by e. The theory of Banach algebras is essential for generalizing classical Banach algebra results, such as the Gelfand-Mazur theorem or the holomorphic functional calculus. Despite their lack of local convexity, many deep results from the theory of Banach algebras have elegant analogues in the p-Banach setting, particularly concerning spectral theory and continuity properties. Several results about the continuity and upper semi-continuity of the boundary spectrum function have already been obtained in Banach algebras [4]. In this paper, we study and extend some of these resuts concerning the continuity and upper semi-continuity of the boundary spectrum function in p-Banach algebras. Also, we give examples to support our results.

1.1. preliminaries.

Definition 1.1. [2] Let X be a linear space. A p-homogeneous norm $\|.\|_p$, $0 < p \leqslant 1$ on X is a non-negative function $x \mapsto \|x\|_p$ satisfying
(i) $\|x\|_p \geqslant 0$, and $\|x\|_p = 0$ if and only if $x = 0$;
(ii) $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$ for all $x, y \in X$;
(iii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$ for all $x \in X$ and $\lambda \in \mathbb{F}$.
By a p-normed algebra $(A, \|.\|_p)$, we mean an algebra A endowed with a p-homogeneous norm $\|.\|_p$ such that $\|xy\|_p \leqslant \|x\|_p \|y\|_p$ for all $x, y \in A$. A complete p-normed algebra is

*Speaker.

called a p-Banach algebra.

Also if A is a p-Banach algebra and $a \in A$, then by Lemma 3.9 of [6], we have

$$r_A(a) = \lim_{n\to\infty} \left(\|a^n\|_p\right)^{\frac{1}{np}} \leq (\|a\|_p)^{\frac{1}{p}}.$$

Let E be a subset of A. Then the topological boundary of E relative to A will be denoted by ∂E.

Definition 1.2. For a p-Banach algebra A, the spectrum $sp_A(a)$ of an element $a \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - a$ is not invertible in A. The spectral radius $r_A(a)$ of an element $a \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

Definition 1.3. Let A be a p-Banach algebra and S be the set of all non-invertible elements of A. Then S is a closed subset of A. Define the set

$$S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda e - a \in \partial S\}.$$

We call this set the boundary spectrum of a in A. This concept was initially introduced in Banach algebras [5].

Definition 1.4. [1] The Hausdorff distance is defined by

$$\Delta(K_1, K_2) = \max\left(\sup_{z\in K_2} \operatorname{dist}(z, K_1), \sup_{z\in K_1} \operatorname{dist}(z, K_2)\right)$$

for K_1, K_2 compact subsets of $\mathbb{C}$. Let $r > 0$ and K be a compact subset of $\mathbb{C}$. If $K + r$ denotes $\{z : \operatorname{dist}(z, K) \leq r\}$, then obviously $K_1 \subseteq K_2 + \Delta(K_1, K_2)$ and $K_2 \subseteq K_1 + \Delta(K_1, K_2)$. Moreover, the set of nonempty compact subsets of the complex plane $\mathbb{C}$ is denoted by $\mathbf{K}(\mathbb{C})$.

As in [1, p.48], we have the following definition.

Definition 1.5. Let A be a p-Banach algebra. The function $x \longmapsto S_\partial(x)$ from A into $\mathbf{K}(\mathbb{C})$ is said to be continuous at $a \in A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - a\|_p < \delta$ implies $\Delta(S_\partial(x), S_\partial(a)) < \epsilon$. Also, the function $x \longmapsto S_\partial(x)$ is upper semi-continuous on A if for every $a \in A$ and every open set U containing $S_\partial(a)$, there exists a $\delta > 0$ such that if $\|x - a\|_p < \delta$, then U contains $S_\partial(x)$.

2. Main results

In this section, we generalize some results of Banach algebras into p-Banach algebras. Using a similar argument in [3], we have the following inequality.

$$|r_A(x) - r_A(y)| \leq \Delta(sp_A(x), sp_A(y)), \text{ for all } x, y \in A, \tag{1}$$

where A is a p-Banach algebra.

Theorem 2.1. *Let A be a p-Banach algebra, then continuity of sp_A implies continuity of r_A.*

In general, the map $x \longmapsto S_\partial(x)$ from A into $\mathbf{K}(\mathbb{C})$ is not continuous even in Banach algebras [1, p. 49]. However, in the case of commutativity , we have the following result.

Proposition 2.2. *Let a and b be commuting elements of a p-Banach algebra A. Then*

$$\mathbb{C}\backslash sp_A(a) \cap S_\partial(b) \subseteq S_\partial(a) + r_A(a - b). \tag{2}$$

From [6, Lemma 3.9] and Proposition 2.2, we have:

Corollary 2.3. *Let a and b be commuting elements of a p-Banach algebra A such that $S_\partial(a) \cap sp_A(b) = \emptyset = S_\partial(b) \cap sp_A(a)$, Then*

$$\Delta(S_\partial(a), S_\partial(b)) = r_A(a-b) \leq \|a-b\|_p^{\frac{1}{p}}. \tag{3}$$

Theorem 2.4. *Let A be a commutative p-Banach algebra such that*

$$S_\partial(a) \cap sp_A(b) = \emptyset = S_\partial(b) \cap sp_A(a).$$

Then the function $a \longmapsto S_\partial(a)$ is continuous on A.

The following result yields the upper semi-continuity of the map $a \longmapsto S_\partial(a)$ in p-Banach algebras.

Theorem 2.5. *Let a be an element of a p-Banach algebra A and U be an open set containing $S_\partial(a)$. Then there exists a $\delta > 0$ such that if $\|x-a\| < \delta$, then U contains $S_\partial(x)$.*

Corollary 2.6. *Let a be an element of a p-Banach algebra A and U be an open set such that $sp_A(a) \cap \partial U = \emptyset$ and $S_\partial(a) \cap U \neq \emptyset$. Then there exists a $\delta > 0$ such that if $\|x-a\| < \delta$, then $S_\partial(x) \cap U \neq \emptyset$.*

Example 2.7. The algebra H_p, $0 < p < 1$ of holomorphic functions in the unit disc

$$x(\lambda) = \sum_{i=0}^{\infty} x_n \lambda^n$$

such that

$$\|x\|_p = \sum_{i=0}^{\infty} |x_n|^p < \infty,$$

with the pointwise multiplication is a p-Banach algebra which is not a Banach algebra.

Example 2.8. Consider the Frechet algebra $C(\mathbb{R})$ of all continuous complex-valued functions on $\mathbb{R}$ with the sequence $\{p_n\}$ of seminorms defined by

$$p_n(f) = \sup_{|x| \leq n} |f(x)|$$

is not a p-Banach algebra.

3. Conclusion

We obtained some continuity properties of the boundary spectrum function in p-Banach algebras.

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A note on CQ-algebras

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Abstract. In this paper, we obtain some results by using the usual and singular spectrum in CQ-algebras.

Keywords: CQ-algebras, Topological divisor of zero, Singular spectrum.

AMS Mathematics Subject Classification [2020]: 46H05, 46H20

1. Introduction

Throughout this paper, all algebras will be assumed to be unital with unity e. Non-normed topological algebras were initially introduced around the year 1950 for the investigation of certain classes of these algebras that appeared naturally in mathematics and physics [2]. An important class of non-normed topological algebras namely Q-algebras have interesting properties regarding the spectral theory. Hence, we can study and generalize some properties of Banach algebras to CQ-algebras.

1.1. preliminaries.

Definition 1.1. [1] Let A be a Hausdorff topological algebra . An element $a \in A$ is said to be a left (right) topological zero divisor if there exists a sequence $\{x_n\}$ in A such that $x_n \not\to 0$ *and* $ax_n \to 0$ $(x_n a \to 0)$ *as* $n \to \infty$. A topological zero divisor is both a left and a right topological zero divisor. If A is commutative, then every left topological zero divisor or right topological zero divisor is a topological zero divisor.

Remark 1.2. Let A be a topological algebra. The set of all invertible elements of A is denoted by $Inv(A)$. Assume that E is a subset of A. Then the topological boundary of E relative to A is denoted by ∂E.

Definition 1.3. Let A be a topological algebra. The usual and singular spectrum of $x \in A$ are denoted by $sp_A(x)$ *and* $\sigma_A(x)$ respectively, and defined in the following:

$$sp_A(x) = \{\lambda \in \mathbb{C} : \ x - \lambda e \notin Inv(A)\},$$
$$\sigma_A(x) = \{\lambda \in \mathbb{C} : \ x - \lambda e \ \text{ is a topological zero divisor}\}.$$

Definition 1.4. A topological algebra is called a Q-algebra if $Inv(A)$ is open.

*Speaker.

Definition 1.5. [1] A topological algebra is called a C-algebra or a continuous algebra if the map $x \longmapsto x^{-1}$ of $Inv(A) \longrightarrow Inv(A)$ is continuous.

Definition 1.6. [1] A Q-algebra which is also a continuous algebra is called a CQ-algebra.

2. Main results

In this section, we obtained some results concerning the usual and singular spectrum in CQ-algebras.
In some topological algebras such as Banach algebras, every boundary point of the set of all invertible elements is a topological divisor of zero [4]. Here we present a similar result for CQ-algebras.

Theorem 2.1. *Let A be a CQ-algebra. If $x \in \partial Inv(A)$, then x is a topological divisor of zero.*

Corollary 2.2. *Let A be a CQ-algebra. If $\lambda \in \partial sp_A(x)$, then $x - \lambda e$ is a topological divisor of zero.*

It is well known that the Gelfand-Mazur theorem holds true for Banach algebras :

Theorem 2.3. *[4, Theorem 42.10] Let A be a Banach algebra. If A has no non-zero topological zero divisor, then $A = \mathbb{C}e$.*

However, [3, Example 3.6] shows that the Gelfand-Mazur theorem may be false for CQ-algebra in general.

Lemma 2.4. *Let A be a Hausdorff topological algebra, then a topological divisor of zero is not invertible.*

Now we use this lemma and get the following result.

Theorem 2.5. *Let A be a CQ-algebra. Then the following statements hold:*
(i) $\partial sp_A(x) \subseteq \sigma_A(x) \subseteq sp_A(x)$,
(ii) $\sigma_A(x)$ is a compact set.

Theorem 2.6. *Let A and B be two CQ-algebras. If $\theta : A \to B$ is a injective continuous homomorphism such that $\theta(e_A) = e_B$, then $\sigma_A(x) \subseteq \sigma_B(\theta(x))$, for all $x \in A$.*

Theorem 2.7. *Let A and B be two complete metrizable CQ-algebras. Also let B be commutative and semisimple. If $\theta : A \to B$ is a injective homomorphism such that $\theta(e_A) = e_B$, then $\sigma_A(x) \subseteq \sigma_B(\theta(x))$, for all $x \in A$.*

Remark 2.8. There exist Q-algebras which are not C-algebras and there are C-algebras which are not Q-algebras. For instance, (see [1, Remark 3.6.19]).

3. Conclusion

In this paper, we obtained some results concerning the usual and singular spectrum in CQ-algebras.

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Boundedness of composition operators on weighted variable exponent Bergman spaces on polydisk

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Abstract. **Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let be a holomorphic function from disk $\mathbb{D}^n$ into $\mathbb{D}^n$. We study the composition operator C_ϕ on the weighted variable exponent Bergman space with classical radial weight and give a suffcient condition for the boundedness of this operator on $A^{p(\cdot)}_\alpha(\mathbb{D}^n)$.**

Keywords: composition operator, Bergman space, weighted variable exponent Lebesgue space, polydisk

AMS Mathematics Subject Classification [2020]: 47B33, 30H20, 46B50

1. Introduction and Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. We mean by a *variable exponent*, a measurable function $p : \Omega \to [1, \infty)$. We shall write

$$p_+ = p^+_\Omega := ess\ sup_{x\in\Omega}\ p(x),$$
$$p_- = p^-_\Omega := ess\ inf_{x\in\Omega}\ p(x).$$

Let $\mathcal{P}(\Omega)$ denote the set of all variable exponents $p(\cdot)$ for which $p_+ < \infty$. For a complex-valued measurable function $f : \Omega \to \mathbb{C}$ and the weight $\omega : \Omega \to [0, \infty)$ we define the modular $\rho_{p(\cdot),\omega}$ by

$$\rho_{p(\cdot),\omega}(f) := \int_\Omega \omega(x)|f(x)|^{p(x)} d\mu(x)$$

where μ is Lebesgue measure on Ω. The *Luxemburg-Nakano* norm induced by this modular is given by

$$\|f\|_{L^{p(\cdot)}_\omega} := inf_\lambda \left\{ \lambda > 0 : \rho_{p(\cdot),\omega}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

*Speaker.

The weighted variable exponent Lebesgue space $L^{p(\cdot)}_{\omega}(\Omega)$ consists of all complex-valued functions $f : \Omega \to \mathbb{C}$ for which $\rho_{p(\cdot),\omega}(f) < \infty$. It is well-known that if $p(\cdot) \in \mathcal{P}$ then $L^{p(\cdot)}_{\omega}$ equipped with the above norm is a Banach space [2]. Moreover, the dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ where the conjugate exponent $p'(\cdot)$ satisfies

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

It can also be verified that the conjugate exponent satisfies the following equalities:

$$p'(\cdot)_+ := \left(p'(\cdot)\right)_+ = (p_-)',$$
$$p'(\cdot)_- := \left(p'(\cdot)\right)_- = (p_+)'.$$

A function $p : \Omega \to \mathbb{R}$ is said to be locally log-holder continuous on Ω if there exist a positive constant C such that for all $x, y \in \Omega$ that $|x - y| < \frac{1}{2}$, we have

$$LH_0 : |p(x) - p(y)| \leq \frac{C}{\log(\frac{1}{|x-y|})}. \tag{1}$$

and is log-holder continuous at infinity if there exists p_∞,$1 < p_\infty < \infty$, such that

$$LH_\infty : |p(x) - p_\infty| \leq \frac{C}{\log\left(e + |x|\right)}. \tag{2}$$

We denote by $\mathcal{P}^{log}(\Omega)$ the set of all locally log-holder continuous functions in Ω for which $1 < p_- \leq p_+ < \infty$.

Given $f \in L^1_{loc}(\Omega)$, the weighted Hardy-Littlewood maximal function of f,that is denoted by Mf, is defined for every $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| d\mu(y), \quad x \in \Omega.$$

It is known that for which $f : \Omega \to \mathbb{C}$ we have

$$|f(x)| \leq Mf(x). \tag{3}$$

The weight ω is said to be of class A_1 and denoted by $\omega \in A_1$ if

$$[\omega]_{A_1} := esssup_{z \in \mathbb{D}^n} \frac{M\omega(z)}{\omega(z)} < \infty.$$

THEOREM 1.1. *([2]) Given $P(\cdot) \in \mathcal{P}(\Omega)$ suppose $1 < p_- \leq p_+ < \infty$ then Hardy-Littlewood maximal function M is bounded on $L^p(\cdot)(\Omega)$ if and only if M is bounded on $L^{p'(\cdot)}(\Omega)$.*

THEOREM 1.2. *(Rubio de Francia extrapolation)([2]): Suppose that for some $p_0 \geq 1$ the family $\mathcal{F}$ (family of non-negative measurable function) is such that for all $\omega \in A_1$ there exists $C_0 > 0$ such that*

$$\int_\Omega F(x)^{p_0} \omega(x) dx \leq C_0 \int_\Omega G(x)^{p_0} \omega(x) dx; \quad F, G \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_0 \leq p_- \leq p_+ < \infty$ and maximal operator is bounded on $L^{\left(\frac{p(.)}{p_0}\right)'}(\Omega)$. Then there exists $C_{p(\cdot)}$ such that

$$\|F\|_{L^{p(\cdot)}} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}}.$$

We now turn to the n-dimentional complex plane, and consider the unit polydisk

$$\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \times ... \times \mathbb{D} = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n\}$$

in $\mathbb{C}^n$. From now on, we assume that $\Omega = \mathbb{D}^n$, so that we can use all the above arguments with $x \in \Omega$ replaced by $z \in \mathbb{D}^n$.

Definition 1.3. For $p \in \mathcal{P}(\mathbb{D}^n)$ and $-1 < \alpha < \infty$, the weighted variable exponent Bergman space $A_\alpha^{p(\cdot)}$ consist of all holomorphic functions $f : \mathbb{D}^n \in \mathbb{C}$ for which

$$\int_{\mathbb{D}^n} |f(z)|^{p(z)} dv_\alpha(z) < \infty$$

where

$$dv_\alpha(z) = dv_\alpha(z_1, z_2, ..., z_n) = dA_\alpha(z_1)...dA_\alpha(z_n)$$

that here

$$dA_\alpha(z) = \frac{\alpha + 1}{\pi}(1 - |z_k|^2)^\alpha dx_k dy_k, \quad z_k = x_k + iy_k.$$

Note that with $\Omega = \mathbb{D}^n$ the maximal operator M takes form

$$Mf(x) := sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(z)| dv_\alpha(z), z \in \mathbb{D}^n$$

where the supremum is taken over all cubes Q (with sides parallel to the coordinate axes) that contain z.

2. Composition operators and Carleson measure

We fix two positive integers m and n. We consider a holomorphic function $\varphi : \mathbb{D}^m \to \mathbb{D}^n$ given by

$$\varphi(z_1, z_2, \cdots, z_m) = (\varphi_1(z_1, z_2, \cdots, z_m), \varphi_2(z_1, z_2, \cdots, z_m), \cdots, \varphi_n(z_1, z_2, \cdots, z_m))$$

where for $1 \leq k \leq n$, φ_k is holomorphic function. We study the composition operator

$$C_\varphi : A_\alpha^{p(\cdot)}(\mathbb{D}^n) \to A_\beta^{p(\cdot)}(\mathbb{D}^m)$$

defind by

$$C_\varphi(f) = f \circ \varphi \quad , \quad f \in A_\alpha^{p(\cdot)}(\mathbb{D}^n).$$

Definition 2.1. Let μ be a positive Borel measure on $\mathbb{D}^n$. We say that μ is a Carleson Measure for $A^{p(\cdot)}(\mathbb{D}^n)$ if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}^n} |f(z)|^{p(z)} d\mu(z) \leq C \int_{\mathbb{D}^n} |f(z)|^{p(z)} dA(z).$$

It is well-known that the Carleson measure does not depend on the exponent.

Theorem 2.2. *([1, 3]): suppose $\alpha > -1$ and μ is a positive Borel measure on $\mathbb{D}^n$. Then, the following two conditions are equivalent.*

$$(a)\ \mu \text{ is a Carleson measure for } A_\alpha^p(\mathbb{D}^n) \text{ for some } p > 0$$

$$(b)\ \mu \text{ is a Carleson measure for } A_\alpha^p(\mathbb{D}^n) \text{ for every } p > 0.$$

This result was used by Stessin and Zhu to prove the following theorem concerning the boundedness of composition operators on the classical (constant-exponent) Bergman spaces of the polydisk. Indeed, Stessin and Zhu proved that the operator C_φ maps $A_\alpha^p \mathbb{D}^n$ boundedly into $A_\alpha^p(\mathbb{D}^n)$ if and only if $\mu_{\varphi,\alpha}$ is Carleson measure for $A_\alpha^p(\mathbb{D}^n)$; here $\mu_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} dv_\alpha$.

THEOREM 2.3. *([5, 6]): Suppose $p > 0$, $\alpha > -1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic self-map of the unit disk. Then the operator $C_\varphi : A^p_\alpha(\mathbb{D}^n) \to A^p_\alpha(\mathbb{D}^n)$ is bounded.*

For different weights, they even determined the best possible constants characterizing these weights, namely α and β.

THEOREM 2.4. *([5]): For any $p > 0$ and $\alpha > -1$ the composition operator C_φ maps $A^p_\alpha(\mathbb{D}^n)$ boundedly into $A^p_\beta(\mathbb{D}^n)$ .where $\beta = n(2 + \alpha) - 2$.*

Morovatpoor and Abkar provided a sufficient condition for the boundedness of composition operators on variable exponent Bergman spaces defined on disk, which we shall refer to in the following.

THEOREM 2.5. *[4] Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D})$. If for every $z \in \mathbb{D}$ we have $p(z) < p(\varphi(z))$, then the composition operator*

$$C_\varphi : A^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$$

is bounded.

3. main result

The main result of this paper is to generalize some of the above results to variable exponent Bergman space $A^{p(\cdot)}_\alpha(\mathbb{D}^n)$. To do this we need to study the Muckenhoupt weight classes, and its relevance to Carleson measures in this new setting. Indeed, we prove that

THEOREM 3.1. *Suppose that $\varphi : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ is an analytic function and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{D}^n)$ such that, for every $z \in \mathbb{D}^n$, we have $p(z) < p(\varphi(z))$. Then the composition operator*

$$C_\varphi : A^{p(\cdot)}_\alpha(\mathbb{D}^n) \longrightarrow A^{p(\cdot)}_\beta(\mathbb{D}^n)$$

is bounded, where α and β satisfy the relation

$$\beta = n(\alpha + 2/p_0) - 2/p_0$$

for at least one value of $1 < p_0 < \infty$.

4. Conclusion

(e) In this paper, the authors have presented a sufficient condition for the boundedness of composition operators on weighted variable exponent Bergman spaces. As a potential direction for further research, one may investigate sufficient or necessary conditions for the compactness of such operators on these spaces.

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On weakly interpolative Kannan contractions

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Abstract. In this article, besides other things, we introduced the concept of weakly and non-weakly interpolative Kannan type contraction mappings, and we will prove and generalize some recently results in the setting of $w-$distances.

Keywords: Fixed point, $w-$distance, weakly interpolative Kannan type contraction, non-weakly interpolative Kannan type contraction.

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1. Introduction and Preliminaries

One of the generalizations of the well-known Banach theorem from 1922 is the introduction after 75 years of the so-called w-distance p in the given metric space (X, d). Thus we obtained an ordered triple (X, d, p) where (X, d) is the given metric space and p is a function from $X \times X$ in $[0, +\infty)$ that satisfies the following three axioms:

p1) $p(x, z) \leq p(x, y) + p(y, z)$ *for all* $x, y, z \in X$;

p2) *For any* $x \in X$, *the function* $p(x, \cdot) : X \to [0; +\infty)$ *is* $d-$*lower semi-continuous;*

p3) *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *imply* $d(x, y) \leq \varepsilon$. *Then,* p *is called a* w*-distance on* X.

The last two axioms are new compared to those known for metric space. The second represents the lower semi-continuity in the second variable and the third connects the metric d and the w-distance p.

A typical example of a w-distance is the metric d itself defined on a nonempty set X. Actually, p1) is fulfilled as a triangle relation. Since the metric d is a continuous function with 2 variables, it is also semi-continuous from below in the second variable. Indeed, if y_n is a sequence in X that converges to y by metric d then $p(x, y) = d(x, y) = \lim_{n\to+\infty} d(x, y_n) = \liminf_{n\to+\infty} d(x, y_n)$ and p2) is fulfilled. Assuming that $\delta = \frac{\varepsilon}{2}$, we get that **p3**) is fulfilled, because $p(x, y) = d(x, y) \leq d(x, z) + d(y, z) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Now we list several typical examples of w-distances, some of which were also mentioned in the first paper on w-distances.

1. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, +\infty)$ defined by $p(x, y) = c$ for every $x, y \in X$ is a $w-$distance on X, where c is a positive real number.

*Speaker.

2. Let X be a normed linear space with norm $\|\cdot\|$. Then a function $p : X \times X \to [0, +\infty)$ defined by $p(x, y) = \|x\| + \|y\|$ for every $x, y \in X$ is a $w-$distance on X.

3. Similarly to example 3, only that the function $p : X \times X \to [0, +\infty)$ is defined by $p(x, y) = \|x\|$ for all $x, y \in X$.

For several examples of $w-$distances see [**2**], pages 382, 383, 384. An important note about **p1)** and **p3)**. Since Example 1.3. from [**3**] (see also Example 4 from [**3**]) shows that the w-distance in the general case is not symmetric, i.e., it is not true that $p(x, y) = p(y, x)$ for every $x, y \in X$, then the triangle relation as well as the axiom **p3)** should be understood as the introduced order $x, z; x, y; y, z$ i.e., $z, x; z, y$ and finally x, y.

The following Lemma is one of the most important that is used in the study of w-distance metric spaces. It relates metric convergence to w-distance convergence. It is also important because it gives us the information (a sufficient condition) when the sequence x_n is Cauchy in the metric space (X, d). In the sequel, we denote by $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{N}$, the sets of positive real numbers, real numbers and natural numbers, respectively.

Lemma 1.1. *([2], Lemma 1.) Let X be a metric space with metric d and let p be a $w-$distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. in particular, if $p(x, y) = 0$ and $p(x, z) = 0$,then $y = z$;

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;

(iii) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(iv) if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

2. Main results

In section 5 of their paper [**2**] the authors proved one theorem and three corollaries. In all four results, they assume that the contraction coefficient k belongs to the set $[0, 1)$ or $[0, 1/2)$. It is easy to see that this is imprecise and that the assumption must be that k belongs to $(0, 1)$ or $(0, 1/2)$. This is the difference obtained when, under the known contractive conditions of metric spaces, the metric d is replaced by the w-distance p. For example, in Theorem 4 from [**2**], if $k = 0$ is set, $p(Tx, T^2x) = 0$ is obtained. Whence it does not have to follow as with metric spaces that then $Tx = T^2x$ because the equality $p(a, b) = 0$ does not necessarily follow $a = b$.

In this article, we will check, among other things, whether the known theorems (results) remain valid if the metric d is replaced by the w-distance p. We are talking about results such as Banach's contraction principle, Kannan's theorem, Chatterjea's, Reich's and Hardy-Rogers' theorems. Then the Boyd-Wong and Meir-Keeler theorems and many others.

By doing so, we would obtain generalizations of the above results because each metric is a w-distance. One of the following sufficient conditions will be used for the existence of a fixed point:

(i) The mapping of T from X to X is continuous;

(ii) The number set $\{p(x, y) + p(x, Tx) : x \in X, y \in X$ but such that y is different from $Ty\}$ has a positive infimum.

First, we will formulate and prove the Hardy-Rogers theorem within metric spaces with w-distance p.

Theorem 2.1. *Let (X, d, p) be a complete metric space with w-distance p, T a mapping from X to X and let there exist non-negative constants A, B, C, D and E such that $A +$*

$B+C+D+E<1$ and that for every x,y from X it holds $p(Tx,Ty)\leq Ap(x,y)+Bp(x,Tx)+Cp(y,Ty)+Dp(x,Ty)+Ep(y,Tx)$.

Then T has a unique fixed point say $z\in X$ such that $p(z,z)=0$ if at least one of the above conditions (i) or (ii) holds.

Let $p(x,y)$ be a given w-distance on the metric space (X,d). For details see [**2**]. According to **p2)** $p(x,\cdot)$ is semi-continuous from below in the second variable. The w-distance p is symmetric if $p(x,y)=p(y,x)$ for all $x,y\in X$. For such a w-distances, if $a\neq b$ then $p(a,b)>0$, i.e., from $p(a,b)=0$ follows $a=b$. In many contractive conditions, symmetry of the w-distance can be assumed. This is achieved by introducing a new function q from $X\times X$ in $[0,+\infty)$ defined by $q(x,y)=\max\{p(x,y),p(y,x)\}$. It is easy to show that q is a w-distance.

The following results are natural in the framework of complete metric spaces with $w-$distance, and for mapping $T:X\to X$, the next will be assumed either T is continuous or the infimum of the number set
$\{p(x,y)+p(x,Tx):x\in X\}$ where y is not a fixed point of T, is positive.

$p-$Banach contraction theorem: $p(Tx,Ty)\leq k\cdot p(x,y),k\in(0,1)$

$p-$Kannan contraction: $p(Tx,Ty)\leq l\cdot[p(x,Tx)+p(y,Ty)],l\in\left(0,\frac{1}{2}\right)$

$p-$Chatterjea contraction: $p(Tx,Ty)\leq m\cdot[p(x,Ty)+p(y,Tx)],m\in\left(0,\frac{1}{2}\right)$

$p-$Reich contraction: $p(Tx,Ty)\leq A\cdot p(x,y)+B\cdot p(x,Tx)+C\cdot p(y,Ty),A,B,C\geq 0$ and $A+B+C<1$

$p-$Hardy-Rogers contraction: $p(Tx,Ty)\leq A\cdot p(x,y)+B\cdot p(x,Tx)+C\cdot p(y,Ty)+D\cdot p(x,Ty)+E\cdot p(y,Tx),A,B,C,D,E\geq 0$ and $A+B+C+D+E<1$

$$p-\acute{C}iri\acute{c}(I)p(Tx,Ty)\leq$$
$$\lambda_1\cdot\max\left\{p(x,y),\frac{p(x,Tx)+p(y,Ty)}{2},\frac{p(x,Ty)+p(y,Tx)}{2}\right\},\lambda_1\in(0,1)$$

$$p-\acute{C}iri\acute{c}(II)p(Tx,Ty)\leq$$
$$\lambda_2\cdot\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\right\},\lambda_2\in(0,1)$$

$$p-\acute{C}iri\acute{c}(III)p(Tx,Ty)\leq$$
$$\lambda_3\cdot\max\left\{p(x,y),\frac{p(x,Tx)+p(y,Ty)}{2},p(x,Ty),p(y,Tx)\right\},\lambda_3\in(0,1)$$

$$p-\acute{C}iri\acute{c}(IV)p(Tx,Ty)\leq$$
$$\lambda_4\cdot\max\{p(x,y),p(x,Tx),p(y,Ty),p(x,Ty),p(y,Tx)\},\lambda_4\in(0,1)$$

$p-$Bryant contraction: $p(T^nx,T^ny)\leq r\cdot p(x,y),r\in(0,1),n\in\mathbb{N}$

If (X,d) is compact metric space and if

$p-$Nemytzki contraction: $p(Tx,Ty)<p(x,y)$ whenever $x\neq y$.

$p-$Browder contraction: $p(Tx,Ty)\leq\phi(p(x,y)),\phi$ nondecreasing and continuous from the right function from $(0,+\infty)$ into$(0,+\infty)$ such that $\phi(t)<t$.

$p-$Boyd-Wong contraction $p(Tx,Ty)\leq\phi(p(x,y)),\phi$ is a real function, upper semi-continuous from the right, satisfying $\phi(t)<t$ for $t>0$.

$p-$Meir-Keeler contraction: For all $\varepsilon>0$ there exists $\delta>0$ such that

$\varepsilon \leq p(x,y) < \varepsilon + \delta$ implies $p(Tx,Ty) < \varepsilon$
Further there are Jungck, Fisher and many other contractions.

3. p-interpolative Kannan type contraction

In 1969, Kannan [**4**] proved the following fixed point theorem.

THEOREM 3.1. *Let $f : X \to X$ be a Kannan contraction mapping, i.e., $d(fx;fy) \leq k(d(x;fx) + d(y;fy))$ for all $x;y \in X$ and some $0 \leq k < \frac{1}{2}$, of a f-orbitally complete metric space. Then f has a unique fixed point.*

Afterwards, T. Suzuki published a nice paper [**8**] in which generalized Kannan's result in two new ones. He introduced the concept of weakly Kannan contraction mappings and non-weakly Kannan contraction mappings. For more details, see Section 4 and 5 in that paper.

Recently, the concept of interpolative Kannan type contraction mappings was introduced by E. Karapinar in [**5**]; and he proved the following theorem:

THEOREM 3.2. *Let (X,d) be a complete metric space. Suppose that $f : X \to X$ is a interpolative Kannan type contraction self-map; i.e. if there exists a constant $\alpha \in (0;1)$ and $k \in [0;1)$ such that either of the followings hold:*

$$d(fx;fy) \leq k[d(x;fx)]^{\alpha}[d(y;fy)]^{1-\alpha} \tag{5}$$

for all $x,y \in X$ with $x \neq fx$. Then f has a unique fixed point in X.

In this paper, we introduced the concept of weakly and non-weakly interpolative Kannan type contraction mappings, and we will prove and generalize Karapinar's theorem in the setting of $w-$distances.

We know that a w-distance p is not symmetric; i.e. $p(x;y)$ is not equal to $p(y;x)$ in general. So, we can define weakly interpolative Kannan type contractions as follows.

DEFINITION 3.3. Let $(X;d)$ be a metric space. The mapping $f : X \to X$ is said to be a weakly interpolative Kannan type contraction or p-interpolative Kannan type contraction, if there exist a constant $\alpha \in (0;1)$ and $k \in [0;1)$ such that either of the followings hold for all $x;y \in X$:

$$p(fx,fy) \leq k[p(fx,x)]^{\alpha}[p(fy,y)]^{1-\alpha}, \tag{6}$$

or

$$p(fx,fy) \leq k[p(fx,x)]^{\alpha}[p(y,fy)]^{1-\alpha}, \tag{7}$$

or

$$p(fx,fy) \leq k[p(x,fx)]^{\alpha}[p(y,fy)]^{1-\alpha}, \tag{8}$$

or

$$p(fx,fy) \leq k[p(x,fx)]^{\alpha}[p(fy,y)]^{1-\alpha}. \tag{9}$$

If $p = d$ then f is called interpolative Kannan type contraction [**5**].

The following example shows that the class of weakly interpolative Kannan type contraction is larger than interpolative Kannan type contraction.

Example 3.4. Let $X = \{x; y; z\}$. Consider the metric d and the w-distance p on X, as follows:

$$d(x;x) = d(y;y) = d(z;z) = 0; d(x;y) = d(y;x) = 3;$$
$$d(x;z) = d(z;x) = 0; d(y;z) = d(z;y) = \frac{3}{2}.$$
$$p(x;x) = p(x;z) = p(z;x) = 0;\ p(y;x) = p(y;z) = 4$$
$$p(x;y) = p(z;y) = 3; p(z;z) = p(y;y) = 1.$$

Also, define $f : X \to X$ by $f(x) = f(z) = x$ and $f(y) = z$. Then for each $\alpha \in (0;1)$ and for each $k \in (\frac{1}{3}, 1)$ the contractions (6)-(9) are true for p, while these contractions are not true for d. For example,

$$d(fx; fy) = d(x; z) = 2 > kd(fx; x)^{\alpha} d(fy; y)^{1-\alpha} = 0.$$

Obviously x is the unique fixed point of f.

Remark 3.5. Note that if p is a symmetric w-distance, then for each $x = y$ we have $p(x; y) > 0$. Since if $p(x; y) = 0$, then

$$p(x; x) \leq p(x; y) + p(y; x) = 2p(x; y) = 0.$$

Therefore $p(x; x) = 0 = p(x; y)$ imply that $x = y$, a contradiction.

Therefore, we have the following theorem.

Theorem 3.6. *Let p be a w-distance on a complete metric space$(X; d)$. Suppose that $f : X \to X$ is a weakly interpolative Kannan type contraction self-map. Then f has a unique fixed point x in X such that $p(x; x) = 0$.*

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A subclass of p-valent meromorphic functions associated by defined q-differential operator

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Abstract. In this paper, we introduce a new class of meromorphic functions $M_{\lambda,q}[p,m,A,B]$ in the punctured open disk $\mathbb{D}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$ by q-derivative.
we give a neccessary and sufficient condition for function in this new subclass.Furthermore, coefficient bound , partial sum,derive distortion bounds and coefficient estimates.

Keywords: q-derivative, multivalent function , meromorphic function, coefficient bound, partial sum

AMS Mathematics Subject Classification [2020]: 58E11, 53B30, 53C50 (at least 1 and at most 3)

1. Introduction

Let $M(p)$ denote the class of meromorphic functions of the type

$$h(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} a_k z^k, \ (a_k \geq 0 \ , \ p \in \mathbb{N}) \tag{1}$$

which are analytic in the punctured open disk

$$\mathbb{D}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}. \tag{2}$$

In particular, if $p = 1$, then

$$h(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^k$$

If $h \in M(p)$ is given by (1) and $g \in M(p)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} b_k z^k, \tag{3}$$

*Speaker.

then the Hadamard product (or convolution) $h * g$ of h and g is defined by

$$(h * g)(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} a_k b_k z^k. \tag{4}$$

Also, Jacksons q- derivative and q- integral of a function $h(z)$ defined on a subset of $\mathbb{C}$ are, respectively, given (see Gasper and Rahman[3,pp. 19-22])

$$D_{q,z}h(z) = \frac{h(qz) - h(z)}{(q-1)z} \ (z \neq 0, q \neq 1) \tag{5}$$

It is clear to observe that

$$D_{q,z}(\sum_{k=1}^{+\infty} a_k z^k) = \sum_{k=1}^{+\infty} [k,q] a_k z^{k-1} \tag{6}$$

The definition of the q- number shift factorial is

$$[k,q]! = \begin{cases} 1 & , \ k = 0 \\ [1,q][2,q][3,q]...[k,q] & , \ k \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases} \tag{7}$$

The q- generalized pochhammer for $n \in \mathbb{R}$ is defined by :

$$[n,q]_m = \begin{cases} 1 & , \ m = 0 \\ [n,q][n+1,q][n+2,q] \cdots [n+m-1,q] & , \ m \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases} \tag{8}$$

(see for instance, [1],[2],[4], [9])

2. Main results

Definition 2.1. The differential operator $\Omega_{\lambda,q} : M(p) \longrightarrow M(p)$ and $h(z) \in M(p)$ by

$$\Omega_{\lambda,q}h(z) = (1 + \lambda[p,q])h(z) + \lambda q^p z D_q h(z) \tag{9}$$

where $\lambda \geq 0$.
Now using(1),we get

$$\Omega_{\lambda,q}h(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} (1 + \lambda[p,q] + \lambda q^p [k,q]) a_k z^k, \tag{10}$$

Definition 2.2. We define this operator in such a way that

$$\Omega^0_{\lambda,q}h(z) = h(z) \tag{11}$$

and

$$\Omega^2_{\lambda,q}h(z) = \Omega_{\lambda,q}(\Omega_{\lambda,q}h(z)) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} (1 + \lambda[p,q] + \lambda q^p [k,q])^2 a_k z^k, \tag{12}$$

In the identical way, for $m \in \mathbb{N} = \{1, 2, \cdots\}$.

$$\Omega^m_{\lambda,q}h(z) = \Omega_{\lambda,q}(\Omega^{m-1}_{\lambda,q}h(z)) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} (1 + \lambda[p,q] + \lambda q^p [k,q])^m a_k z^k, \tag{13}$$

Remark 2.3. From(9)and (13) after some simplification,we get the identity

$$\Omega_{\lambda,q}^{m+1}h(z) = \lambda q^p z D_q \Omega_{\lambda,q}^m h(z) + (1+\lambda[p,q])\Omega_{\lambda,q}^m h(z). \tag{14}$$

Inspired by the above-mentioned works ,we now define the new subclass $M_{\lambda,q}[p,m,A,B]$ of meromophic functions $M(p)$ using the idea of the operator $\Omega_{\lambda,q}^m$ as follows.

Definition 2.4. Let $-1 \leq B < A \leq 1$and $0 < q < 1$. Then a function $h \in M(p)$ given(1) is said to be in the class $M_{\lambda,q}[p,m,A,B]$ if it satisfies the inequality

$$\left|\frac{\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + 1]}{B\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + A]}\right| < 1. \tag{15}$$

Theorem 2.5. *Let $h \in M(p)$,of the form (1) then $h \in M_{\lambda,q}[p,m,A,B]$ if and only if*

$$\begin{aligned} &\sum_{k=p+1}^{+\infty} ((1+B)q^p[k,q]) + (1+A)[p,q])(1+\lambda[p,q]+\lambda q^p[k,q])^m \left|a_k\right| \\ \leq\ & [p,q](A-B). \end{aligned} \tag{16}$$

Proof. To validate the sufficiency condition,we aim to verify that subrdination inequality(15) is indeed satisfied under the given assumptions.

$$\begin{aligned} &\left|\frac{\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + 1]}{B\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + A]}\right| \\ =\ & \left|\frac{q^p z D_q \Omega_{\lambda,q}^m h(z) + [p,q]\Omega_{\lambda,q}^m h(z)]}{A[p,q]\Omega_{\lambda,q}^m h(z) + Bq^p z D_q \Omega_{\lambda,q}^m h(z)}\right| \\ =\ & \left|\frac{\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (q^p[k,q]+[p,q])a_k z^k}{\frac{(A-B)[p.q]}{z^p} + \sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (A[p,q]+Bq^p[k,p])a_k z^k}\right| \\ =\ & \left|\frac{\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (q^p[k,q]+[p,q])a_k z^{k+p}}{(A-B)[p.q] + \sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (A[p,q]+Bq^p[k,p])a_k z^{k+p}}\right| \\ \leq\ & \frac{\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (q^p[k,q]+[p,q])|a_k|}{(A-B)[p,q] - (1+\lambda[p,q]+\lambda q^p[k,q])^m (A[p,q]+Bq^p[k,p])|a_k|} \\ \leq\ & 1. \end{aligned}$$

using inequality(16) we can get the direct part the proof.
For the converse part, let $h \in M_{\lambda,q}[p,m,A,B]$ be given (1). Then from(15), for $z \in \mathbb{U}^*$,we have

$$\begin{aligned} &\left|\frac{\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + 1]}{B\frac{q^p z D_q \Omega_{\lambda,q}^m h(z)}{[p,q]\Omega_{\lambda,q}^m h(z)} + A]}\right| \\ =\ & \left|\frac{\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (q^p[k,q]+[p,q])a_k z^{k+p}}{(A-B)[p.q] + \sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m (A[p,q]+Bq^p[k,q])a_k z^{k+p}}\right| \end{aligned}$$

Since $Re(z) \leq |z|$ for all z, therefore

$$Re\{\frac{\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m(q^p[k,q]+[p,q])a_k z^{k+p}}{(A-B)[p.q]+\sum_{k=p+1}^{+\infty}(1+\lambda[p,q]+\lambda q^p[k,q])^m(A[p,q]+Bq^p[k,q])a_k z^{k+p}}\}<1.$$

Clearing the denominator in (17) and $z \to \overline{1}$, through real values, we obtain (16). □

3. Conclusion

In our present paper, we were essentially motivated by the recent research going on in this field of study and we have first introduced a class of meromorphic multivalent function with $-q$ qdifferential operator.We recall the attention of curious readers to the prospect influenced by Srivastava's [7] newly published survey-cum-expository review paper that the (p, q)-extension would be a relatively minor and unimportant change, as the new parameter p is redundant (for details, see Srivastava [7, p. 340]). Furthermore, in light of Srivastava's recent result [5] and[8], the interested reader's attention is brought to further investigation of the (k, s)-extension of the Riemann–Liouville fractional integral.

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Certaion subclass of p-valent meromorphic multivalent q-starlike and q-convex sunctions with$q$-differential operator

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Abstract. The principal aim of this paper introduced and defined a new subclass of meromorphic multivalent function $S_{\lambda,q}[p,m,A,B]$ using the $q-$ differential operator in punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$. Furthermore, we obtain some radii of starlikeness and convexity and investigate about partial sum property.

Keywords: $q-$ differential, meromorphic function,starlikeness , convexity

AMS Mathematics Subject Classification [2020]: 58E11, 53B30, 53C50 (at least 1 and at most 3)

1. Introduction

Let S_p denote the class of meromorphic functions of the type

$$f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} a_k z^k, \ (a_k \geq 0 \ , \ p \in \mathbb{N}) \ , \ \mathbb{U}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\} \tag{1}$$

Also, Jacksons q- derivative and q- integral of a function $f(z)$ defined on a subset of $\mathbb{C}$ are, respectively, given (see [1-2-4-5])

$$D_{q,z} f(z) = \frac{f(z) - f(qz)}{(1-q)z} \ (z \neq 0, q \neq 1) \tag{2}$$

The q- generalized pochhammer for $n \in \mathbb{R}$ is defined by :

$$[n,q]_m = \begin{cases} 1 & , \ m = 0 \\ [n,q][n+1,q][n+2,q] \cdots [n+m-1,q] & , \ m \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases} \tag{3}$$

*Speaker.

2. Main results

Definition 2.1. The differential operator $\phi_{\theta,q} : S_p \longrightarrow S_p$ and $f(z) \in S_p$ by

$$\phi_{\theta,q} f(z) = \theta q^p z D_q f(z) + (1 + [p,q]\theta) f(z) \ , \ \theta \geq 0. \tag{4}$$

Now using(1),we get

$$\phi_{\theta,q} f(z) = \frac{1}{z^p} + \sum\nolimits_{k=p+1}^{+\infty} \theta q^p [k,q] a_k z^k + (1 + [p,q]\theta), \tag{5}$$

Definition 2.2. We define this operator in such a way that for $m \in \mathbb{N} = \{1, 2, \cdots\}$.

$$\phi^m_{\theta,q} f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} (1 + \theta[p,q] + \theta q^p [k,q])^m a_k z^k, \tag{6}$$

We create a new subclass $S_{\theta,q}[p, m, A, B]$ of S_p by utilizing the operator $\phi^m_{\theta,q}$ as follows.

Definition 2.3. Let $-1 \leq B < A \leq 1$and $0 < q < 1$. Then a function $f \in S_p$ given(1) is said to be in the class $S_{\theta,q}[p, m, A, B]$ if it satisfies the inequality

$$\Big| \frac{q^p z D_q \phi^m_{\theta,q} f(z) + [p,q] \phi^m_{\theta,q} f(z)}{B q^p z D_q \phi^m_{\theta,q} f(z) + A[p,q] \phi^m_{\theta,q} f(z)} \Big| < 1. \tag{7}$$

Theorem 2.4. *Let $f \in S_p$, then $f \in S_{\theta,q}[p, m, A, B]$ if and only if*

$$\sum_{k=p+1}^{+\infty} ((1 + B) q^p [k,q]) + (1 + A)[p,q])(1 + \theta[p,q] + \theta q^p [k,q])^m \Big| a_k \Big| \leq [p,q](A - B).$$

Proof. Using relationship (2) and (6) we get the direct part the proof. □

Theorem 2.5. *Let $f \in S_{\theta,q}[p, m, A, B]$ and has the form (1).Then for $0 < r = |z| < 1$*

$$r^p \Big(\frac{1}{r^{2p}} - \frac{[p,q](A - B)}{(q^{p+1}[k,q](1 + B) + (1 + A)[p + 1, q])(1 + \theta[p + 1, q] + \theta q^{p+1}[k,q])^m} \Big) \leq |f(z)|$$

$$\leq r^p \Big(\frac{1}{r^{2p}} + \frac{[p,q](A - B)}{(q^{p+1}[k,q](1 + B) + (1 + A)[p + 1, q])(1 + \theta[p + 1, q] + \theta q^{p+1}[k,q])^m} \Big)$$

Proof. Since $f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} a_k z^k$ and $a_k \geq 0$, we have

$$|f(z)| = \Big| \frac{1}{z^p} + \sum_{k=p+1}^{+\infty} a_k z^k \Big| \leq \frac{1}{|z|^p} + \sum_{k=p+1}^{+\infty} a_k |z|^k \leq r^p \Big(\frac{1}{r^{2p}} + \sum_{k=p+1}^{+\infty} a_k r^{k-p} \Big)$$

$$|f(z)| \leq \Big(r^p \Big(\frac{1}{r^{2p}} + \sum_{k=p+1}^{+\infty} |a_k| \Big). \ (r^k \leq r^p, \ r \leq 1, \ k \geqslant p + 1, \ |z| = r < 1) \tag{8}$$

Similarly, we have

$$|f(z)| \geq r^p \Big(\frac{1}{r^{2p}} - \sum_{k=p+1}^{+\infty} |a_k| \Big). \tag{9}$$

Thus from theorme(2.4),we obtain

$$((1+B)q^{p+1}[k,q])+(1+A)[p+1,q])(1+\theta[p+1,q]+\theta q^{p+1}[k,q])^m \sum_{k=p+1}^{+\infty}|a_k|$$
$$\leq \sum_{k=p+1}^{+\infty}((1+B)q^{p}[k,q])+(1+A)[p,q])(1+\theta[p,q]+\theta q^{p}[k,q])^m|a_k|.$$

Similarly, we have

$$\sum_{k=p+1}^{+\infty}((1+B)q^{p+1}[k,q])+(1+A)[p+1,q])(1+\theta[p+1,q]+\theta q^{p+1}[k,q])^m|a_k|$$
$$\leq \quad [p,q](A-B)$$

which also can be written as

$$\sum_{k=p+1}^{+\infty}|a_k| \leq \frac{[p,q](A-B)}{((1+B)q^{p+1}[k,q])+(1+A)[p+1,q])(1+\theta[p+1,q]+\theta q^{p+1}[k,q])^m}$$

Now by putting this value into(8)and (9), we get the reuired result. □

3. Radii Condition and partial sum property

In this section we obtain radii of starlikeness and investigate convexity and convexity about partial sum property.
A function $f \in CS_p(\alpha)$ drenote the popular classes of meromorphic multivalent convex functions of the order α if $f \in S_p$ satisfies the inequality

$$Re\Big(-\Big(1+\frac{zf''(z)}{f'(z)}\Big)>\alpha\Big)\Big). \ (0 \leqslant \alpha < p).$$

THEOREM 3.1. *If $f \in S_{\theta,q}[p,m,A,B]$, then $f \in CS_p(\alpha)$ for in disk $|z|<R_1$, where*

$$R_1=\{\frac{p(p-\alpha)(((1+B)q^{p+k}[k,q])+(1+A)[p+k,q])(1+[p+k,q]\theta+\theta q^{p+k}[k,q])^m)}{(p+k)(p+k+\alpha)(A-B)[p-q]}\}^{\frac{1}{k+2p}}$$

PROOF. Assume that $f \in S_{\theta,q}[p,m,A,B]$. In ordeer to establish that $f \in CS_p(\alpha)$, demonstrating that following condition is sufficient for the proof

$$|\frac{zf''(z)+(p+1)f'(z)}{zf''(z)+(1+2\alpha-p)f'(z)}|<1,$$

Using (1) after some simple computation,we get

$$\sum_{k=1}^{+\infty}\frac{(p+k)(p+k+\alpha)}{p(p+\alpha)}|a_{k+p}||z|^{k+2p} \leq 1, \tag{10}$$

Now inequality (10) and theorem (2.4) will hold if

$$\sum_{k=1}^{+\infty}\frac{(p+k)(p+k+\alpha)}{p(p+\alpha)}|a_{k+p}||z|^{k+2p} \leq$$

$$\sum_{k=1}^{+\infty}\frac{(((1+B)q^{p+k}[k,q])+(1+A)[p+n,q])(1+[p+k,q]\theta+\theta q^{p+k}[k,q])^m)}{(A-B)[p,q]}|a_{k+p}|,$$

which implies that

$$|z|^{k+2p} \leq \frac{p(p-\alpha)(((1+B)q^{p+k}[k,q]) + (1+A)[p+k,q])(1+[p+k,q]\theta + \theta q^{p+k}[k,q])^m)}{(p+k)(p+k+\alpha)(A-B)[p,q]},$$

and thus

$$|z| \leq \{\frac{p(p-\alpha)(((1+B)q^{p+k}[k,q]) + (1+A)[p+k,q])(1+[p+k,q]\theta + \theta q^{p+k}[k,q])^m)}{(p+k)(p+k+\alpha)(A-B)[p,q]}\}^{\frac{1}{k+2p}}$$

from which we get the desired condition. □

A function $f \in CS_p^*(\alpha)$ drenote the popular classes of meromorphic multivalent starlike functions of the order α if $f \in S_p$ satisfies the inequality

$$Re\Big(-\frac{zf'(z)}{f(z)}\Big) > \alpha, \ (0 \leqslant \alpha < p).$$

Remark 3.2. Let $f \in S_{\theta,q}[p,m,A,B]$ then $f \in CS_p^*(\alpha)$ if and only if

$$\Big|\frac{zf'(z)+pf(z)}{zf'(z)+(2\alpha-p)f(z)}\Big| \leq 1, \ (0 \leqslant \alpha < p). \tag{11}$$

4. Conclusion

In our present paper, We next investigate some useful properties such as coefficient estimates, growth and distortion theorems, radii of starlikeness and convexity, convex linear combination, extreme points theorem and the arithmetic mean for the present subclass of meromorphic multivalent functions. Then we characterized these functions with the help of some useful their properties like suffciency criteria, distortion bounds, coeffcient estimates,radius of starlikness, radius of convexity, inclusion property, and convex combinations,(see [3-6-8-9]).

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On the Ma-Minda Starlikeness of the Polynomial $(1+xz)^3$

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Abstract. We investigate the geometric properties of the cubic family $F_x(z) = (1+xz)^3$ for $0 < x \leq 1$ in the unit disk. We establish sharp bounds for univalence, starlikeness, and convexity, derive extremal modulus estimates, and introduce a new Ma-Minda type class $\mathcal{ST}^x$ with the Limaçon domain as comparison region.

Keywords: Univalent functions, Starlike function,

AMS Mathematics Subject Classification [2020]: 30C45, 30C80

1. Introduction

Throughout, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc and

$$\mathcal{A} = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\}$$

is the class of normalized holomorphic functions on $\mathbb{D}$. The subclass $\mathcal{S} \subset \mathcal{A}$ consists of the univalent members. A function $f \in \mathcal{S}$ is called

- **starlike**, written $f \in \mathcal{ST}$, if $f(\mathbb{D})$ is star-shaped with respect to the origin, or equivalently
$$\Re \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D};$$
- **convex**, written $f \in \mathcal{CV}$, if $f(\mathbb{D})$ is a convex domain, or equivalently
$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \quad z \in \mathbb{D}.$$

For $f, g \in \mathrm{Hol}(\mathbb{D})$, we write $f \prec g$ and say that f is subordinate to g if there exists an analytic self-map $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f = g \circ \omega$. When g is univalent, $f \prec g$ is equivalent to the two conditions (see for e.g. [1])

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

*Speaker.

Ma and Minda [2] proposed a unifying scheme: fix an analytic model function φ mapping $\mathbb{D}$ onto a domain symmetric about the real axis, starlike with respect to $\varphi(0) = 1$, and with $\varphi'(0) > 0$. Using φ as a comparison domain, one introduces

$$\mathcal{P}(\varphi) = \{p \in \mathrm{Hol}(\mathbb{D}) : p(0) = 1,\ p \prec \varphi\}$$

and then defines the φ-starlike and φ-convex classes

$$\mathcal{ST}(\varphi) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}(\varphi)\right\}, \qquad \mathcal{CV}(\varphi) = \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}(\varphi)\right\}.$$

The present paper focuses on the cubic family

$$\begin{aligned} F_x(z) &= (1 + xz)^3, \\ &= 1 + 3xz + 3x^2z^2 + x^3z^3, \qquad 0 < x \le 1. \end{aligned} \tag{1}$$

The present paper shows that this single one-parameter family already encodes the full spectrum of behaviors-univalence, convexity, starlikeness, strong starlikeness, extremal growth-that are usually discussed with far more complicated constructions. We determine the sharp bounds on x that guarantee each property, compute the extremal modulus on every circle $|z| = r < 1$, and use F_x as a model domain to introduce a new Ma–Minda type class $\mathcal{ST}^x$ whose geometry is completely dictated by a Limaçon region [3].

1.1. Univalence of F_x.

Theorem 1.1. *Let the functions F_x be defined by (1). Then F_x is univalent on the open unit disk $\mathbb{D}$ if and only if*

$$0 < x \le \frac{\sqrt{3}}{2}.$$

1.2. Real Part Estimates.

Theorem 1.2. *Let the functions F_x be defined by (1). Then for all $z \in \mathbb{D}$ and $0 < x \le 1$,*

$$\min_{z\in\mathbb{D}} \Re\{F_x(z)\} = \begin{cases} (1-x)^3 & \text{for} \quad 0 < x \le \frac{1}{3}, \\ \left(\frac{1}{2} - x\right)(1+x)^2 & \text{for} \quad \frac{1}{3} < x \le 1, \end{cases}$$

and

$$\max_{z\in\mathbb{D}} \Re\{F_x(z)\} = (1+x)^3.$$

Theorem 1.3. *Let the functions F_x be defined by (1). Then F_x has positive real part in the open unit disk $\mathbb{D}$ if and only if*

$$0 < x \le \frac{1}{2}.$$

1.3. Convexity of the Normalized Map.

Theorem 1.4. *Let the function f_x be defined by*

$$f_x(z) := \frac{F_x(z) - 1}{3x}, \qquad z \in \mathbb{D}. \tag{2}$$

Then f_x is convex in the open unit disk $\mathbb{D}$ if and only if

$$0 < x \le \frac{1}{3}.$$

1.4. Starlikeness of the Normalized Map.

THEOREM 1.5. *Let the function f_x given by (2). Then f_x is starlike in the open unit disk $\mathbb{D}$ if and only if*

$$0 < x \leq \sqrt{\frac{5}{7}}.$$

1.5. Extremal Modulus on Concentric Circles.

THEOREM 1.6. *Let the functions F_x be defined by (1). Then, for any $r \in [0,1)$ and $|z| = r$, we have*

$$\min_{|z|=r} |F_x(z)| = F_x(-r) \quad \text{and} \quad \max_{|z|=r} |F_x(z)| = F_x(r).$$

2. The Class $\mathcal{ST}^x$ and Its Properties

LEMMA 2.1. *Let F_x be defined by (1), where $0 < x \leq \frac{1}{2}$. Then F_x belongs to the Ma–Minda admissible class. In particular, the following properties hold:*

(i) *F_x is univalent in $\mathbb{D}$ and satisfies $\Re\{F_x(z)\} > 0$ for all $z \in \mathbb{D}$;*
(ii) *F_x is starlike with respect to the point $F_x(0) = 1$;*
(iii) *The image $F_x(\mathbb{D})$ is symmetric with respect to the real axis;*
(iv) *$F_x'(0) = 3x > 0$.*

Consequently, F_x is a conformal map whose image domain is bounded by a Limaçon curve.

DEFINITION 2.2. Let $\mathcal{ST}^x$ denote the subclass of $\mathcal{A}$ consisting of analytic functions f that satisfy the subordination condition

$$\frac{zf'(z)}{f(z)} \prec F_x(z), \qquad z \in \mathbb{D}, \quad 0 < x \leq \frac{1}{2}.$$

From Theorem 1.2, it follows that for every $f \in \mathcal{ST}^x$, the following real part condition holds:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \begin{cases} (1-x)^3, & \text{for } 0 < x \leq \dfrac{1}{3}, \\ \left(\dfrac{1}{2} - x\right)(1+x)^2, & \text{for } \dfrac{1}{3} < x \leq \dfrac{1}{2}, \end{cases} \qquad z \in \mathbb{D}.$$

We define the functions ψ_x in $\mathcal{ST}^x$ by the relation

$$\frac{z\psi_x'(z)}{\psi_x(z)} = F_x(z),$$

which gives

$$\begin{aligned} \psi_x(z) &= z \exp\left(\int_0^z \frac{F_x(t) - 1}{t}\, \mathrm{d}t\right) \\ &= z \exp\left(3xz + \frac{3x^2z^2}{2} + \frac{x^3z^3}{3}\right) \end{aligned} \tag{3}$$

The largest open Euclidean disk centered at 1 that is entirely contained in the image $F_x(\mathbb{D})$ is

$$R_{\max} = \{\, w \in \mathbb{C} : |w - 1| < r_{\max} \,\}, \qquad r_{\max} = 1 - (1-x)^3.$$

The smallest open Euclidean disk centered at 1 that contains the entire image is

$$R_{\min} = \{\, w \in \mathbb{C} : |w - 1| < r_{\min} \,\}, \qquad r_{\min} = (1+x)^3 - 1.$$

The image satisfies the sharp two-sided inclusion

$$\{\, w : |w-1| < 1-(1-x)^3 \,\} \;\subset\; F_x(\mathbb{D}) \;\subset\; \{\, w : |w-1| < (1+x)^3 - 1 \,\}.$$

2.1. Argument Estimates and Strong Starlikeness.

THEOREM 2.3. *Let the functions F_x be defined by (1). Then, for all $z \in \mathbb{D}$ and $0 < x \leq \frac{1}{2}$, the following inequalities hold:*

$$|\arg(F_x(z))| < 3 \arcsin x,$$

and consequently, the inclusion

$$\mathcal{ST}^x \subseteq \mathcal{ST}_{\frac{6}{\pi} \arcsin x}$$

is valid.

3. Conclusion

We have conducted a comprehensive geometric analysis of the cubic family $F_x(z) = (1+xz)^3$, determining sharp bounds for univalence, convexity, starlikeness, and other geometric properties. Our results reveal a clear hierarchy of conditions on the parameter x:

$$0 < x \leq \frac{1}{2} \implies \Re(F_x(z)) > 0$$

$$0 < x \leq \frac{\sqrt{3}}{2} \implies F_x \text{ is univalent}$$

$$0 < x \leq \frac{1}{3} \implies f_x \text{ is convex}$$

$$0 < x \leq \sqrt{\frac{5}{7}} \implies f_x \text{ is starlike}$$

Acknowledgement

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Fixed Point approximations for Mappings in $CAT(0)$ spaces

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Abstract. In this paper, we study fixed point approximations for mappings in geodesic spaces, and we prove some stability results in fixed point theory for contraction mappings in geodesic spaces. we also consider some extention theorems for these spaces.[1]

1. Introduction and Preliminary

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subseteq R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(x1, x2, x3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.
A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

Fixed point theorems in $CAT(0)$ spaces have been developed in several recent papers including [2, 3, 4, 5]. The existence of fixed points for nonexpansive mappings in a complete $CAT(0)$ space was proved by Kirk ([2]) as follows:

Theorem 1.1. *Suppose K is a nonempty, bounded, closed and convex subset of complete $CAT(0)$space and suppose $f : K \longrightarrow K$ is nonexpansive. Then f has a fixed point.*

2. Main results

Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X, d). Consider

$$A := \{A : A \textit{ is a contraction selfmap on } K \textit{ with } Fix(A) \neq \emptyset\},$$

with

$$\rho(A, B) = \sup\{d(Ax, Bx) : x \in K\}$$

for every $A, B \in A$. Then (A, ρ) is a complete metric space.

Definition 2.1. Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $CAT(0)$ space (X, d). A map $A : K \to K$ is said to have the stable fixed point property if there exist $x \in Fix(A)$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni (B \in A, \rho(A, B) \leq \delta) \Longrightarrow$$
$$\exists z \in K \ni (Bz = z, d(z, x) \leq \varepsilon).$$

Theorem 2.2. *Let $K \subseteq X$ be a nonempty, bounded, closed and convex subset of a complete $CAT(0)$ space (X, d), $x \in X$, and $A \in A$. Soppose $\lambda : X \to [0, 1]$ be a continuous map. Define B on K by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Then B is contraction and $Bz \in K$ for all $z \in K$.*

Proof. Since K is a closed convex subset of a complete $CAT(0)$ space, $Bz \in K$ for all $z \in K$. To see that B is contraction, Let $z_1, z_2 \in K$, Then by definition of B ,we have

$$\begin{aligned}
&d(\lambda(z_1)x \oplus (1 - \lambda(z_1))Az_1, \lambda(z_2)x \oplus (1 - \lambda(z_2))Az_2) \\
&\leq d(\lambda(z_1)x \oplus (1 - \lambda(z_1))Az_1, \lambda(z_1)x \oplus (1 - \lambda(z_2))Az_2) \\
&+d(\lambda(z_1)x \oplus (1 - \lambda(z_2))Az_2, \lambda(z_2)x \oplus (1 - \lambda(z_2))Az_2) \\
&\leq |\lambda(z_1) - \lambda(z_2)|d(Az_1, Az_2) + |\lambda(z_1) - \lambda(z_2)|d(x, x) \\
&\leq |\lambda(z_1) - \lambda(z_2)|\alpha d(z_1, z_2).\square
\end{aligned}$$

Theorem 2.3. *Let $A \in A$ and $\varepsilon > 0$. then there exists $\delta > 0$ such that for each $B \in A$ satisfying $\rho(A, B) \leq \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in Fix(A)$ such that $d(x, y) \leq \varepsilon$.*

Proof. In contrary, suppose,

$$\exists \varepsilon > 0 \quad \forall n \quad B_n \in A \quad \textit{such that} \quad \rho(A, B_n) \leq \frac{1}{n}$$

and

$$\exists x_n \in K \quad \textit{such that} \quad B_n x_n = x_n$$

and $d(x_n, y) > \varepsilon$ for all $y \in Fix(A)$. Since K is compact, we may assume without loss of generality that there exists $x \in K$ such that $x_n \to x$ as $k \to \infty$. So

$$\begin{aligned}
d(Ax, x) &\leq d(Ax, Ax_n) + d(Ax_n, B_n x_n) + d(B_n x_n, x_n) + d(x_n, x) \\
&\leq d(Ax, Ax_n) + \frac{1}{n} + d(x_n, x).
\end{aligned}$$

Therefore $Ax = x$ hence $x \in Fix(A)$ and $d(x_n, x) > \varepsilon$ for all n. This is a contradiction.$\square$

In view of this result, It is natural to ask,does for each $A \in A$, A have the stable fixed point property?

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall B \in A$ satisfying $\rho(A, B) \leq \delta$, $\exists y \in Fix(A)$ such that $By = y$, and $d(x, y) \leq \varepsilon$.

Example 2.4. Put $X := \mathbb{R}$, $K := [0,1]$ and $Ax = x$ for all $x \in K$ so $Fix(A) = K$. and let $A_n x = (1 - \frac{1}{n})x$ and $B_n x = \min\{x + \frac{1}{n}, 1\}$ for all n. Therefore $A_n, B_n \to A$ and $Fix(A_n) = \{0\}$ and $Fix(B_n) = [1 - \frac{1}{n}, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this paper that for a typical $A \in A$ the answer is positive.

Theorem 2.5. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Then there exist $B \in A$ and $\delta > 0$ such that $\rho(A,B) \leq \varepsilon$ and $Bz = x$ for each $z \in K$ satisfying $d(z,x) \leq \delta$.*

Proof. By continuity of A there exists $\delta > 0$ such that $d(z,x) \leq \delta$ implies that $d(Az,x) \leq \varepsilon$ for all $z \in K$. Since (X,d) is normal, by Urysohn's theorem, there exists a continuous map $\lambda : X \to [0,1]$ such that

$$\begin{cases} 1 & \forall z \in X : d(z,x) \leq \frac{\delta}{4}; \\ 0, & \forall z \in X : d(z,x) \geq \frac{\delta}{2}. \end{cases}$$

Define $B : K \to K$ by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Since K is a closed convex subset of a complete $\mathbb{R}$-tree, $Bz \in K$ for all $z \in K$. In particular, B is continuous, by theorem (2.1). $B(K)$ is contained in a compact subset of X and $Bx = x$.

(1) For all $z \in K$ which $d(z,x) \geq \frac{\delta}{2}$ according to $Bz = Az$ we have

$$d(Az, Bz) = 0$$

(2) For all $z \in K$ which $d(z,x) \leq \frac{\delta}{4}$ according to $Bz = x$ we have

$$d(Az, Bz) = d(Az, x) < \varepsilon$$

(3) For all $z \in K$ which $d(z,x) \leq \frac{\delta}{2}$ we have

$$\begin{aligned} d(Bz, Az) &= d(\lambda(z)x \oplus (1 - \lambda(z))Az, Az) \\ &\leq \lambda(z)d(x, Az) + (1 - \lambda(z))d(Az, Az) \\ &\leq d(x, Az) \leq \varepsilon. \square \end{aligned}$$

Theorem 2.6. *Let (X,d) be a geodesically bounded normal complete $CAT(0)$ space, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Let $B \in A$ and $\delta > 0$ be as guaranteed by theorem (2.5). Then for each $C \in A$ which $\rho(C,B) \leq \delta$, there exists $y \in K$ such that $Cy = y$ and $d(y,x) \leq \rho(B,C)$.*

Proof. By theorem (2.5), $\rho(A,B) \leq \varepsilon$ and $Bz = x$, for all $z \in K$ satisfies $d(z,x) \leq \delta$. Assume that $C \in A$ satisfies $\rho(C,B) \leq \delta$. Set

$$\Gamma := \{z \in K : d(z,x) \leq \rho(C,B)\}.$$

Clearly, Γ is closed convex set, thus

$$\begin{aligned} d(x, Cz) &\leq d(x, Bz) + d(Bz, Cz) \\ &= d(Bz, Cz) \\ &\leq \rho(C,B), \end{aligned}$$

$Cz \in \Gamma$ for all $z \in \Gamma$. So $C(\Gamma) \subseteq \Gamma$, clearly $C(\Gamma) \subseteq C(X)$ is contained in a compact subset of X. Now by theorem (1.7) there exists $y \in K$ such that $Cy = y$.$\square$

Theorem 2.7. *Let (X, d) be a geodesically bounded normal complete $CAT(0)$ space. Then there exists a subset F of A which is a countable intersection of open subsets of (A, ρ) so that for each $A \in F$, A have the stable fixed point property.*

Proof. Let $A \in A$ and $\varepsilon > 0$. By theorem (2.5) and (2.6) there exist $A_\varepsilon \in \Gamma$, $x_{A,\varepsilon} \in K$ and $\delta_{A,\varepsilon} \in (0, 1)$ such that

$$\rho(A, A_\varepsilon) \leq \varepsilon, \quad A_\varepsilon z = x_{A,\varepsilon},$$

for all $z \in K$ satisfying $d(z, x_{A,\varepsilon}) \leq \delta_{A,\varepsilon}$ and for all $C \in A$ satisfying $\rho(C, A_\varepsilon) \leq \delta_{A,\varepsilon}$ there exists $y \in K$ such that $Cy = y$ and $d(y, x_{A,\varepsilon}) \leq \rho(C, A_\varepsilon)$. For each integer $i \geq 1$, set $U(A, \varepsilon, i) := \{C \in A : \rho(C, A_\varepsilon) < \frac{\delta_{A,\varepsilon}}{i}\}$. Define

$$F := \bigcap_{i=1}^{\infty} \bigcup_{A \in A, \varepsilon \in (0,1)} U(A, \varepsilon, i).$$

Clearly F is a countable intersection of open subsets of (A, ρ).
Let $B \in F$, for all $i \geq 1$ there exists $A_i \in \Gamma$ and $\varepsilon_i \in (0, 1)$ such that $B \in U(A_i, \varepsilon_i, i)$, then for all $i \geq 1$ there exists $y_i \in K$ such that

$$By_i = y_i, \quad d(y_i, x_{A_i,\varepsilon_i}) \leq \rho(B, (A_i)_{\varepsilon_i}) \leq \frac{\delta_{A,\varepsilon_i}}{i},$$

since $\{y_i\}_i \subseteq B(K)$ so there exists subsequence $\{y_{i_k}\}_k$ which $y_{i_k} \to x$ for some $x \in K$. Let $\varepsilon > 0$ so there exists $k \in \mathbb{N}$ such that

$$\frac{1}{i_k} < \frac{\varepsilon}{8}, \quad d(y_{i_k}, x) \leq \frac{\varepsilon}{8},$$

then it follows from above

$$d(y_{i_k}, x_{A_{i_k},\varepsilon_{i_k}}) \leq \frac{1}{i_k} < \frac{\varepsilon}{8},$$

and

$$d(x, x_{A_{i_k},\varepsilon_{i_k}}) \leq d(x, y_{i_k}) + d(y_{i_k}, x_{A_{i_k},\varepsilon_{i_k}}) \leq \frac{\varepsilon}{4}.$$

Let $C \in U(A_{i_k}, \varepsilon_{i_k}, i_k)$ then there exists $z \in K$ such that $Cz = z$ and

$$d(z, x_{A_{i_k},\varepsilon_{i_k}}) \leq \rho(C, (A_{i_k})_{\varepsilon_{i_k}}) \leq \frac{1}{i_k} \leq \frac{\varepsilon}{8},$$

this implies that

$$d(z, x) \leq d(z, x_{A_{i_k},\varepsilon_{i_k}}) + d(x_{A_{i_k},\varepsilon_{i_k}}, x) \leq \frac{\varepsilon}{2}. \square$$

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Hybrid Algorithm for Generalized Mixed Equilibrium Problems and Fixed Points of W-Mappings in Banach Spaces

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Abstract. We propose a hybrid iterative algorithm for solving generalized mixed equilibrium problems and fixed point problems of countable nonexpansive mappings in Banach spaces. The method integrates W-mappings with a hybrid projection scheme. Under suitable assumptions, strong convergence to the solution is proved. The results generalize existing theorems and provide a unified approach for these problems.

Keywords: Generalized mixed equilibrium problem, W-Mappings, Strong convergence.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X with norm $\|\cdot\|$ and X^* be the dual of X. Assume that $A : C \to 2^{X^*}$ is a nonlinear mapping, $\phi : C \to \mathbb{R}$ is a real-valued function and $\psi : C \times C \to \mathbb{R}$ is a bifunction. The generalized mixed equilibrium problem (GMEP) is: Find $x \in C$ such that

$$\psi(x, y) + \langle Ax, y - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C. \tag{2.1}$$

The solution set of (2.1) is denoted by GMEP(ψ, ϕ, A).

The duality mapping J_X^p on X is defined by

$$J_X^p(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in X$. If X is a p-uniformly convex and uniformly smooth, then J_X^p is single valued, one-to-one and satisfies $J_X^p = (J_X^*)^{-1} = (J_X^q)^{-1}$, where $J_X^* = J_X^q$ is the duality mapping of X (see [1]). If $p = 2$, then $J_X = J_2 = J$ is the normalized duality mapping. It is well known that if X is a reflexive, strictly convex and smooth Banach space and $J_X^* : X^* \to 2^X$ is the duality mapping on X^*, then $J_X^{-1} = J_X^*$.

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The function $\varphi : X \times X \to \mathbb{R}$ is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, J_X y\rangle + \|y\|^2, \quad \forall x, y \in X.$$

Let $\psi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $\psi(x, x) = 0$ for all $x \in C$,
(A2) ψ is monotone, i.e. $\psi(x, y) + \psi(y, x) \leq 0$, for all $x, y \in C$,
(A3) $\lim_{t\downarrow 0} \psi(tz + (1 - t)x, y) \leq \psi(x, y)$, for all $x, y, z \in C$,
(A4) The function $y \mapsto \psi(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 1.1. *[2] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space X. Let $A : C \longrightarrow X^*$ be a continuous and monotone mapping, $\phi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and ψ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Then for all $r > 0$ the following hold:*

(1) *For $x \in X$, there exists $u \in C$ such that*

$$\psi(u, y) + \langle Au, y - u\rangle + \phi(y) - \phi(u) + \frac{1}{r}\langle y - u, J_X u - J_X x\rangle \geq 0, \quad \forall y \in C,$$

(2) *If we define a mapping $K_r : C \longrightarrow C$ by*

$$K_r(x) = \{u \in C : \psi(u, y) + \langle Au, y - u\rangle + \phi(y) - \phi(u) + \frac{1}{r}\langle y - u, J_X u - J_X x\rangle \geq 0, \ \forall y \in C\},$$

then, the mapping K_r has the following properties:

(a) *K_r is single-valued,*
(b) *K_r is a firmly nonexpansive mapping, i.e. for all $x, y \in X$,*

$$\langle K_r x - K_r y, J_X K_r x - J_X K_r y\rangle \leq \langle K_r x - K_r y, J_X x - J_X y\rangle,$$

(c) $F(K_r) = \hat{F}(K_r) = \mathrm{GMEP}(\psi, \phi, A)$,
(d) $\mathrm{GMEP}(\psi, \phi, A)$ *is a closed convex,*
(e) $\varphi(p, K_r z) + \varphi(K_r z, z) \leq \varphi(p, x), \quad \forall p \in F(K_r), z \in X.$

Let C be a nonempty, closed and convex subset of X. Let $\{S_n\}$ be a family of mappings of C into itself and let $\{\delta_k : k \in \mathbb{N}\}$ be a sequence of real numbers such that $0 \leq \delta_i \leq 1$ for every $i \in \mathbb{N}$. Shimoji et al. [3], introduced a mapping W_n of C into itself for each $n \in \mathbb{N}$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \delta_n S_n U_{n,n+1} + (1 - \delta_n)I, \\
U_{n,n-1} &= \delta_{n-1} S_{n-1} U_{n,n} + (1 - \delta_{n-1})I, \\
&\vdots \\
U_{n,k} &= \delta_k S_k U_{n,k+1} + (1 - \delta_k)I, \\
&\vdots \\
U_{n,2} &= \delta_2 S_2 U_{n,3} + (1 - \delta_2)I, \\
W_n = U_{n,1} &= \delta_1 S_1 U_{n,2} + (1 - \delta_1)I. \qquad (2.11)
\end{aligned}$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping generated by S_n, S_{n-1}, ..., S_1 and δ_n, δ_{n-1}, ..., δ_1. Let C nonempty subset of a Banach space X

and $S : C \to C$ a mapping. Then S is said to be asymptotically regular on C if for all $x \in C$, $S^n x - S^{n+1}x \to 0$.

PROPOSITION 1.2 ([3]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and let $\{\delta_i\}$ be a real sequence such that $0 \leq \delta_i \leq b < 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\delta_n, \delta_{n-1}, \ldots, \delta_1$. Then*

(i) *W_n is asymptotically regular and nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$, for all $n \in \mathbb{N}$.*
(ii) *For every $x \in C$ and for each positive integer j, $\lim_{n\to\infty} U_{n,j}x$ exists.*
(iii) *The mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$, is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $\{S_i\}_{i\in\mathbb{N}}$ and $\{\delta_i\}_{i\in\mathbb{N}}$.*

PROPOSITION 1.3 ([4]). *Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $\{\delta_i\}$ be a real sequence such that $0 \leq \delta_i \leq b < 1$ for all $i \in \mathbb{N}$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

2. Main results

We assume that the proposed Algorithm 1 satisfies the following conditions:

(1) X is a 2-uniformly convex and uniformly smooth real Banach space that satisfies the Opial condition.
(2) C is a nonempty closed and convex subset of X.
(3) g is a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies the conditions (A1)-(A4).
(4) $\{S_i\}_{i\in\mathbb{N}}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$.
(5) $\{\delta_i\}$ is a real sequence such that $0 \leq \delta_i \leq b < 1$ for every $i \in \mathbb{N}$.
(6) For any $n \in \mathbb{N}$, W_n is the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\delta_n, \delta_{n-1}, \ldots, \delta_1$.
(7) $B : C \to X^*$ is an α-inverse strongly monotone operator.
(8) $\psi : C \to \mathbb{R}$ is a lower semi-continuous and convex function.

Algorithm 1 Iterative Algorithm

Initialization: Let $r_n \in [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$. Let $x_0 \in C$ be initial point.

Step 1: Find $y_n \in C$ such that

$$g(y_n, y) + \langle By_n, y - y_n\rangle + \psi(y) - \psi(y_n) + \frac{1}{r_n}\langle y - y_n, J_{X_1}y_n - J_{X_1}x_n\rangle \geq 0, \quad \forall y \in C,$$

Step 2: Compute $z_n = \Pi_C J_{X_1}^{-1}[\beta_n J_{X_1} W_n y_n + (1 - \beta_n) J_{X_1} x_n]$,

Step 3: Construct $C_{n+1} = \{z \in C_n : \phi(z, z_n) \leq \phi(z, x_n)\}$,

Step 4: Set $C_0 = C$ and compute $x_{n+1} = \Pi_{C_{n+1}} x_0$.

where $r_n \in [a, \infty)$ for some $a > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$.

THEOREM 2.1. *Suppose the conditions (A1)-(A4) hold. Let* $\Omega = (\bigcap_{i=1}^{\infty} F(S_i)) \cap GMEP(g, \psi, B) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to* $\Pi_\Omega x_0$.

3. Conclusion

This paper presented a hybrid iterative algorithm for finding common solutions to generalized mixed equilibrium problems and fixed points of countable families of nonexpansive mappings in Banach spaces. The method combines W-mappings with a hybrid projection technique. Under appropriate conditions, strong convergence of the sequence to the projection onto the solution set was established.

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The Stable Fixed Point Property for Continuous Mappings in Metric Trees

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Abstract. In this lecture, we introduce the concept of the stable fixed point property for mappings, and we prove the existence of the stable fixed point property for continuous mappings defined on a compact and convex subset of a complete metric tree. Also, inspired of the Kirk's proof of the fixed point property for continuous mappings in complete metric trees, we generalize some theorems in such spaces.[1]

1. Introduction and Preliminary

Let (X, d) be a metric space. A geodesic segment or a metric segment from x to y is the image of an isometric embedding $\alpha : [a, b] \to X$ of a closed interval $[a, b]$ of $\mathbb{R}$ such that $\alpha(a) = x$ and $\alpha(b) = y$. When there exists a unique geodesic segment from x to y, it will be denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic ray is a subset of X isometric to the half-line $[0, \infty)$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. For each $t \in [0, 1]$, let $(1-t)x \oplus ty$ denote the unique member of $[x, y]$ satisfying $d(x, (1-t)x \oplus ty) = td(x, y)$ and $d((1-t)x \oplus ty, y) = (1-t)d(x, y)$.

Definition 1.1. An $\mathbb{R}$-tree is a nonempty metric space X satisfying:

(1) Any two points $x, y \in X$ are the endpoints of a unique metric segment $[x, y]$.
(2) If $x, y, z \in X$ then $[x, y] \bigcap [x, z] = [x, w]$ for some $w \in X$.
(3) If $x, y, z \in X$ and $[x, y] \bigcap [y, z] = \{y\}$ then $[x, y] \bigcup [y, z] = [x, z]$.

The existence of fixed points for continuous mappings in a geodesically bounded complete $\mathbb{R}$-tree was proved by Kirk ([3]) as follows:

Theorem 1.2. *([2]) Suppose X is a geodesically bounded complete $\mathbb{R}$-tree. Then every continuous mapping $f : X \longrightarrow X$ has a fixed point.*

2. Main results

Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $\mathbb{R}$-tree (X, d). Consider

$$A := \{A : A \textit{ is a continuous selfmap on } K \textit{ with } Fix(A) \neq \emptyset\},$$

with

$$\rho(A, B) = \sup\{d(Ax, Bx) : x \in K\}$$

for every $A, B \in A$. Then (A, ρ) is a complete metric space.

Definition 2.1. Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $\mathbb{R}$-tree (X, d). A map $A : K \to K$ is said to have the stable fixed point property if there exist $x \in Fix(A)$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 \ni (B \in A, \rho(A, B) \leq \delta) \Longrightarrow$$
$$\exists z \in K \ni (Bz = z, d(z, x) \leq \varepsilon).$$

Theorem 2.2. *Let $K \subseteq X$ be a nonempty, compact and convex subset of a complete $\mathbb{R}$-tree (X, d), $x \in X$, and $A \in A$. Soppose $\lambda : X \to [0, 1]$ be a continuous map. Define B on K by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Then B is continuous and $Bz \in K$ for all $z \in K$.*

Proof. Since K is a closed convex subset of a complete $\mathbb{R}$-tree, $Bz \in K$ for all $z \in K$. To see that B is continuous, We let $z_0 \in K$ and $\varepsilon > 0$, By continuity of A and λ, there exists $\delta > 0$ such that $d(z, z_0) \leq \delta$ implies that $d(Az, Az_0) \leq \frac{\varepsilon}{2}$ and $|\lambda(z) - \lambda(z_0)| \leq \frac{\varepsilon d(x, Az_0)}{2(1+d(x, Az_0))}$ for all $z \in K$. Then for each $z \in B_d(z_0, \delta)$, we have

$$\begin{aligned}
&d(\lambda(z)x \oplus (1 - \lambda(z))Az, \lambda(z_0)x \oplus (1 - \lambda(z_0))Az_0) \\
&\leq d(\lambda(z)x \oplus (1 - \lambda(z))Az, \lambda(z)x \oplus (1 - \lambda(z))Az_0) \\
&+d(\lambda(z)x \oplus (1 - \lambda(z))Az_0, \lambda(z_0)x \oplus (1 - \lambda(z_0))Az_0) \\
&\leq (1 - \lambda(z))d(Az, Az_0) + |\lambda(z) - \lambda(z_0)|d(x, Az_0) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon d(x, Az_0)}{2(1 + d(x, Az_0))} = \varepsilon. \square
\end{aligned}$$

Theorem 2.3. *Let $A \in A$ and $\varepsilon > 0$. then there exists $\delta > 0$ such that for each $B \in A$ satisfying $\rho(A, B) \leq \delta$ and each $x \in K$ satisfying $Bx = x$, there exists $y \in Fix(A)$ such that $d(x, y) \leq \varepsilon$.*

Proof. In contrary, suppose,

$$\exists \varepsilon > 0 \quad \forall n \quad B_n \in A \quad \textit{such that} \quad \rho(A, B_n) \leq \frac{1}{n}$$

and

$$\exists x_n \in K \quad \textit{such that} \quad B_n x_n = x_n$$

and $d(x_n, y) > \varepsilon$ for all $y \in Fix(A)$. Since K is compact, we may assume without loss of generality that there exists $x \in K$ such that $x_n \to x$ as $k \to \infty$. So

$$\begin{aligned}
d(Ax, x) &\leq d(Ax, Ax_n) + d(Ax_n, B_n x_n) + d(B_n x_n, x_n) + d(x_n, x) \\
&\leq d(Ax, Ax_n) + \frac{1}{n} + d(x_n, x).
\end{aligned}$$

Therefore $Ax = x$ hence $x \in Fix(A)$ and $d(x_n, x) > \varepsilon$ for all n. This is a contradiction.$\square$

In view of this result, It is natural to ask,does for each $A \in A$, A have the stable fixed point property?

Example 2.4. Put $X := \mathbb{R}$, $K := [0,1]$ and $Ax = x$ for all $x \in K$ so $Fix(A) = K$. and let $A_n x = (1 - \frac{1}{n})x$ and $B_n x = \min\{x + \frac{1}{n}, 1\}$ for all n. Therefore $A_n, B_n \to A$ and $Fix(A_n) = \{0\}$ and $Fix(B_n) = [1 - \frac{1}{n}, 1]$.

This example shows that in general the answer to our question is negative. Nevertheless, we show in this paper that for a typical $A \in A$ the answer is positive.

Theorem 2.5. *Let (X,d) be a geodesically bounded normal complete $\mathbb{R}$-tree, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Then there exist $B \in A$ and $\delta > 0$ such that $\rho(A,B) \leq \varepsilon$ and $Bz = x$ for each $z \in K$ satisfying $d(z,x) \leq \delta$.*

Proof. By continuity of A there exists $\delta > 0$ such that $d(z,x) \leq \delta$ implies that $d(Az,x) \leq \varepsilon$ for all $z \in K$. Since (X,d) is normal, by Urysohn's theorem, there exists a continuous map $\lambda : X \to [0,1]$ such that

$$\begin{cases} 1 & \forall z \in X : d(z,x) \leq \frac{\delta}{4}; \\ 0, & \forall z \in X : d(z,x) \geq \frac{\delta}{2}. \end{cases}$$

Define $B : K \to K$ by $Bz = \lambda(z)x \oplus (1 - \lambda(z))Az$. Since K is a closed convex subset of a complete $\mathbb{R}$-tree, $Bz \in K$ for all $z \in K$. In particular, B is continuous, by theorem (2.1). $B(K)$ is contained in a compact subset of X and $Bx = x$.

(1) For all $z \in K$ which $d(z,x) \geq \frac{\delta}{2}$ according to $Bz = Az$ we have

$$d(Az, Bz) = 0$$

(2) For all $z \in K$ which $d(z,x) \leq \frac{\delta}{4}$ according to $Bz = x$ we have

$$d(Az, Bz) = d(Az, x) < \varepsilon$$

(3) For all $z \in K$ which $d(z,x) \leq \frac{\delta}{2}$ we have

$$\begin{aligned} d(Bz, Az) &= d(\lambda(z)x \oplus (1 - \lambda(z))Az, Az) \\ &\leq \lambda(z)d(x, Az) + (1 - \lambda(z))d(Az, Az) \\ &\leq d(x, Az) \leq \varepsilon. \square \end{aligned}$$

Theorem 2.6. *Let (X,d) be a geodesically bounded normal complete $\mathbb{R}$-tree, $A \in A$, $\varepsilon > 0$ and $x \in Fix(A)$. Let $B \in A$ and $\delta > 0$ be as guaranteed by theorem (2.5). Then for each $C \in A$ which $\rho(C,B) \leq \delta$, there exists $y \in K$ such that $Cy = y$ and $d(y,x) \leq \rho(B,C)$.*

Proof. By theorem (2.5), $\rho(A,B) \leq \varepsilon$ and $Bz = x$, for all $z \in K$ satisfies $d(z,x) \leq \delta$. Assume that $C \in A$ satisfies $\rho(C,B) \leq \delta$. Set

$$\Gamma := \{z \in K : d(z,x) \leq \rho(C,B)\}.$$

Clearly, Γ is closed convex set, thus

$$\begin{aligned} d(x, Cz) &\leq d(x, Bz) + d(Bz, Cz) \\ &= d(Bz, Cz) \\ &\leq \rho(C,B), \end{aligned}$$

$Cz \in \Gamma$ for all $z \in \Gamma$. So $C(\Gamma) \subseteq \Gamma$, clearly $C(\Gamma) \subseteq C(X)$ is contained in a compact subset of X. Now by theorem (1.7) there exists $y \in K$ such that $Cy = y$.$\square$

Theorem 2.7. *Let (X,d) be a geodesically bounded normal complete $\mathbb{R}$-tree. Then there exists a subset F of A which is a countable intersection of open subsets of (A,ρ) so that for each $A \in F$, A have the stable fixed point property.*

Proof. Let $A \in A$ and $\varepsilon > 0$. By theorem (2.5) and (2.6) there exist $A_\varepsilon \in \Gamma$, $x_{A,\varepsilon} \in K$ and $\delta_{A,\varepsilon} \in (0,1)$ such that

$$\rho(A, A_\varepsilon) \leq \varepsilon, \quad A_\varepsilon z = x_{A,\varepsilon},$$

for all $z \in K$ satisfying $d(z, x_{A,\varepsilon}) \leq \delta_{A,\varepsilon}$ and for all $C \in A$ satisfying $\rho(C, A_\varepsilon) \leq \delta_{A,\varepsilon}$ there exists $y \in K$ such that $Cy = y$ and $d(y, x_{A,\varepsilon}) \leq \rho(C, A_\varepsilon)$. For each integer $i \geq 1$, set $U(A, \varepsilon, i) := \{C \in A : \rho(C, A_\varepsilon) < \frac{\delta_{A,\varepsilon}}{i}\}$. Define

$$F := \bigcap_{i=1}^{\infty} \bigcup_{A \in A, \varepsilon \in (0,1)} U(A, \varepsilon, i).$$

Clearly F is a countable intersection of open subsets of (A, ρ).
Let $B \in F$, for all $i \geq 1$ there exists $A_i \in \Gamma$ and $\varepsilon_i \in (0,1)$ such that $B \in U(A_i, \varepsilon_i, i)$, then for all $i \geq 1$ there exists $y_i \in K$ such that

$$By_i = y_i, \quad d(y_i, x_{A_i,\varepsilon_i}) \leq \rho(B, (A_i)_{\varepsilon_i}) \leq \frac{\delta_{A,\varepsilon_i}}{i},$$

since $\{y_i\}_i \subseteq B(K)$ so there exists subsequence $\{y_{i_k}\}_k$ which $y_{i_k} \to x$ for some $x \in K$. Let $\varepsilon > 0$ so there exists $k \in \mathbb{N}$ such that

$$\frac{1}{i_k} < \frac{\varepsilon}{8}, \quad d(y_{i_k}, x) \leq \frac{\varepsilon}{8},$$

then it follows from above

$$d(y_{i_k}, x_{A_{i_k},\varepsilon_{i_k}}) \leq \frac{1}{i_k} < \frac{\varepsilon}{8},$$

and

$$d(x, x_{A_{i_k},\varepsilon_{i_k}}) \leq d(x, y_{i_k}) + d(y_{i_k}, x_{A_{i_k},\varepsilon_{i_k}}) \leq \frac{\varepsilon}{4}.$$

Let $C \in U(A_{i_k}, \varepsilon_{i_k}, i_k)$ then there exists $z \in K$ such that $Cz = z$ and

$$d(z, x_{A_{i_k},\varepsilon_{i_k}}) \leq \rho(C, (A_{i_k})_{\varepsilon_{i_k}}) \leq \frac{1}{i_k} \leq \frac{\varepsilon}{8},$$

this implies that

$$d(z, x) \leq d(z, x_{A_{i_k},\varepsilon_{i_k}}) + d(x_{A_{i_k},\varepsilon_{i_k}}, x) \leq \frac{\varepsilon}{2}. \square$$

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Existence of coupled best approximation point for almost cyclic contractive mappings

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Abstract: This paper focuses on exploring and establishing conditions for the existence and uniqueness of coupled best approximation point but not coupled best proximity point using a specific type of mapping called almost cyclic contraction ordered pair mappings. Also it is to be noted we use generalized form of cyclic contraction ordered pair defined by Zlatanov and connect to almost cyclic contraction. Indeed, it is interesting that, the class of almost cyclic contractions includes cyclic contractions.

Keywords: Coupled best approximation point, coupled best proximity point, almost cyclic contraction, $\mathcal{UC}$ property, uniformly convex Banach space.

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1. INTRODUCTION AND PRELIMINARIES

A first approximation result concerning the convergence of iterative sequences to the best proximity point for cyclic contractions was presented in [6]. This finding was subsequently broadened to encompass coupled best proximity points in [3]. In [5], B. Zlatanov extended existing findings on coupled best proximity points for ordered pairs of cyclic contraction mappings $(\mathcal{F}, \mathcal{G})$.

On the other hand, we introduce ordered pairs of almost cyclic mappings and investigate the existence of coupled best approximation point for such mappings. These mappings are defined as follows:

$$\mathcal{F}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_a,\ \mathcal{f}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_b,\ \mathcal{G}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_a \text{ and } \mathcal{g}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_b$$

where $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty subsets of the metric space $(\mathcal{W}, \rho)$.

This section will present essential definitions and concepts for subsequent discussions. Consider two non-empty subsets, $\mathcal{A}$ and $\mathcal{B}$, within a metric space $(\mathcal{W}, \rho)$. Let mapping $T: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$, satisfying the following inequality:

$$\rho(Ta, Tb) \leq k\, \rho(a, b) + (1-k)\, \rho(\mathcal{A}, \mathcal{B}),$$

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For all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where k is constant in the interval $(0,1)$. The term $\rho(\mathcal{A}, \mathcal{B})$ denotes the infimum of distances between points in $\mathcal{A}$ and $\mathcal{B}$, formally defined as

$$\rho(\mathcal{A}, \mathcal{B}) := inf\{\, \rho(a, b) : a \in \mathcal{A} \text{ and } b \in \mathcal{B} \,\}.$$

If T maps $\mathcal{A}$ into $\mathcal{B}$ and $\mathcal{B}$ into $\mathcal{A}$, then T is termed a cyclic contraction map. For a cyclic contraction map a point $\xi \in \mathcal{A} \cup \mathcal{B}$ is considered a best proximity point if $\rho(\xi, T\xi) = \rho(\mathcal{A}, \mathcal{B})$.

Definition 1.2. Given non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of a metric space, a mapping $T: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be an almost cyclic contraction if the following conditions hold:

(i) T is a cyclic mapping.

(ii) There exists a real number $k \in [0,1)$ such that

$$\rho(Ta, Tb) \leq k\, \rho(a, b) + (1-k)\, \rho(b, \mathcal{A}),$$

For all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Definition 1.3. Let $\mathcal{A}$ be a non-void subset of a metric space $\mathcal{W}$. A point ξ^* in $\mathcal{A}$ is called a best approximation or a best approximant of a mapping $T: \mathcal{A} \to \mathcal{W}$ if

$$\rho(\xi^*, T\xi^*) = \rho(T\xi^*, \mathcal{A}).$$

It is easy to see that every best proximity point of a mapping is also a best approximant of the mapping but the converse does not hold.

Definition 1.4. [4] Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of a metric space $(\mathcal{W}, \rho)$. We say that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the $\mathcal{UC}$ property if the following condition holds:
Whenever sequences $\{a_n\}$ and $\{a'_n\}$ in $\mathcal{A}$ and a sequence $\{b_n\}$ in $\mathcal{B}$ satisfy

$$\lim_{n\to\infty} \rho(a_n, b_n) = \lim_{n\to\infty} \rho(a'_n, b_n) = \rho(\mathcal{A}, \mathcal{B}),$$

Then it follows that $\lim_{n\to\infty} \rho(a_n, a'_n) = 0$.

Definition 1.5. The modulus of convexity of a Banach space $\mathcal{W}$ is the function $\delta_{\mathcal{W}}: [0,2] \to [0,1]$ defined by

$$\delta_{\mathcal{W}}(\varepsilon) = inf\left\{1 - \left\|\frac{u+p}{2}\right\| : \|u\| \leq 1, \|p\| \leq 1, \|u-p\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_{\mathcal{W}}(\varepsilon) > 0$ for all $\varepsilon > 0$. The space $(\mathcal{W}, \|.\|)$ Is called uniformly convex Banach space.

2. Main results

Definition 2.1. Let $\mathcal{A}_a, \mathcal{A}_b, \mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty subsets of the metric space $(\mathcal{W}, \rho)$. Let $\mathcal{F}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_a$ and $\mathcal{f}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_b$. An order pair $(a^*, b^*) \in \mathcal{A}_a \times \mathcal{A}_b$ is called a coupled best approximation point of $(\mathcal{F}, \mathcal{f})$ if $\rho\big(a^*, \mathcal{F}(a^*, b^*)\big) = \rho(\mathcal{F}(a^*, b^*), \mathcal{A}_a)$ and $\rho\big(b^*, \mathcal{f}(a^*, b^*)\big) = \rho(\mathcal{A}_b, \mathcal{f}(a^*, b^*))$.

Definition 2.2. [5] Let $\mathcal{A}_a, \mathcal{A}_b, \mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty subsets of the metric space $(\mathcal{W}, \rho)$. Let $\mathcal{F}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_a$ and $\mathcal{f}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_b$, $\mathcal{G}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_a$ and $\mathcal{g}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_b$. For any pair $(a, b) \in \mathcal{A}_a \times \mathcal{A}_b$, let the sequences $\{a_n\}$ and $\{b_n\}$ be defined with initial values $a_0 = a$, $b_0 = b$ and

$$a_{2n+1} := \mathcal{F}(a_{2n}, b_{2n}), \quad a_{2n+2} := \mathcal{G}(a_{2n+1}, b_{2n+1}),$$
$$b_{2n+1} := \mathcal{f}(a_{2n}, b_{2n}), \quad b_{2n+2} := \mathcal{g}(a_{2n+1}, b_{2n+1}).$$

Definition 2.3. Let $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty subsets of the metric space $(\mathcal{W}, \rho)$. Let $\mathcal{F}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_a$ and $\mathcal{f}: \mathcal{A}_a \times \mathcal{A}_b \to \mathcal{B}_b$, $\mathcal{G}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_a$ and $\mathcal{g}: \mathcal{B}_a \times \mathcal{B}_b \to \mathcal{A}_b$. The ordered pair of ordered pairs $\big((\mathcal{F}, \mathcal{f}), (\mathcal{G}, \mathcal{g})\big)$ is said to be an almost cyclic contraction ordered pair if there exist non-negative numbers α, β such that $\xi = max\{\alpha, \beta\} < 1$ and there holds the inequality

$$\rho\big(\mathcal{F}(a,b),\mathcal{G}(u,p)\big)+\rho\big(\mathfrak{f}(a,b),\mathfrak{g}(u,p)\big)$$
$$\leq \alpha\rho(a,u)+(1-\alpha)\rho\big(\mathcal{A}_a,\mathcal{F}(a,b)\big)+\beta\rho(b,p)+(1-\beta)\rho\big(\mathcal{A}_b,\mathfrak{f}(a,b)\big).$$

For all $(a\,,b)\in\mathcal{A}_a\times\mathcal{A}_b$ and $(u,p)\in\mathcal{B}_u\times\mathcal{B}_p$.

Theorem 2.4. Let $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty convex subsets of a uniformly convex Banach space $(\mathcal{W},\|.\|)$ Such that $\rho(\mathcal{A}_a,\mathcal{B}_a)>0$ and $\rho(\mathcal{A}_b,\mathcal{B}_b)>0$. Let $\mathcal{F}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_a$ and $\mathfrak{f}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_b$, $\mathcal{G}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_a$ and $\mathfrak{g}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_b$. Suppose that $\big((\mathcal{F},\mathfrak{f}),(\mathcal{G},\mathfrak{g})\big)$ is an almost cyclic contraction ordered pair. Then $(\mathcal{F},\mathfrak{f})$ has a unique coupled best approximation point $(a^*,b^*)\in\mathcal{A}_a\times\mathcal{A}_b$ and $(\mathcal{G},\mathfrak{g})$ has a unique coupled best approximation point $(a^{**},b^{**})\in\mathcal{B}_a\times\mathcal{B}_b$ such that $\mathcal{F}(a^*,b^*)=a^{**}$, $\mathfrak{f}(a^*,b^*)=b^{**}$, $\mathcal{G}(a^{**},b^{**})=a^*$ and $\mathfrak{g}(a^{**},b^{**})=b^*$. Moreover for every arbitrary point $(a\,,b)\in\mathcal{A}_a\times\mathcal{A}_b$ there hold

$$a_{2n}\to a^*,\qquad b_{2n}\to b^*,\qquad a_{2n+1}\to a^{**},\qquad b_{2n+1}\to b^{**}.$$

3. Conclusion

Since every cyclic contraction is an almost cyclic contraction so we can conclude the following definition.

Definition 3.1. Let $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty subsets of the metric space $(\mathcal{W},\rho)$. Let $\mathcal{F}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_a$ and $\mathfrak{f}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_b$, $\mathcal{G}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_a$ and $\mathfrak{g}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_b$. The ordered pair of ordered pairs $\big((\mathcal{F},\mathfrak{f}),(\mathcal{G},\mathfrak{g})\big)$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers α,β such that $\xi=max\{\alpha,\beta\}<1$ and there holds the inequality

$$\rho\big(\mathcal{F}(a,b),\mathcal{G}(u,p)\big)+\rho\big(\mathfrak{f}(a,b),\mathfrak{g}(u,p)\big)$$
$$\leq \alpha\rho(a,u)+(1-\alpha)\rho(\mathcal{A}_a,\mathcal{B}_u)+\beta\rho(b,p)+(1-\beta)\rho\big(\mathcal{A}_b,\mathcal{B}_p\big).$$

For all $(a\,,b)\in\mathcal{A}_a\times\mathcal{A}_b$ and $(u,p)\in\mathcal{B}_u\times\mathcal{B}_p$.

Theorem 3.2. [5] Let $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{B}_a$ and $\mathcal{B}_b$ are non-empty convex subsets of a uniformly convex Banach space $(\mathcal{W},\|.\|)$ Such that $\rho(\mathcal{A}_a,\mathcal{B}_a)>0$ and $\rho(\mathcal{A}_b,\mathcal{B}_b)>0$. Let $\mathcal{F}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_a$ and $\mathfrak{f}:\mathcal{A}_a\times\mathcal{A}_b\to\mathcal{B}_b$, $\mathcal{G}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_a$ and $\mathfrak{g}:\mathcal{B}_a\times\mathcal{B}_b\to\mathcal{A}_b$. Suppose that $\big((\mathcal{F},\mathfrak{f}),(\mathcal{G},\mathfrak{g})\big)$ is a cyclic contraction ordered pair. Then

(i) $(\mathcal{F},\mathfrak{f})$ has a unique coupled best approximation point $(a^*,b^*)\in\mathcal{A}_a\times\mathcal{A}_b$ and $(\mathcal{G},\mathfrak{g})$ has a unique coupled best approximation point $(a^{**},b^{**})\in\mathcal{B}_a\times\mathcal{B}_b$ such that $\mathcal{F}(a^*,b^*)=a^{**}$, $\mathfrak{f}(a^*,b^*)=b^{**}$, $\mathcal{G}(a^{**},b^{**})=a^*$ and $\mathfrak{g}(a^{**},b^{**})=b^*$. Moreover for every arbitrary point $(a\,,b)\in\mathcal{A}_a\times\mathcal{A}_b$ there hold

$$a_{2n}\to a^*,\qquad b_{2n}\to b^*,\qquad a_{2n+1}\to a^{**},\quad b_{2n+1}\to b^{**}.$$

If in addition $(\mathcal{W},\|.\|)$ has a modulus of convexity of power type with constants $\mathcal{C}>0$ and $q>1$, then

(ii) A priori error estimates hold

$$\|a^*-a_{2m}\|\leq\frac{\mathcal{P}_{0,1}(a\,,b)}{1-\sqrt[q]{\xi^2}}\sqrt[q]{\frac{\mathcal{P}_{0,1}(a\,,b)-\rho}{c\rho(A_a\,,B_a)}}\left(\sqrt[q]{\xi}\right)^{2m},$$

$$\|b^*-b_{2m}\|\leq\frac{\mathcal{P}_{0,1}(a\,,b)}{1-\sqrt[q]{\xi^2}}\sqrt[q]{\frac{\mathcal{P}_{0,1}(a\,,b)-\rho}{c\rho(A_b\,,B_b)}}\left(\sqrt[q]{\xi}\right)^{2m};$$

(iii) A posteriori error estimates hold

$$\|a_{2n} - a^*\| \leq \frac{\mathcal{P}_{2n,2n-1}(a\,,b)}{1-\sqrt[q]{\xi^2}} \sqrt[q]{\frac{\mathcal{P}_{2n,2n-1}(a\,,b)-\rho}{c\rho(A_a\,,B_a)}}\,\sqrt[q]{\xi},$$

$$\|\,b_{2n} - \,b^*\| \leq \frac{\mathcal{P}_{2n,2n-1}(a\,,b)}{1-\sqrt[q]{\xi^2}} \sqrt[q]{\frac{\mathcal{P}_{2n,2n-1}(a\,,b)-\rho}{c\rho(A_b\,,B_b)}}\,\sqrt[q]{\xi}.$$

where

$\rho := \rho(A_a\,,B_a) + \rho(A_b\,,B_b)$,

$\mathcal{P}_{0,1}(a\,,b) := \|a_0 - a_1\| + \|b_0 - b_1\|$,

$\mathcal{P}_{2n,2n-1}(a\,,b) := \|a_{2n} - a_{2n-1}\| + \|b_{2n} - b_{2n-1}\|$.

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The class of meromorphic $(\alpha;\varphi)$-starlike functions for p-valency

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Abstract. We introduce and studied the class of meromorphic p-valent functions f the punctured open unit disk $\mathbb{D}^* = \{z \in \mathbb{C} \colon 0 < |z| < 1\}$ with $f(z) \neq 0$ which satisfy the condition

$$\frac{1}{p}\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(1+\alpha)\frac{zf'(z)}{f(z)}\right] \prec \varphi(z),$$

where α be real, $\prec$ is the subordination relation and φ is an analytic function in the open unit disk $\mathbb{D}$ with positive real part whose range is symmetric with respect to the real axis. Further an integral representation for functions belonging to this class is shown.
meromorphic p-valent functions, meromorphic $(\alpha;\varphi)$-starlike functions, Subordination, Extreme function

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Let $\mathcal{H}$ denote the class of holomorphic functions in the open unit disc $\mathbb{D} := \{z \colon |z| < 1\}$ on the complex plane $\mathbb{C}$, and let $\mathcal{H}[a,n]$ denote the subclass of the functions $\mathfrak{p} \in \mathcal{H}$ of the form

$$\mathfrak{p}(z) = a + c_n z^n + c_{n+1} z^{n+1} + \cdots, \qquad (a \in \mathbb{C},\ n \in \mathbb{N})$$

A function $f(z)$ which is analytic in domain $\mathbb{D}$ is called *p-valent*, if

- for every complex number ω, the equation $f(z) = \omega$ have at most p roots in $\mathbb{D}$, and
- there exits a complex number ω_0 such that the set $f^{-1}(\{\omega_0\})$, has exactly p element in $\mathbb{D}$.

Let Σ_p denote the class of all meromorphic functions $h \in \mathcal{H}$ of the following form:

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in \mathbb{N}), \tag{1}$$

which are analytic in the punctured open unit disk $\mathbb{D}^* = \{z \in \mathbb{C} \colon 0 < |z| < 1\} = \mathbb{D} \setminus \{0\}$ with a pole of order p at $z = 0$. The class Σ_1 denoted by Σ.

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A function $h \in \Sigma_p$ is called p-valent meromorphic starlike of order β; $(0 \leq \beta < 1)$, if $h(z) \neq 0$ on $\mathbb{D}^*$ satisfies the conditions

$$-\Re\left\{\frac{1}{p}\frac{zh'(z)}{h(z)}\right\} > \beta \qquad (z \in \mathbb{D}), \tag{2}$$

and called p-valent meromorphic convex of order β; $(0 \leq \beta < 1)$, if $h(z) \neq 0$ on $\mathbb{D}^*$ satisfies the conditions

$$-\Re\left\{\frac{1}{p}\left[1+\frac{zh''(z)}{h'(z)}\right]\right\} > \beta \qquad (z \in \mathbb{D}), \tag{3}$$

We denote by $\mathcal{MS}_p^*(\beta)$ and $\mathcal{MK}_p(\beta)$ the classes of p-valent meromorphic starlike and convex of order β, respectively. Let α be a real number. Define the differential operator $\mathcal{G}_p^{\alpha,1-\alpha}\colon \Sigma_p \longrightarrow \mathcal{H}[1,1]$ as follows:

$$\mathcal{G}_p^{\alpha,1+\alpha}[h](z) := \left[\frac{z^{p+1}h'(z)}{-p}\right]^{\alpha}\left[\frac{1}{z^p h(z)}\right]^{1+\alpha} = \frac{1}{z^p h(z)}\left[-\frac{1}{p}\frac{zh'(z)}{h(z)}\right]^{\alpha},$$

for some $h \in \Sigma_p$ with $z^{2p+1}h(z)h'(z) \neq 0$ on $\mathbb{D}^*$ given by (1). Here and hereafter, all powers are mean as principal values. For more details about differential operator $\mathcal{G}_p^{\alpha,1+\alpha}$ see [**?**].

Definition 0.1 ([1]). Let f and g be analytic in $\mathbb{D}$. Then the function f is said to *subordinate* to g in $\mathbb{D}$ written by $f(z) \prec g(z)$, if there exists a function $\omega(z)$ which is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$; $(z \in \mathbb{D})$, and such that $f(z) = g(\omega(z))$; $(z \in \mathbb{D})$. From the definition of the subordinations, it is easy to show that the subordination $f(z) \prec g(z)$ implies that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ In particular, if $g(z)$ is univalent in $\mathbb{D}$, then the subordination $f(z) \prec g(z)$ is equivalent to the condition $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Lemma 0.2 ([4, Theorem 3.2a, p. 81]). *Let $\mathfrak{q}$ be convex function in $\mathbb{D}$, with $\Re\{\beta\mathfrak{q}(z)+\gamma\} > 0$. If $\mathfrak{p}$ is analytic in $\mathbb{D}$, with $\mathfrak{p}(0) = \mathfrak{q}(0)$, then*

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z)+\gamma} \prec \mathfrak{q}(z) \Longrightarrow \mathfrak{p}(z) \prec \mathfrak{q}(z).$$

Rosihan and Ravichandran [2] proposed two unifying way to treat subclasses of p -valent starlike and convex functions respectively, as follows:

$$\mathcal{MS}_p^*(\varphi) = \left\{h \in \Sigma_p\colon\ -\frac{1}{p}\frac{zh'(z)}{h(z)} \prec \varphi(z)\right\},$$

$$\mathcal{MK}_p(\varphi) = \left\{h \in \Sigma_p\colon\ -\frac{1}{p}\left[1+\frac{zh''(z)}{h'(z)}\right] \prec \varphi(z)\right\}.$$

Where analytic functions φ has positive real part with $\varphi(0) = 1$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and maps the unit disk $\mathbb{D}$ onto a region satrlike with respect to 1.

This work is planned as a struggle at defining a new subfamily of $\mathcal{A}_p$ subordinate to analytic functions φ, which is mentioned.

1. Extremal functions for classes $\mathcal{MS}_p^*(\varphi)$ and $\mathcal{MK}_p(\varphi)$

We first introduce some functions in either $\mathcal{MS}_p^*(\varphi)$ or $\mathcal{MK}_p(\varphi)$, which turn out to be extremal for certain functionals. We observe that $\varphi(z^n)$ is contained in $\varphi(\mathbb{D})$ for

$n=1,2,3,\ldots$. Define $R_{p,\varphi,n}\in\Sigma_p$ by

$$\left[\frac{1}{R_{p,\varphi,n}(z)}\right]^{\frac{1}{p}}\Bigg|_{z=0}=\left(\left[\frac{1}{R_{p,\varphi,n}(z)}\right]^{\frac{1}{p}}\right)'\Bigg|_{z=0}-1=0\qquad\text{and}\qquad-\frac{1}{p}\left[\frac{zR'_{p,\varphi,n}(z)}{R_{p,\varphi,n}(z)}\right]=\varphi\left(z^n\right),$$

which is clearly in the class $\mathcal{MS}_p^*(\varphi)$. we write $R_{p,\varphi,1}$ simply as $R_{p,\varphi}$ which plays the role of the Koebe function for the family $\mathcal{MS}_p^*(\varphi)$. Clearly

$$R_{p,\varphi,n}(z)=z^{-p}\exp\left(p\int_0^z\frac{1-\varphi(\tau^n)}{\tau}\,\mathrm{d}\tau\right)\qquad(z\in\mathbb{D}^*)\,. \tag{4}$$

Similarly, we define $S_{p,\varphi,n}\in\mathcal{H}$ by

$$\left[\frac{1}{S_{p,\varphi,n}(z)}\right]^{\frac{1}{p}}\Bigg|_{z=0}=\left(\left[\frac{1}{S_{p,\varphi,n}(z)}\right]^{\frac{1}{p}}\right)'\Bigg|_{z=0}-1=0\qquad\text{and}\qquad-\frac{1}{p}\left[1+\frac{zS''_{p,\varphi,n}(z)}{S'_{p,\varphi,n}(z)}\right]=\varphi\left(z^n\right),$$

Then $S_{p,\varphi,n}$ belongs to $\mathcal{MK}_p(\varphi)$ and $S_{p,\varphi,1}$, which we simply denote by $S_{p,\varphi}$, plays the role of the Koebe function for the family $\mathcal{MK}_p(\varphi)$. Note that $-\frac{1}{p}zS'_{p,\varphi,n}(z)=R_{p,\varphi,n}(z)$. Clearly

$$S'_{p,\varphi,n}(z)=-pz^{-p-1}\exp\left(p\int_0^z\frac{1-\varphi(\tau^n)}{\tau}\,\mathrm{d}\tau\right),\qquad(z\in\mathbb{D}^*) \tag{5}$$

The structural formula for functions in the classes $\mathcal{MS}_p^*(\varphi)$ and $\mathcal{MK}_p(\varphi)$ are as follows:

Lemma 1.1. *A function g is in the class $\mathcal{MS}_p^*(\varphi)$, if and only if there exists an analytic function $\mathfrak{p}$; $\mathfrak{p}\prec\varphi$, such that*

$$g(z)=z^{-p}\exp\left(p\int_0^z\frac{1-\mathfrak{p}(t)}{t}\,\mathrm{d}t\right),\qquad(z\in\mathbb{D}^*)\,. \tag{6}$$

Using the same notation and the same reasoning as in the proof of Lamma 1.1, we get Lemma 1.2.

Lemma 1.2. *A function g is in the class $\mathcal{MK}_p(\varphi)$ if and only if there exists an analytic function $\mathfrak{p}$; $\mathfrak{p}\prec\varphi$, such that*

$$g'(z)=-pz^{-p-1}\exp\left(p\int_0^z\frac{1-\mathfrak{p}(t)}{t}\,\mathrm{d}t\right),\qquad(z\in\mathbb{D}^*)\,.$$

2. The class $\mathcal{M}_p(\alpha;\varphi)$ and its properties

Definition 2.1. Let $\mathcal{M}_p(\alpha;\varphi)$ denote the subfamily of Σ_p consisting of the functions f, satisfying the condition $f(z)\neq0$ and

$$\frac{1}{p}\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(1+\alpha)\frac{zf'(z)}{f(z)}\right]\prec\varphi(z),\qquad(z\in\mathbb{D},\ \alpha\in\mathbb{R})$$

The function $f\in\Sigma_p^*(\alpha;\varphi)$ is called p-valent meromorphic $(\alpha;\varphi)$-*starlike* function.

Remark 2.2. Let $f\in\Sigma_p$, $\varphi(z)=(1+z)/(1-z)$ and

$$-\frac{1}{p}\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right]\prec\frac{1+z}{1-z},\qquad(z\in\mathbb{D})\,.$$

we say that f is p-valent meromorphic α-convex function and denoted by $\Sigma_p^*(\alpha)$. Also, $\Sigma^*(\alpha):=\Sigma_1^*(\alpha)$.

To obtain the structural formula for functions in the classes $\Sigma_p^*(\alpha;\varphi)$, we state the following theorem.

THEOREM 2.3. *Let* $\alpha > 0$. *Then* $\Sigma_p^*(\alpha;\varphi) \subset \mathcal{MS}_p^*(\varphi)$.

PROOF. Let $f \in \Sigma_p^*(\alpha;\varphi)$. For function $\mathfrak{p}$ defined by $\mathfrak{p}(z) := -zf'(z)/(pf(z))$, we have $\mathfrak{p}(0) = 1$ and

$$\mathfrak{p}(z) + \frac{\alpha}{p}\frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} = \frac{1}{p}\left[\alpha\left(1 + \frac{zg''(z)}{g'(z)}\right) - (1+\alpha)\frac{zg'(z)}{g(z)}\right] \prec \varphi(z)$$

Taking into account the properties of function φ and Lemma 0.2, we deduce $\mathfrak{p}(z) \prec \varphi(z)$. □

THEOREM 2.4. *A function* g *is in the class* $\Sigma_p^*(\alpha;\varphi)$; $(\alpha \geq 0)$ *if and only if there exists an analytic function* ψ; $\psi \prec \varphi$, *such that*

$$\mathcal{G}_p^{\alpha,1-\alpha}[g](z) = z^p g(z)\left[-\frac{zg'(z)}{pg(z)}\right]^{\alpha} = \exp\left(p\int_0^z \frac{1-\psi(\tau)}{\tau}\,\mathrm{d}\tau\right). \tag{7}$$

Special Cases in Theorem 2.4 for $\alpha = 0$ and $\alpha = 1$ are given in Lemma 1.1 and Lemma 1.2, respectively. Also, using Lemma 1.1 and Theorem 2.4, we can obtain the following corollary.

COROLLARY 2.5. *Let* $f \in \Sigma_p$. *The function* $f \in \Sigma_p^*(\alpha;\varphi)$; $(\alpha > 0)$ *if and only if the function* $F(z)$ *defined by*

$$F(z) = z^{-p}\,\mathcal{G}_p^{\alpha,1-\alpha}[f](z) = f(z)\left[\frac{-zf'(z)}{pf(z)}\right]^{\alpha} \quad \text{or} \quad f(z) = \left[-\frac{p}{\alpha}\int_0^z \frac{[F(\tau)]^{\frac{1}{\alpha}}}{\tau}\,\mathrm{d}\tau\right]^{\alpha}. \tag{8}$$

belongs to $\mathcal{MS}_p^*(\varphi)$.

From (8) and the fact that the function $H_{p,\varphi}$ is extremal for p-valent starlike $\mathcal{MS}_p^*(\varphi)$, The function $\kappa_{p,\varphi}$ defined by

$$z^{-p}\,\mathcal{G}_p^{\alpha,1-\alpha}\left[\kappa_{p,\varphi}\right](z) = \kappa_{p,\varphi}(z)\left[-\frac{z\kappa'_{p,\varphi}(z)}{p\kappa_{p,\varphi}(z)}\right]^{\alpha} = z^{-p}\exp\left(p\int_0^z \frac{1-H_{p,\varphi}(\tau)}{\tau}\,\mathrm{d}\tau\right),$$

or

$$\kappa_{p,\varphi}(z) = \left[-\frac{p}{\alpha}\int_0^z \frac{[H_{p,\varphi}(\tau)]^{\frac{1}{\alpha}}}{\tau}\,\mathrm{d}\tau\right]^{\alpha}. \tag{9}$$

is extremal function for several problems in the class $\Sigma_p^*(\alpha;\varphi)$ and called meromorphic $(\alpha;\varphi)$-starlike Koebe function.

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Remarks on the paper: "Best proximity point theorems for some classes of contractions"

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Abstract. In this article, we examine best proximity point theorems in spaces where the pair (A, B) is either a proximally complete space or a fairly complete space. These spaces do not necessarily need to be complete or possess the UC property. This work generalizes and improves some of the theorems presented in the paper by Sadiq Basha [Best proximity point theorems for some classes of contractions, Acta Mathematica Hungarica, 156, 2018, 336–360]. Motivated by the results of Sadiq Basha concerning K-proximal and C-proximal contractions of the first kind, we introduce and study the concept of a KC-proximal ϑ-contraction of the first kind.

Keywords: Best proximity point, KC-proximal ϑ-contraction of the first kind, proximally complete space, fairly complete space.

AMS Mathematics Subject Classification [2020]: 47H09, 47H10.

1. Introduction and Preliminaries

This section serves to revisit existing concepts and introduce new ideas. Unless stated otherwise, we will assume throughout this section that A and B are non-empty subspaces within a metric space.

$$d(A, B) := \inf\{d(x, y) : \quad x \in A, y \in B\};$$
$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\};$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 1.1. [4, Definition 2.1] Given two sequence $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$, we say that $\{(x_n, y_n)\} \subseteq A \times B$ is a cyclically Cauchy sequence in (A, B) if and only if for every $\epsilon > 0$, there exists a positive integer N such that $d(x_m, y_n) < d(A, B) + \epsilon$, for all $m, n \geq N$.

Definition 1.2. [4, Definition 2.2] Given two sequence $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$, we say that $\{(x_n, y_n)\} \subseteq A \times B$ is a fairly Cauchy sequence in (A, B) if and only if the following conditions are satisfied:

*Speaker.

- $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences;
- $\{(x_n, y_n)\}$ is a cyclically Cauchy sequence in (A, B).

Definition 1.3. [4, Definition 2.3] The pair (A, B) is said to be a fairly complete space if and only if for every fairly Cauchy sequence $\{(x_n, y_n)\}$ in $A \times B$, the sequences $\{x_n\}$ and $\{y_n\}$ converge in A and B, respectively.

Definition 1.4. [4, Definition 2.4] We say that the pair (A, B) is a proximally complete space [3] if and only if for every cyclically Cauchy sequence $\{(x_n, y_n)\}$ in $A \times B$, the sequences $\{x_n\}$ and $\{y_n\}$ have subsequences that converge in A and B, respectively.

Definition 1.5. [2,6] Let A and B be nonempty subsets of the metric space (X, d). Then (A, B) is said to satisfy the property UC, if $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n) = dist(A, B)$, then $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

Definition 1.6. [4, Definition 2.12] Given a non-self mapping $T : A \to B$, the subspace A is said to have uniform T-approximation in the subspace B if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left.\begin{array}{l} d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \\ d(u_1, u_2) < \delta \end{array}\right\} \Longrightarrow d(Tx_1, Tx_2) < \epsilon,$$

for all $x_1, u_1, x_2, u_2 \in A$.

Definition 1.7. [4, Definition 4.1] Given non-void subsets A and B of a metric space, a mapping $T : A \to B$ is called a K-proximal contraction of the first kind if and only if there is a non-negative number $\alpha < \frac{1}{2}$ such that

$$d(u_1, Tx_1) = d(A, B) \quad \text{and} \quad d(u_2, Tx_2) = d(A, B),$$

implyes that

$$d(u_1, u_2) \leq \alpha[d(x_1, u_1) + d(x_2, u_2)],$$

for all $x_1, x_2, u_1, u_2 \in A$.

Theorem 1.8. *[4, Theorem 4.3] Let t A and B are non-void closed subsets of a metric space such that A_0 and B_0 are non-void, (A, B) is a fairly complete space. Let $T : A \to B$ be such that the subset A has uniform T-approximation in the subset B. Moreover, assume the following:*

- *T is a strong K-proximal contraction of the first kind;*
- *$T(A_0) \subseteq B_0$.*

Then, there exists a unique element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Further, for any given initial element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the preceding best proximity point x^* of the mapping T.

Definition 1.9. [4, Definition 4.2] Given non-void subsets A and B of a metric space, a mapping $T : A \to B$ is called a C-proximal contraction of the first kind if and only if there is a non-negative number $\alpha < \frac{1}{2}$ such that

$$d(u_1, Tx_1) = d(A, B) \quad \text{and} \quad d(u_2, Tx_2) = d(A, B),$$

implyes that

$$d(u_1, u_2) \leq \alpha[d(u_1, x_2) + d(u_2, x_1)],$$

for all $x_1, x_2, u_1, u_2 \in A$.

THEOREM 1.10. *[4, Theorem 4.5] Let A and B are non-void closed subsets of a metric space such that A_0 and B_0 are non-void, (A, B) is a fairly complete space. Let $T : A \to B$ be such that the subset A has uniform T-approximation in the subset B. Moreover, assume the following:*

- *T is a strong C-proximal contraction of the first kind;*
- *$T(A_0) \subseteq B_0$.*

Then, there exists a unique element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Further, for any given initial element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the aforesaid best proximity point x^* of the mapping T.

2. KC-proximal ϑ-contractions of the first kind

From now on for the strictly increasing function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ we assume that $I - \vartheta$ is also increasing. Let $C(\mathbb{R}_+, \mathbb{R}_+)$ be a space of continuous functions acting from $\mathbb{R}_+$ to $\mathbb{R}_+$ and

$$\mathcal{K} := \{\varphi \in C(\mathbb{R}_+, \mathbb{R}_+) : \varphi \text{ is strictly increasing function and } I - \varphi \text{ is increasing}\}.$$

Given nonempty subsets A and B of a metric space (X, d), we set $d^*(x, y) := d(x, y) - dist(A, B)$ for every $x, y \in X$.

DEFINITION 2.1. Given non-void subsets A and B of a metric space (X, d), a mapping $T : A \to B$ is said to be a KC-proximal ϑ-contraction of the first kind if there exist $\vartheta \in \mathcal{K}$ such that

$$d(u_1, Tx_1) = d(A, B) \quad \text{and} \quad d(u_2, Tx_2) = d(A, B),$$

implyes that

$$d(u_1, u_2) \leq (I - \vartheta)\big(\max\{d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), d(x_1, u_2), d(x_2, u_1)\}\big). \tag{1}$$

for all $x_1, x_2, u_1, u_2 \in A$.

It is easy to see that a KC-proximal ϑ-contraction of the first kind is not necessarily continuous. Further, the requirement for a self-mapping $T : A \to A$ to be a KC-proximal ϑ-contraction of the first kind is that

$$d(Tx_1, Tx_2) \leq (I - \vartheta)\big(\max\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2), d(Tx_2, u_1)\}\big),$$

for all $x_1, x_2 \in A$.

EXAMPLE 2.2. A Ćirić's quasi-contraction map in the sense of Theorem 2 of [**?**], is a KC-proximal ϑ-contraction of the first kind with $A = B = X$ and $\vartheta(t) = (1 - \lambda)t$ for $t \geq 0$ and $\lambda \in [0, 1)$.

EXAMPLE 2.3. Suppose that (X, d) be a metric space and $T : X \to X$ be a generalized φ-contraction in the sence of [1]. From Example 3.4 of [5], we can conclude that T is a KC-proximal ϑ-contraction of the first kind with $A = B = X$ and $\vartheta(t) := t - \varphi(t, t, t, t, t)$ for $t \geq 0$.

Theorem 2.4 is extend of Theorems 4.3 and 4.5 of [4]

THEOREM 2.4. *Let us suppose that A and B are non-void subsets of a metric space such that A_0 and B_0 are non-void, (A, B) is a fairly complete space. Let the mapping $T : A \to B$ be such that the subset A has uniform T-approximation in the subset B. Moreover, let us assume that the mapping $T : A \to B$ satisfies the following:*

(a) *T is a KC-proximal ϑ-contraction of the first kind;*
(b) *$T(A_0) \subseteq B_0$.*

Then

(i) *there exists a unique element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$;*
(ii) *for any given initial element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by*
$$d(x_{n+1}, Tx_n) = d(A, B),$$
converges to the preceding best proximity point x^* of the mapping T;
(iii) *a priori error estimate holds:*
$$d(x_n, x^*) \leq (I - \vartheta)^n \left(\vartheta^{-1}\big(d(x_0, x_1)\big)\right).$$

Theorem 2.5 is extend of Theorems 4.7 and 4.8 of [4]

THEOREM 2.5. *Let A and B be non-empty closed subsets of a metric space such that (A, B) is a proximally complete space. Further, suppose that A_0 and B_0 are non-void. Let $T : A \to B$ satisfy following conditions:*

(a) *T is a KC-proximal ϑ-contraction of the first kind;*
(b) *$T(A_0) \subseteq B_0$.*

Then

(i) *there exists a unique element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$;*
(ii) *for any given initial element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by*
$$d(x_{n+1}, Tx_n) = d(A, B),$$
converges to the preceding best proximity point x^* of the mapping T;
(iii) *a priori error estimate holds:*
$$d(x_n, x^*) \leq (I - \vartheta)^n \left(\vartheta^{-1}\big(d(x_0, x_1)\big)\right).$$

3. Conclusion

This work generalizes and improves some of the theorems presented in the paper by Sadiq Basha [4].

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Some fixed point theorems for (ζ, η)-weakly cyclic contraction mappings in b-metric spaces

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Abstract. In the present paper, we investigate some fixed point results for (ζ, η)-weakly cyclic contraction mappings in the framework of b-metric spaces. Our findings extend and generalize a large number of relevant fixed point results from the current literature. Examples are provided to support of our main results.

Keywords: altering distance function, cyclic map, (ζ, η)-weakly cyclic contraction
AMS Mathematics Subject Classification [2020]: 47H10, 54H25

1. Introduction

The Banach contraction principle is one of the methods used, which states that if (X, d) is a complete metric space and $T : X \to X$ is self-mapping with contraction, then T has a unique fixed point [2]. With the help of this theory, many scholars have been interested in examining the presence and distinctiveness of fixed points as well as common fixed points. The principle of Banach contraction was demonstrated by numerous researchers in various generalized metric spaces. The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions. Such as, $b-$ metric spaces and generalized metric spaces. Bakhtin introduced the concept of b-metric space in [1] as a generalization of metric space. By extending the Banach contraction concept in metric space, he demonstrated the contraction mapping idea in b-metric spaces. Following the work of Bakhtin, numerous papers have been written regarding the theory of fixed points in b-metric space (for further elaboration, one can see [3] and the references from there).

The fixed point theorem of Banach was extended to the class of cyclic mappings by Kirk et al. [4] in their 2003 publication, which also introduced the idea of cyclical contractive mappings. They demonstrated that cyclical contractive mappings possess a distinct fixed point and extended the notion of contractive mappings to encompass these mappings. In addition to offering a new method for researching the presence and distinctiveness of fixed points in many mathematical contexts, this expanded the range of mappings for which a fixed point can be guaranteed. Following the study of Kirk et al., several authors stated numerous fixed point results for cyclic mappings, satisfied different contractive conditions.

*Speaker.

The readers may refer to [5] and the references therein. In this work, we extend the contraction condition (1) for cyclic mappings in complete b-metric spaces. By imposing different conditions on the control function η, we investigate fixed point results in the setting of b-metric spaces. The current article falls into 2 parts. In Section 2, the basic concepts and facts needed in the remainder of this work are collected. In Section 3, we investigate some fixed point results for (ζ, η)-weakly cyclic contraction mappings in the framework of b-metric spaces. Finally, we present some illustrative examples to show the usefulness of the obtained results.

2. Preliminaries and Auxiliary facts

Definition 2.1. [4] Let A and B be non-empty subsets of a metric space (X, d) and $f : A \cup B \to A \cup B$ be a mapping. Then f is said to be a cyclic mapping if $f(A) \subset B$ and $f(B) \subset A$.

The following fundamental theorem for a cyclic mapping was given in [4].

Theorem 2.2. *[4] Let (X, d) be a complete metric space, A and B be non-empty closed subsets of X, $f : A \cup B \to A \cup B$ be a cyclic mapping such that $f(dx, dy) \leq kd(x, y)$ for all $x \in A$, $y \in B$ and for some $k \in (0, 1)$. Then f posses a distinct fixed point in $A \cap B$.*

Note that although a contraction is continuous, a cyclic contraction need not be (see [5]). Khan et al. proposed the idea of altering distance function in 1984 and presented the following definition.

Definition 2.3. [5] Let $\zeta : [0, \infty) \to [0, \infty)$ be a function satisfying the following conditions.
1. ζ is strictly increasing and continuous,
2. $\zeta(t) = 0$ if and only if $t = 0$.
Then ζ is known as an altering distance function.

For instance $\zeta(t) = t^2$ is an altering distance function.

Definition 2.4. [5] Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function with parameter $s \geq 1$ that fulfills the following conditions.
1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s(d(x, z) + d(x, y))$,
for each $x, y, z \in X$. Then, d is known as b-metric on X, and (X, d) is referred to as b-metric space.

Example 2.5. [5] Let (X, d) be a metric space and $\varpi(x, y) = (d(x, y))^p$ for a real number $p > 1$. Then, ϖ is b-metric with $s = 2^{p1}$.

Definition 2.6. [5] Let (X, d) and $(Y, \acute{d})$ be two metric spaces. Then, a mapping $f : X \to Y$ is said to be continuous at $x \in X$ if $\lim_{n\to\infty} \acute{d}(f(t_n), f(t_0)) = 0$, whenever $\{t_n\} \subset X$ with $\lim_{n\to\infty} d(t_n, t_0) = 0$.

It is observed that while a metric is typically continuous, a b-metric may be discontinuous in both of its involved variables.

Definition 2.7. [5] Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called a lower semicontinuity (for short LSC) at a point $t_0 \in X$ if for every sequence $\{t_n\} \subset X$ such that $\{t_n\}$ converges to t_0, then either $\liminf_{t_n\to t_0} f(t_n) = \infty$ or $f(t_0) \leq \liminf_{t_n\to t_0} f(t_n)$.

Definition 2.8. [5] Let (X, d) be a b-metric space.
1. A sequence $\{t_n\}$ in X is said to be convergent to t_0 in X if $\lim_{n\to\infty} \acute{d}(t_n, t_0) = 0$.
2. A sequence $\{t_n\}$ in X is called Cauchy in X, if $\lim_{n,m\to\infty} \acute{d}(t_n, t_m) = 0$.
3. The b-metric space (X, d) is said to be complete if all of its Cauchy sequences are convergent.

3. Main results

In this section, we investigate some fixed point results for (ζ, η)-weakly cyclic contraction mappings in the framework of b-metric spaces. Finally, we present some illustrative examples to show the usefulness of the obtained results.

Definition 3.1. Let (X, d) be b-metric space with parameter $s \geq 1$. Let $r \in \mathbb{N}$, $V_1, ..., V_r$ be non-empty subsets of X and $\Xi = \bigcap_{i=1}^{n} V_i$. Then, an operator $f : \Xi \to \Xi$ is a (ζ, η)-weakly cyclic contraction if
1. $\Xi = \bigcap_{i=1}^{n} v_i$ is a cyclic representation of Ξ concerning f,
2. For any $(x, y) \in V_i \times V_{i+1}$, $i = 1, 2, ..., r$ (with $V_{r+1} = V_1$),

$$\zeta(d(f(x), f(y))) \leq \xi(x, y) - \eta(x, y), \tag{1}$$

where $\zeta : [0, \infty) \to [0, \infty)$ is altering distance function and $\eta : [0, \infty) \to [0, \infty)$ is lower semi-continuous with $\eta(t) = 0$ if and only if $t = 0$.

Theorem 3.2. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$, and $\Xi = \bigcap_{i=1}^{r} V_i$ $r \in$, $V_1, ..., V_r$ are non-empty closed subsets of X. Assume $f : \Xi \to \Xi$ is a (ζ, η)-weakly cyclic contraction. Then, f has a distinct fixed point in $\bigcap_{i=1}^{r} V_i$.*

By assuming the control function η is continuous, we obtain an immediate consequence of Theorem 3.2 as follows:

Theorem 3.3. *(X, d) be a complete b-metric space with some $s \geq 1$, and $\Xi = \bigcap_{i=1}^{r} V_i$ $r \in$, $V_1, ..., V_r$ be non-empty closed subsets of X. Assume $f : \Xi \to \Xi$ is a (ζ, η)-weakly cyclic contraction, where η is a continuous mapping. Then, f has distinct fixed point in $\bigcap_{i=1}^{r} V_i$.*

Now, we provide an example that supports the validity of our first result given in Theorem 3.2.

Example 3.4. Let $X = [1, 1]$ and a b-metric defined by $d(x, y) = |xy|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. We consider two non-empty closed subsets $B = [1, 0]$ and $D = [0, 1]$ of X. Clearly, $B \cap D = \emptyset$ and $B \cup D = X$. Consider the mapping $f : B \cup D \to B \cup D$ defined by $f(t) = -\frac{t}{3}$ for all $t \in B \cup D$. Then, $f(B) \subset D$ and $f(D) \subset B$, and this indicates that $X = B \cup D$ is a cyclic representation of X concerning f.
Define $\zeta(t) = \frac{t}{2}$ and

$$\eta(t) = \begin{cases} 0, & \text{if } t = 0, \\ t^2 + 1, & \text{if } t \in (0, 1]. \end{cases}$$

Clearly η is lower semi-continuous, since $\eta(0) = 0 \leq \liminf_{t\to 0} \eta(t) = 1$. We can see that the conditions of Theorem 3.2 hold. Thus the mapping f has a distinct fixed point $t = 0$ in $B \cup D$.

Next, we provide an example that supports our second result given in Theorem 3.3.

Example 3.5. Consider $X = \mathbb{R}$ and a b-metric that is defined by $d(x, y) = |xy|^2$ for all $x, y \in X$. Then $s = 2$ and (X, d) is a complete b-metric space. Let $r = 2$, $V_1 = [0, \infty)$ and $V_2 = [-\infty, 0)$. Then we have that $V_1 \cap V_2 = \emptyset$ and $X = V_1 \cup V_2$. Now, consider the mapping $f : V_1 \cup V_2 \to V_1 \cup V_2$ defined by $f(t) = -\frac{t}{4(1+t^2)}$ for all $t \in V_1 \cup V_2$. Then, we can see that $f(V_1) \subset V_2$ and $f(V_2) \subset V_1$. This implies that $X = V_1 \cup V_2$ is a cyclic representation with respect to f. Define $\zeta(t) = \frac{t}{4}$ and $\eta(t) = \frac{t}{3}$ for each $t \geq 0$. Thus, from the above cases, we observe that each of the conditions of Theorem 3.3 is satisfied. Therefore, f has a distinct fixed point $0 \in V_1 \cup V_2$.

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Independence Property of Parseval Continuous K-g-Frames

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Abstract. In this article, we stablish a continuous K-g-dual for Parseval continuous K-g-frames in Hilbert spaces. Furthermore, we explore the independent relation between a Parseval continuous K-g-frame and its continuous K-g-dual.

Keywords: Parseval continuous K-g-frame, continuous K-g-dual, continuous g-Bessel sequence

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1. Introduction

The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer [9], when they studied some problems in nonharmonic Fourier series in 1952. Daubechies [7] reintroduced the frames as a generalization of orthonormal bases. Frames are very important and useful in signal processing, image processing, sampling theory and coding theory; for more details see [4, 6, 8].
Note that a sequence $\{f_n\}_{n=1}^{\infty}$ in a separable Hilbert space H is called a Bessel sequence if there exists $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad f \in H.$$

By Theorem 3.2.3 of [5], $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence if and only if the mapping, definde by

$$T : l^2 \longrightarrow H, \quad T(\{c_n\}_{n=1}^{\infty}) := \sum_{n=1}^{\infty} c_n f_n,$$

is a well-defined bounded linear operator with $\| T \| \leq \sqrt{B}$. Then T is called the synthesis operator of $\{f_n\}_{n=1}^{\infty}$. So, the adjoint operator of T is given by

$$T^*(f) = \{\langle f, f_n\rangle\}_{n=1}^{\infty}, \qquad f \in H,$$

and is called the analysis operator of $\{f_n\}_{n=1}^{\infty}$.
A sequence $F = \{f_n\}_{n=1}^{\infty}$ in H is called a frame for H, if there exist constants A, $B > 0$

*Speaker.

such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad f \in H.$$

The constants A and B are called the lower and upper frame bounds, respectively. We say that a frame is tight (resp. Parseval) if one can choose $A = B$ (resp. $A = B = 1$). The frame operator of a frame $\{f_n\}_{n=1}^{\infty}$ is defined by

$$S : H \longrightarrow H, \quad Sf = TT^*f = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n, \qquad f \in H,$$

which is a bounded, positive and invertible operator; see Lemma 5.1.5 of [5]. In 2006, Sun [11] introduced g-frames as a generalization of discreate frames. He gave a characterization of g-frames and studied g-Riesz bases, g-orthonormal bases and dual g-frames.
In 2012, K-frames was introduced by Găvruta [10], as a new generalization of frames. It is a frame for the range of a bounded linear operator K on a Hilbert space H, that reconstruct the members of range of K. Recently, Abtahi et al [1] introduced and studied the space of all operators $K \in B(H)$, such that for a given Bessel sequence $\{f_n\}_{n=1}^{\infty}$, to be a K-frame. Alizadeh et al in [2] gave the equivalent characterization of continuous K-g-frames (c-K-g-frames) for H and they showed that $\{\Lambda_\omega K^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H, where $K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. Moreover, they proved that $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-TK-g-frame for H, if $T, K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.
In this article, we remeber the concept of c-K-g-duals and obtain some results about them. Throughout this article, (Ω, μ) is a measure space, H, H_1 and H_2 are separable complex Hilbert spaces and $B(H_1, H_2)$ is the Banach space of all bounded operators from H_1 into H_2. We use $B(H)$ as the abbreviation of $B(H, H)$. Also, $\{H_\omega\}_{\omega\in\Omega}$ is a family of Hilbert spaces. At first, we review some definitions and a lemma.

Definition 1.1. (see [2]) *A mapping $F = \{f_\omega\}_{\omega\in\Omega}$ from a measure space Ω into $\bigcup_{\omega\in\Omega} H_\omega$ is called strongly measurable, if F as a mapping from Ω to $\bigoplus_{\omega\in\Omega} H_\omega$ to be measurable.*

Definition 1.2. (see [2]) *Let $K \in B(H)$. A family $\{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$ is called a continuous K-g-frame (c-K-g-frame) for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(i) *$\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable, for each $f \in H$;*
(ii) *There exist constants $0 < A \leq B < \infty$, such that*

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \qquad f \in H.$$

In particular, if $K = I$ is the identity operator on H, then Λ is called a continuous g-frame (c-g-frame) for H.

As usual, the constants A, B are called the lower and upper c-K-g-frame bounds, respectively. If $A\|K^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)$, for all $f \in H$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and every tight c-K-g-frame with $A = 1$, is called Parseval c-K-g-frame.
For a c-g-Bessel family (or c-K-g-frame) $\{\Lambda_\omega\}_{\omega\in\Omega}$, the c-g (or frame) operator $S : H \longrightarrow H$ is defined by

$$\langle Sf, g\rangle = \int_\Omega \langle {\Lambda_\omega}^* \Lambda_\omega f, g\rangle d\mu(\omega), \qquad f, g \in H.$$

It is obvious that $\langle Sf, f\rangle = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega)$, for all $f \in H$, and so S is a positive operator. In [2], the authors characterized a c-K-g-frame by frame operator.

LEMMA 1.3. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-g-Bessel family for H. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K-g-frame for H with frame operator S if and only if there exists a constant $A > 0$ such that $S = AKK^*$.*

2. c-K-g-Duals and Independent Property of Them

In this section, it is introduced the c-K-g-dual frame of a given c-K-g-frame. The following theorem is an important result in the study of the c-K-g-duals. We refer to [2,3] for details.

THEOREM 2.1. *Let $K \in B(H)$. Then the following statements are equivalent.*

(i) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

(ii) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and there exists a c-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ such that,*

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda_\omega{}^*\Gamma_\omega f, h\rangle d\mu(\omega), \qquad f, h \in H.$$

So, for any c-K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$, we have a c-g-Bessel family

$$\{\Gamma_\omega \in B(H, H_\omega) : \omega \in \Omega\}$$

such that

$$Kf = \int_\Omega \Lambda_\omega{}^*\Gamma_\omega f d\mu(\omega), \qquad f \in H,$$

and

$$K^*f = \int_\Omega \Gamma_\omega{}^*\Lambda_\omega f d\mu(\omega), \qquad f \in H.$$

The family $\{\Gamma_\omega\}_{\omega\in\Omega}$ is called a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and it is a c-K^*-g-frame. It is obvious that, under the above assumptions, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K^*-g-dual of $\{\Gamma_\omega\}_{\omega\in\Omega}$ and if K is a self adjoint operator, $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ are called c-K-g-dual pairs.

THEOREM 2.2. *Suppose that $K \in B(H)$ such that K^* is surjective and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame for H. Then $\{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Moreover, $\{\Lambda_\omega(K^*)^\dagger\}_{\omega\in\Omega}$ is both a Parseval c-$K^\dagger K$-g-frame and Parseval c-g-frame.*

DEFINITION 2.3. *A c-g-Bessel sequence $\{\Lambda_\omega\}_{\omega\in\Omega}$ in H is independent, if for a c-g-Bessel sequence $\{\Psi_\omega\}_{\omega\in\Omega}$ in H and all $f \in H$,*

$$\int_\Omega \Lambda_\omega{}^*\Psi_\omega f d\mu(\omega) = 0,$$

implies that $\Psi_\omega = 0$ a.e.

In the followig, we inspect the independent relation between a Parseval c-K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Lambda_\omega(K^\dagger)^*\}_{\omega\in\Omega}$ as its c-K-g-dual.

THEOREM 2.4. *Suppose that $K \in B(H)$ such that K^* is surjective and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame for H. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent if and only if its c-K-g-dual $\{\Lambda_\omega(K^\dagger)^*\}_{\omega\in\omega}$ is independent.*

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Resulted Sharp for a New Subclass of Ma-Minda Starlike Functions

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Abstract. In this paper, we generalize the q-derivative operator, which plays an essential role in quantum calculus. In fact, by using the Hadamard product and extended Koebe function we define the following $(\alpha, \beta, \gamma)-$derivative operator

$$d_{\alpha,\beta,\gamma} f(z) = \frac{1}{z} \left\{ f(z) * \mathcal{L}_{\alpha,\beta,\gamma}(z) \right\},$$

where $\mathcal{L}_{\alpha,\beta,\gamma}(z) = \dfrac{2(1-\gamma)z}{(1-\alpha z)(1-\beta z)}$, $\alpha, \beta \in [-1, 1]$, $\alpha\beta \neq \pm 1$ and $\gamma \in [0, 1)$. So, by using subordination relation, the operator$d_{\alpha,\beta,\gamma} f(z)$ and a function $\phi_\delta(z) = 1 + \delta z/\exp(\delta z)$, $(0 < \delta \leq 1)$, we present a new particular Ma-Minda class.

Keywords: Unit disk, Analytic functions, Starlike function, Subordination

AMS Mathematics Subject Classification [2020]: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denotes the family of all analytic and normalized functions in the unit disk. Ma and Minda introduced a certain class of starlike functions using the subordination relation $\prec$ as follows [4]:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \ \frac{zf'(z)}{f(z)} \prec \varphi(z), \ |z| < 1 \right\},$$

where φ is an analytic function with positive real part such that satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$. Note that φ maps the unit disk $|z| < 1$ onto a starlike domain with respect to $\varphi(0) = 1$ which is symmetric with respect to the real axis. Many authors have studied the class $\mathcal{S}^*(\varphi)$ for special cases of φ [1,5,6]. Motivated by [1,5,6], in this paper we introduce a new class of Ma-Minda starlike functions.

This article is organized as follows. In Section 2 we define a new derivative operator and a new subclass of analytic functions. In Section 3 we investigate radius problem and coeficient estimate.

*Speaker.

2. Preliminaries

Throughout this article, we use the following symbols.
- $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$.
- $\partial\mathbb{D} := \{z \in \mathbb{C} : \ |z| = 1\}$ the boundary of $\mathbb{D}$.
- $\overline{\mathbb{D}} := \{z \in \mathbb{C} : \ |z| \leq 1\}$.
- $\mathbb{D}_\rho := \{z \in \mathbb{C} : \ |z| < \rho, \rho > 0\}$.

All analytic and normalized functions f $(f(0) = f'(0) - 1 = 0)$ in $\mathbb{D}$ having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}, \tag{1}$$

with $a_n \in \mathbb{C}$ is denoted by $\mathcal{A}$. A subclass of $\mathcal{A}$ including of all univalent (one-to-one) functions in $\mathbb{D}$ is denoted by $\mathcal{U}$. For two analytic functions f_1 and f_2 in $\mathcal{A}$, we say that f_1 is subordinate to f_2, written as $f_1(z) \prec f_2(z)$ $(z \in \mathbb{D})$ or $f_1 \prec f_2$, if there exists a Schwarz function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ so that for all $z \in \mathbb{D}$, $f_1(z) = f_2(\omega(z))$. Alos, we have the following relation in $\mathbb{D}$:

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \ \& \ f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

Definition 2.1. A function $f \in \mathcal{S}$ is said to be starlike of order α in Δ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{D},$$

where $\alpha \in [0, 1)$.

Definition 2.2. A function $f \in \mathcal{S}$ is a convex function of order α in Δ if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f(z)}\right\} > \alpha, \qquad z \in \mathbb{D}.$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the class of starlike and convex functions of order α, respectively.

Definition 2.3. We say that a function $f \in \mathcal{A}$ is strongly starlike function of order β, denoted by $\mathcal{SS}(\beta)$, if satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta, \qquad z \in \mathbb{D}.$$

Note that $\mathcal{SS}(1) = \mathcal{S}^*$.

In the following, we introduce the $q-$calculus. Quantum calculus ($q-$calculus) was developed by Frank Hilton Jacksonin the early twentieth century [2,3]. The $q-$derivative for $q \in [0, 1]$ was introduced and studied by Jackson as follows:

$$d_q f(z) = \begin{cases} (f(qz) - f(z))/(qz - z); & z \neq 0, 0 \leq q < 1 \\ f'(0); & z = 0 \\ f'(z); & q = 1. \end{cases} \tag{2}$$

It is easy to see that if $f \in \mathbb{A}$, then

$$d_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^n, \qquad [n]_q = \frac{1 - q^n}{1 - q} = \sum_{k=0}^{n-1} q^k, \qquad n = 2, 3, \ldots. \tag{3}$$

Definition 2.4. Let α, β be two complex numbers such that $|\alpha|, |\beta| \leq 1$. Also let γ be a real number so that $\gamma[0,1)$. We define the $(\alpha, \beta, \gamma)-$derivative operator as

$$d_{\alpha,\beta,\gamma} f(z) = \frac{1}{z}\left\{f(z) * \mathcal{L}_{\alpha,\beta,\gamma}(z)\right\}, \tag{4}$$

where $\mathcal{L}_{\alpha,\beta,\gamma}(z) = \dfrac{2(1-\gamma)z}{(1-\alpha z)(1-\beta a)}$.

3. Radius Problems

Lemma 3.1. *Let $\phi_\delta(z) := 1 + \dfrac{\delta z}{e^{\delta z}}$, where $\delta \in (0,1]$ and $z \in \mathbb{D}$. Then $\phi_\delta(z)$ is a convex univalent function in $|z| < r_c(\delta)$, where $r_c(\delta) := (3-\sqrt{5})/2\delta$.*

Lemma 3.2. *Let $\phi_\delta(z) := 1 + \dfrac{\delta z}{e^{\delta z}}$, where $\delta \in (0,1]$. Then*

$$1 - \delta e^{\delta} < \mathrm{Re}\{\phi_\delta(z)\} < 1 + \delta e^{-\delta}.$$

Theorem 3.3. *Let α, β be two complex numbers with $|\alpha|, |\beta| \leq 1$ and γ be a real number such that $\gamma \in [0,1)$. Let $\delta_0 = 0.567143$ be the unique root of the equation $1 - \delta e^{\delta} = 0$, where $\delta \in (0,1]$. If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma)$, then in the disk $\mathbb{D}_{r_s}$:*

$$\mathrm{Re}\left(\frac{z d_{\alpha,\beta,\gamma} f(z)}{f(z)}\right) > 0,$$

where $r_s \in (0, \delta_0)$. The result is sharp.

Theorem 3.4. *The radius of convexity of the class $\mathcal{S}^*(1, 1.1/2, \delta)$ is $r \in (0, 0.17)$ for all $\delta \in (0,1]$.*

4. On coeficients

Theorem 4.1. *Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma)$ such that α, β, γ satisfy the assumption of Theorem 3.3, and $0 < \delta \leq 1$. Then*

$$|a_2| \leq \frac{\delta}{[2]_{\alpha,\beta,\gamma} - 1}, \qquad |a_3| \leq \frac{\delta([2]_{\alpha,\beta,\gamma} - 1) + \delta^2}{([2]_{\alpha,\beta,\gamma} - 1)([3]_{\alpha,\beta,\gamma} - 1)},$$

where

$$[n]_{\alpha,\beta,\gamma} := 2(1-\gamma)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right), \qquad n = 2, 3, \ldots.$$

The result is sharp.

Theorem 4.2. *Let $\delta \in (0,1]$, and α, β, γ satisfy the assumption of Theorem 3.3. If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -4\mu' + 2; & \mu \leq \mu_1 \\ 2; & \mu_1 \leq \mu \leq \mu_2 \\ 4\mu' - 2; & \mu \geq \mu_2, \end{cases} \tag{5}$$

where

$$\begin{aligned} \mu' &= \frac{\mu\delta([3]_{\alpha\beta\gamma} - 1) - [(1+\delta)(1 - [2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{2[2]_{\alpha\beta\gamma} - 1)^2}, \\ \mu_1 &= \frac{[(1+\delta)(1 - [2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{\delta([3]_{\alpha\beta\gamma} - 1)}, \end{aligned}$$

$$\mu_2 \quad := \quad \frac{[(1+\delta)(1-[2]_{\alpha\beta\gamma})+\delta]([2]_{\alpha\beta\gamma}-1)+2([2]_{\alpha\beta\gamma}-1)^2}{\delta([3]_{\alpha\beta\gamma}-1)}.$$

The result is sharp.

Example 4.3. For the function $f_\delta(z)$ in Theorem 3.3, we get $|a_3-\mu a_2^2|=|\mu|$. So, if we put $|\mu|=1$, then $\mu=1$ or $\mu=-1$. On the other hand, if we put $\delta=1$, we will get $\mu_1=-\frac{1}{2}$, $\mu_2=\frac{1}{2}$ and $\mu'=\frac{2\mu+1}{2}$ and for $\mu=-1$, $\mu'=-\frac{1}{2}$ and for $\mu=1$ and $\mu'=\frac{3}{2}$, we will have

$$|a_3-\mu a_2^2|=|\mu|=1<\begin{cases} -4\times\left(-\frac{1}{2}\right)+2=4, & -1=\mu<\mu_1=-\frac{1}{2} \\ 4\times\left(\frac{3}{2}\right)-2=4, & 1=\mu>\mu_2=\frac{1}{2}. \end{cases}$$

And for second inequality, we put $\mu=\frac{1}{4}$, so $|a_3-\mu a_2^2|=|\mu|=\left|\frac{1}{4}\right|<2$ when $-\frac{1}{2}=\mu_1<\frac{1}{4}=\mu<\mu_2=\frac{1}{2}$ if $\mu_1=\mu=\mu_2=1$.

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Bionomial Distribution and its Associated New Ma-Minda Type Class

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Abstract. In this paper, we define a new Ma-Minda type class based on binomial distribution series. Our investigation will be focused on the coeficients of the function f belonging to that class.

Keywords: Convolution or Hadamard product, Ma-Minda type class, Subordination, Binomial distribution

AMS Mathematics Subject Classification [2020]: 30C45, 30C50

1. Introduction

Throughout this article, we use the following symbols.
- $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$.
- $\partial\mathbb{D} := \{z \in \mathbb{C} : \ |z| = 1\}$ the boundary of $\mathbb{D}$.
- $\overline{\mathbb{D}} := \{z \in \mathbb{C} : \ |z| \leq 1\}$.

All analytic and normalized functions f $(f(0) = f'(0) - 1 = 0)$ in $\mathbb{D}$ having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}, \tag{1}$$

with $a_n \in \mathbb{C}$ is denoted by $\mathcal{A}$. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, and f be defined as (1). Then their Hadamard product is defined as $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, where $z \in \mathbb{D}$. We denote by $\mathcal{S}$ the class of all univalent functions in $\mathbb{D}$. For two analytic functions f_1 and f_2 in $\mathcal{A}$, we say that f_1 is subordinate to f_2, written as $f_1(z) \prec f_2(z)$ $(z \in \mathbb{D})$ or $f_1 \prec f_2$, if there exists a Schwarz function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ so that for all $z \in \mathbb{D}$, $f_1(z) = f_2(\omega(z))$. Alos, we have the following relation in $\mathbb{D}$:

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \ \& \ f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

A function $f \in \mathcal{A}$ is starlike in $\mathcal{D}$, if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in \mathbb{D}.$$

*Speaker.

We denote $\mathcal{S}^*$ the class of all starlike functions in $\mathbb{D}$. Also, a function $f \in \mathcal{A}$ is convex in the unit disk $\mathbb{D}$, if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad z \in \mathbb{D}.$$

The class of all convex functions in $\mathbb{D}$ is denoted by $\mathcal{K}$. In it well-known that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathbb{A}$ [2, 5]. There is an other way to define the class of starlike and convex functions. Indeed, applying the subordination relation gives

$$f \in \mathcal{S}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \qquad z \in \mathbb{D},$$
$$f \in \mathcal{K} \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \qquad z \in \mathbb{D}.$$

Definition 1.1. For $\alpha \in (0, 1]$, let ψ_α be defined as

$$\psi_\alpha(z) = \frac{e^{\alpha z} - 1}{\alpha(1 - \alpha z)}, \qquad z \in \mathbb{D}. \tag{2}$$

It easy to check that ψ_α has the following expansion

$$\psi_\alpha(z) = z + \frac{3\alpha}{2}z^2 + \frac{5\alpha^2}{3}z^3 + \ldots, \qquad z \in \mathbb{D}. \tag{3}$$

Lemma 1.2. *The function ψ_α is convex univalent in the unit disk $\mathbb{D}$ for all $\alpha \in (0, 0.32]$.*

2. Binomial Distribution

A binomial distribution is used when a trial has exactly two mutually exclusive outcomes. A successful outcome will be tagged success and a failure will be tagged failure. With a binomial distribution, x successes are calculated in m trials, with p representing the probability of success on a single trial. For all trials, p is assumed to be fixed in a binomial distribution. A binomial probability mass function can be expressed as follows:

$$P(x; p, n) = \binom{m}{x} p^x q^{m-x}, \qquad m \geq 1, x = 0, 1, 2, \ldots, m, \tag{4}$$

where $q = 1 - p$ and $\binom{m}{x} = m!/x!(m-x)!$. Therefore, binomial cumulative probability can be calculated using the following formula:

$$F(x; p, m) = \sum_{i=0}^{x} \binom{m}{i} p^i q^{m-i}. \tag{5}$$

By using (4) we define a new analytic and normalized function whose coeficients are binomial distribution probabilities as follows:

$$\begin{aligned} K(m, p, q; z) &= z + \sum_{n=2}^{\infty} \binom{m}{n-2} p^{n-2} q^{m-n+2} z^n \\ &= z + q^m z^2 + mpq^{m-1} z^3 + \ldots. \end{aligned} \tag{6}$$

It is easy to see that by ratio test, the radius of convergence of the above series $K(m, p, q; z)$ is infinity.

Definition 2.1. Let $\psi_\alpha(z)$ be defined by (2), $\alpha \in (0,1]$, $0 \le p \le 1$, $q = 1-p$ and $m \ge 1$. We say that a function $f(z) \in \mathcal{A}$ belongs to the class $B_\psi(\alpha, m, p, q)$, if it satisfies the following subordination relation:

$$\left(\frac{z(K(m,p,q;z) * f(z))'(z)}{(K(m,p,q;z) * f(z))(z)} - 1\right) \prec \psi_\alpha(z), \qquad z \in \mathbb{D}, \tag{7}$$

where $K(z)$ is defined in (6).

3. Main Results

Theorem 3.1. *Let $\alpha \in (0,1]$, $0 \le p \le 1$, $q = 1-p$ and $m \ge 1$. If a function $f(z)$ is of the form (1) belongs to the class $B_\psi(\alpha, m, p, q)$, then*

$$|a_2| \le \frac{1}{q^m}, \qquad |a_3| \le \frac{3\alpha + 2}{4mpq^{m-1}}.$$

If $\alpha \in (0, 0.625]$, then

$$|a_4| \le \frac{4 + (9\alpha + 6)}{6m(m-1)p^2q^{m-2}}.$$

All inequalities are sharp.

Theorem 3.2. *Let the function $f \in \mathcal{A}$ belong to the class $B_\psi(\alpha, m, p, q)$, where $\alpha \in (0,1]$, $0 \le p \le 1$, $q = 1-p$, and $m \ge 1$. Then for all $\mu \in \mathbb{C}$ we have*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \begin{cases} \dfrac{1}{4kmpq^{m-1}}\left(\dfrac{2(1-k)mp}{kq^{m+1}} - \dfrac{4mp\mu}{kq^{m+1}} + 3\alpha + 2\right), & \mu \le \delta_1 \\ \dfrac{1}{2kmpq^{m-1}}, & \delta_1 \le \mu \le \delta_2 \\ \dfrac{1}{4kmpq^{m-1}}\left(\dfrac{4mp\mu}{kq^{m+1}} - \dfrac{2(1-k)mp}{kq^{m+1}} - 3\alpha - 2\right), & \mu \ge \delta_2. \end{cases}$$

where

$$\begin{aligned} \delta_1 &:= \frac{3\alpha k q^{m+1}}{4mp} - \frac{1}{2}(k-1), \qquad k \ge 1, \\ \delta_2 &:= \frac{kq^{m+1}}{mp} + \frac{3\alpha k q^{m+1}}{4mp} - \frac{1}{2}(k-1), \qquad k \ge 1. \end{aligned}$$

The result is sharp.

Pommerenke [4] was the first to study the Hankel determinant $H_{q,n}(f)$ of a function f given by (1). The Hankel determinant $H_{q,n}(f)$ is given as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Based on difierent values of q and n, Hankel determinants for various orders can be derived. Here is how the above-defined determinant looks when n is equal to 1 and q is equal to 2

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2| \qquad a_1 = 1.$$

In addition, if $q = n = 2$, then we have the second Hankel determinant

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left| a_2 a_4 - a_3^2 \right| .$$

The upper bound of $H_{2,2}(f)$ for diferent subclasses of analytical functions has been studied and investigated by many authors in recent years, see for instance [1, 2, 4, 5]. Motivated by aforementioned works, we estimate $H_{2,1}(f)$ and $H_{2,2}(f)$, where $f \in B_\psi(\alpha, m, p, q)$.

THEOREM 3.3. *Let $0 < p \leq 1, 0 < q \leq 1, m \geq 1$, and $\alpha \in (0, 1]$. Also, let the function f be of the form (1) belong to $B_\psi(\alpha, m, p, q)$. Then*

$$H_{2,1}(f) \leq \frac{1}{2mpq^{m-1}} \left(\frac{3\alpha}{2} + 1 \right) + \frac{1}{q^{2m}}.$$

The result is sharp.

THEOREM 3.4. *Let $0 < p \leq 1, 0 < q \leq 1, m \geq 1$, and $\alpha \in (0, 1]$. If a function f is of the form (1) belongs to $B_\psi(\alpha, m, p, q)$, then*

$$H_{2,2}(f) \leq \frac{2}{3m(m-1)p^2 q^{2(m-1)}} \left(1 + \frac{3}{2} \left(\frac{3\alpha}{2} + 1 \right) \right) + \frac{1}{4m^2 p^2 q^{2(m-1)}} \left(\frac{3\alpha}{2} + 1 \right)^2 .$$

The result is sharp.

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Some fixed point theorems in sequential G-metric spaces and their application to the solvability of a system of integral equations

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Abstract. The goal of this work is to demonstrate some fixed point theorems for JS-contractions over sequential G-metric spaces that are not necessarily continuous in any variable and do not enjoy the triangle inequality. This new space is a generalization of standard G-metric spaces and G_b-metric spaces. Additionally, a related application and an illustrated example are included to verify the accuracy of the findings. .

Keywords: Fixed point, Sequential metric space, JS-contraction, Integral equation..

AMS Mathematics Subject Classification [2020]: Primary: 47H10; Secondary: 54H25.

1. Introduction

If we take the metric as the absolute value metric in $\mathbb{R}$, contraction mapping theorem states that if for both arbitrary points x and y in the domain of the function f, the length of the line segment connecting the points fx and fy is less than or equal to a multiple $(in[0,1))$of the distance between the points x and y, then this function has a unique fixed point. The length of a line segment is one of the geometric concepts that can be changed by perimeter, area, and volume. Contraction mapping theorem in $\mathbb{R}$ with a G-metric (which is a generalization of the ordinary metric) expresses the same geometric fact in which for example, instead of the length of the line segment, the perimeter of the triangle made by the points fx, fy and fz is considered.

Mustafa and Sims introduced the idea of a generalized metric space, known as a G-metric space [**?**]. Several fixed point theorems for mappings that meet distinct contractive requirements on G-metric spaces have been obtained by numerous researchers in recent years. We refer the reader to [**?**, **1**, **2**, **6**], [**?**]- [**?**], [**?**], [**?**], and the references cited therein

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for an overview of fixed point theory, its applications, various contractive conditions, and related subjects in G-metric spaces.

2. Sequential G-metric spaces

Let $\mathfrak{L}$ be a non-empty set and $\underline{\mathcal{G}} : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to [0, \infty]$ be a mapping. For any $\varsigma \in \mathfrak{L}$, let us define the set

$$C(\underline{\mathcal{G}}, \mathfrak{L}, \varsigma) = \{\{\varsigma_n\} \subset \mathfrak{L} : \lim_{n\to\infty} \underline{\mathcal{G}}(\varsigma, \varsigma_n, \varsigma_n) = 0\}. \tag{1}$$

Definition 2.1. (Sequential G-Metric Space) Let $\mathfrak{L}$ be a nonempty set and $\underline{\mathcal{G}} : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}^+$ be a function that satisfies the subsequent criteria:
(G1) $\underline{\mathcal{G}}(\vartheta, \vartheta', \vartheta'') = 0$ yields that $\vartheta = \vartheta' = \vartheta''$;
(G2) $0 < \underline{\mathcal{G}}(\vartheta, \vartheta, \vartheta')$, for all $\vartheta, \vartheta' \in \mathfrak{L}$ with $\vartheta \neq \vartheta'$;
(G3) $\underline{\mathcal{G}}(\vartheta, \vartheta, \vartheta') \leq \underline{\mathcal{G}}(\vartheta, \vartheta', \vartheta'')$, for all $\vartheta, \vartheta', \vartheta'' \in \mathfrak{L}$ with $\vartheta'' \neq \vartheta'$;
(G4) $\underline{\mathcal{G}}(\vartheta, \vartheta', \vartheta'') = \underline{\mathcal{G}}(\vartheta, \vartheta'', \vartheta') = \underline{\mathcal{G}}(\vartheta', \vartheta'', \vartheta) = \cdots$, (symmetry in all three variables);
(G5) there is $\omega > 0$ so that for every $\vartheta, \vartheta', \vartheta'' \in \mathfrak{L}$ and $\{\vartheta_n\} \in C(\underline{\mathcal{G}}, \mathfrak{L}, \vartheta)$,

$$\underline{\mathcal{G}}(\vartheta, \vartheta', \vartheta'') \leq \omega \liminf_{n\to\infty} \underline{\mathcal{G}}(\vartheta_n, \vartheta', \vartheta'').$$

$(\mathfrak{L}, \underline{\mathcal{G}})$ is referred to as a sequential $\underline{\mathcal{G}}$-metric space (SGM space) thereafter.

Furthermore, we refer to it as a symmetric SGM space iff $\underline{\mathcal{G}}(\vartheta, \vartheta, \vartheta') = \underline{\mathcal{G}}(\vartheta', \vartheta', \vartheta)$, for all $\vartheta, \vartheta' \in \mathfrak{L}$.

It is noteworthy that there are many classes of SGM spaces. SGM spaces include, for instance, every G-metric space, every G_b-metric space, and every G_b-metric like space.

Theorem 2.2. *Let $(\mathfrak{L}, \underline{\mathcal{G}})$ be a complete symmetry SGM space and $\nabla : \mathfrak{L} \to \mathfrak{L}$ be a mapping such that:*
(i) ∇ is a $\underline{\mathcal{G}}$-JS-Hardy-Rogers contraction of type I,
(ii) there is $\vartheta_0 \in \mathfrak{L}$ so that

$$\delta(\underline{\mathcal{G}}, \nabla, \vartheta_0) := \sup\left\{\underline{\mathcal{G}}\big(\nabla^i\vartheta_0, \nabla^j\vartheta_0, \nabla^j\vartheta_0\big) : i, j = 1, 2, \cdots\right\} < \infty.$$

This suggests that $\mathfrak{L}$ contains at least one fixed point for ∇. Furthermore, if $(\mathfrak{L}, \underline{\mathcal{G}})$ be a symmetry SGM space and for all fixed point ϑ of ∇ in $\mathfrak{L}$ one has $\underline{\mathcal{G}}(\vartheta, \vartheta, \vartheta) = 0$, then the fixed point of ∇ is unique.

Some fixed point theorems in a SGM space

We further limit the requirements on the control function ν in this section. The set of functions $\nu : [0, \infty) \to [1, \infty)$ is indicated by the symbol Θ where
(ν1) ν is continuous and increasing,
(ν2) for any sequence $\{\vartheta_n\} \subseteq (0, \infty)$

$$\lim_{n\to\infty} \nu(\vartheta_n) = 1 \Leftrightarrow \lim_{n\to\infty} \vartheta_n = 0.$$

Definition 2.3. Let $(\mathfrak{L}, \underline{\mathcal{G}})$ be a SGM space with parameter $\omega \geq 1$. A mapping $\nabla : \mathfrak{L} \to \mathfrak{L}$ is called a $\underline{\mathcal{G}}$-JS-Hardy-Rogers contraction of type I if

$$\begin{aligned} &\nu(\underline{\mathcal{G}}(\nabla\vartheta, \nabla\vartheta', \nabla\vartheta'')) \\ &\leq \big(\nu(\underline{\mathcal{G}}(\vartheta, \vartheta', \vartheta''))\big)^{\chi_1} \big(\nu(\frac{\underline{\mathcal{G}}(\vartheta, \nabla\vartheta, \nabla\vartheta)}{\omega})\big)^{\chi_2} \big(\nu(\underline{\mathcal{G}}(\vartheta', \nabla\vartheta', \nabla\vartheta'))\big)^{\chi_3} \\ &\big(\nu(\underline{\mathcal{G}}(\vartheta'', \nabla\vartheta'', \nabla\vartheta''))\big)^{\chi_4} \end{aligned} \tag{2}$$

$$\big(\nu(\underline{\mathcal{G}}(\vartheta,\nabla\vartheta',\nabla\vartheta'))\big)^{\chi_5}\big(\nu(\underline{\mathcal{G}}(\vartheta',\nabla\vartheta'',\nabla\vartheta''))\big)^{\chi_6}\big(\nu(\underline{\mathcal{G}}(\vartheta'',\nabla\vartheta,\nabla\vartheta))\big)^{\chi_7}$$

for some $\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6,\chi_7 \geq 0$ with $\sum \chi_i < 1$ where $\nu \in \Theta$.

For more details on this type of contractions we refer the eader to [**?**, **?**].

Theorem 2.4. *Let $(\mathfrak{L},\underline{\mathcal{G}})$ be a complete symmetry SGM space and $\nabla : \mathfrak{L} \to \mathfrak{L}$ be a mapping such that:*
(i) ∇ is a $\underline{\mathcal{G}}$-JS-Hardy-Rogers contraction of type I,
(ii) there is $\vartheta_0 \in \mathfrak{L}$ so that

$$\delta(\underline{\mathcal{G}},\nabla,\vartheta_0) := \sup\left\{\underline{\mathcal{G}}\big(\nabla^i\vartheta_0,\nabla^j\vartheta_0,\nabla^j\vartheta_0\big) : i,j = 1,2,\cdots\right\} < \infty.$$

This suggests that $\mathfrak{L}$ contains at least one fixed point for ∇. Furthermore, if $(\mathfrak{L},\underline{\mathcal{G}})$ be a symmetry SGM space and for all fixed point ϑ of ∇ in $\mathfrak{L}$ one has $\underline{\mathcal{G}}(\vartheta,\vartheta,\vartheta) = 0$, then the fixed point of ∇ is unique.

3. Application

The following system of Fredholm integral equations can be considered:

$$\begin{cases} \mathcal{Q}_1(\vartheta) = \int_0^\kappa K(\vartheta,\varsigma)f(\varsigma,\mathcal{Q}_1(\varsigma),\mathcal{Q}_2(\varsigma),...,\mathcal{Q}_n(\varsigma))d\varsigma \\ \mathcal{Q}_2(\vartheta) = \int_0^\kappa K(\vartheta,\varsigma)f(\varsigma,\mathcal{Q}_2(\varsigma),...,\mathcal{Q}_n(\varsigma),\mathcal{Q}_1(\varsigma))d\varsigma \\ \qquad\vdots \\ \mathcal{Q}_n(\vartheta) = \int_0^\kappa K(\vartheta,\varsigma)f(\varsigma,\mathcal{Q}_n(\varsigma),\mathcal{Q}_1(\varsigma),...,\mathcal{Q}_{n-1}(\varsigma))d\varsigma. \end{cases} \tag{3}$$

where $f : [0,\kappa]\times\mathbb{R}^n \to \mathbb{R}$ is a continuous function.

Assume that the set of continuous real functions defined on $I = [0,\kappa]$ is $\mathfrak{L} = C(I,\mathbb{R})$. An existence theorem for a solution of (2) in $\mathfrak{L}$ is now provided.

For all $\mathcal{Q} \in \mathfrak{L}$, define

$$||\mathcal{Q}||_\infty = \sup_{\vartheta\in I}|\mathcal{Q}(\vartheta)|.$$

Here, $(\mathfrak{L},||.||_\infty)$ is a Banach space.

4. Conclusions

In this paper, we presented several fixed point results in the framework of SGM spaces. In fact, we obtained some new fixed point results in the context of sequential G-metric spaces by combining the concepts of Jleli-Samet, Hardy-Rogers, and Prešić contractions. In our novel framework, we finally solved a system of Volterra-type integral equations. Now some questions come to mind which are as follows:

Is it possible to prove the results of this paper about Prešić mappings by replacing the liminf with the limsup in a sequential G-metric space?

Is it possible to remove the symmetry condition of the space from the results obtained in this paper?

Is it possible to prove the existence of a fixed point for mappings that satisfy other contraction conditions such as Wardowski contractions, weak-Wardowski contractions, etc.?

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On Ruscheweyh q-derivative operator associated with fractional integral operator

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Abstract. In this paper, we employ a Ruscheweyh q-derivative operator to perform a q-analogue of certain fractional integral operator defined on an open unite disk. Then, we make use of the Hadamard convolution product to discuss several related results. Also, we investigate the q-fractional integral operator and obtain some applications for the differential subordination.

Keywords: analytic functions, Convolution, q-derivative, Subordination, Ruscheweyh q-derivative, q-fractional operator.

AMS Mathematics Subject Classification [2020]: 30C45, 65L05

1. Introduction

Many investigation find the generalization of many classes of analytic function using q-derivative and q-integral. Jackson [4] first introduced the concept of q-calculus.
Let f be a real or complex valued and U be unite disk $U = \{z : |z| < 1\}$, $0 < q < 1$. Then the q-derivative operator is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{z - qz}, z \in U.$$

Let $\mathcal{A}[a, n]$ be the class of all analytic function of the form $f(z) = a + a_n z^n + \sum_{k=2}^{\infty} a_{k+n} z^{k+n}$ in the open unit disk U and use $\mathcal{A} = \mathcal{A}[0, 1]$. Therefore, the class $\mathcal{A}$ of analytic and univalent function $f(z)$ in the open unite disk U normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

The convolution of two function f and g is a function in $\mathcal{A}$ given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \tag{2}$$

where $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Additionally, we say that the function f subordinate to g and write $f(z) \prec g(z)$ if there exist Schwartz function w in U such that $w(0) = 0$ and

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$|w(z)| < 1$ and $f(z) = g(w(z))$.
From above, for the function $f(z)$ in equation (2), we obtain

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \ k \in \mathbb{N}, \ z \in U,$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + \sum_{i=1}^{k-1} q^i, \ [0]_q = 0.$$

It is clear that $[1]_q = 1$. For $k \in \mathbb{Z}^+$, the q-fractional is given by [1]

$$[k]_q! = \begin{cases} 1, & k = 0, \\ [1]_q[2]_q...[k]_q, & k \in \mathbb{N}. \end{cases}$$

In addition, with $t > 0$, the q-Pochhammer symbol has the form [1]

$$[t]_{q,k} = ([t]_q)_k = \begin{cases} 1, & k = 0, \\ [t]_q[t+1]_q...[t+k-1]_q, & k \in \mathbb{N}. \end{cases}$$

Note that $[t]_{q,k} = [t]_k$ when $q \to 1^-$.
For $t > 0$, the q-analogue of the gamma function is presented as

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \text{ and } \Gamma_q(1) = 1.$$

Now, we define Ruscheweyh q-derivative operator $\mathcal{I}_q^{\mu}(z) : \mathcal{A} \to \mathcal{A}$ as follow

$$\mathcal{I}_q^{\mu} f(z) = \mathcal{I}_q^{\mu}(z) * f(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1]_{q,k}}{[k-1]!_k} a_k z^k, \ \mu > -1, z \in U.$$

where $\mathcal{I}_q^{\mu}(z)$ is defined by the relation [5]

$$\mathcal{I}_q^{\mu}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1]_{q,k}}{[k-1]!_k} z^k, \ \mu > -1, z \in U.$$

Definition 1.1. [2] The q-fractional Rieman-Liouville integral of order λ ($\lambda > 0$) is defined for a function f by

$$D_{q,z}^{-\lambda} f(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z \frac{f(t)}{(1-qt)^{1-\lambda}} d_q t, \ z \in U,$$

where f is an analytic function in U.

2. Preliminary lemmas

Lemma 2.1. *[3] Let $f \in A$ and h be a convex function. If $f \prec h$ then*

$$f * g \prec h * g,$$

for all convex functions $g \in \mathcal{A}$.

Lemma 2.2. *[6] Let f_i, $i = 1, 2$ is analytic function in U from the following form*

$$f_i(z) = 1 + b_{i1} z + b_{i2} z^2 + ..., \ i = 1, 2, z \in U.$$

If $Re\,(f_i) > \beta_i$, $0 \le \beta_i < 1$, then the function $f_1 * f_2$ is an analytic function satisfies the inequality

$$Re(f_1 * f_2) > 1 - 2(1-\beta_1)(1-\beta_2).$$

LEMMA 2.3. *[6] Let f, is analytic function in U from the following form*

$$f(z) = 1 + b_1 z + b_2 z^2 + ..., \ z \in U.$$

If $Ref > \beta$, $0 \leq \beta < 1$, then

$$Re(f(z)) > 2\beta - 1 + \frac{2(1-\beta)}{1+|z|}, \ z \in U.$$

3. Definition and coefficient bounds

We introduce a q-fractional integral of the operator $\mathcal{I}_q^n f(z)$.

DEFINITION 3.1. Let $0 < q \leq 1$, $\lambda \geq 0$ and $n \in \mathbb{N}$. The q-fractional integral of the operator $\mathcal{I}_q^n f$ is defined by the following

$$\begin{aligned} D_{q,z}^{-\lambda}\mathcal{I}_q^n f(z) &= \frac{1}{\Gamma_q(\lambda)}\int_0^z \frac{\mathcal{I}_q^n f(t)}{(1-qt)^{1-\lambda}} d_q t \\ &= \frac{1}{\Gamma_q(\lambda)}\int_0^z \frac{t}{(1-qt)^{1-\lambda}} d_q t + \frac{1}{\Gamma_q(\lambda)}\sum_{k=2}^{\infty}\frac{[\mu+1]_{q,k}}{[k-1]!_k} a_k \int_0^z \frac{t^k}{(1-qt)^{1-\lambda}} d_q t \\ &= \frac{1}{\Gamma_q(\lambda)}\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} z^{1+\lambda} + \frac{1}{\Gamma_q(\lambda)}\sum_{k=2}^{\infty}\frac{[\mu+1]_{q,k}}{[k-1]!_k}\frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} a_k z^{k+\lambda}. \end{aligned}$$

We note that $D_{q,z}^{-\lambda}\mathcal{I}_q^n f(z) \in \mathcal{A}[0, \lambda+1]$ and $D_{q,z}^{0}\mathcal{I}_q^n f(z) = \mathcal{I}_q^n f(z)$.
Now, by taking $q \to 1^-$, we have

$$D_z^{-\lambda} I^n f(z) = \frac{1}{(\lambda)_2 \Gamma^2(\lambda)} z^{1+\lambda} + \frac{1}{\Gamma^2(\lambda)}\sum_{k=2}^{\infty}\frac{[\mu+1]_{q,k}}{[k-1]!_k}\frac{k!}{(\lambda)_{k+1}} a_k z^{k+\lambda}. \tag{3}$$

Now, we study the subclass of analytic function by using the new operator.

DEFINITION 3.2. Let the function $f \in \mathcal{A}$, $0 < q < 1$, $\delta \in [0,1)$ and $\lambda > 0$. We define the subclass $S_n^q(\lambda, \delta)$ of functions f which satisfy the inequality

$$Re\left\{D_q\left(D_{q,z}^{-\lambda}\mathcal{I}_q^n f(z)\right)\right\} > \delta.$$

By allowing $q \to 1^-$ in Definition 3.2, the class $S_n^q(\lambda, \delta)$ is denoted by $S_n(\lambda, \delta)$.

THEOREM 3.3. *Let $0 < q < 1$, $\delta \in [0,1)$ and $\lambda > 0$. Then, the class $S_n^q(\lambda, \delta)$ is convex.*

By taking $q \to 1^-$ into Theorem 3.3 leads to the following corollary.

COROLLARY 3.4. *Let $\delta \in [0,1)$ and $\lambda > 0$. Then, the class $S_n(\lambda, \delta)$ is convex.*

LEMMA 3.5. *Let $0 < q < 1$, $h, s \in S_q^n(\lambda, \delta)$, $h(z) = s(z) + \frac{q^c}{[c]_q} z D_q s(z)$ and $c \in \mathbb{N}$ such that*

$$\Lambda_q^c(z) = z + \sum_{k=2}^{\infty}\frac{q^c + [c]_q}{q^c[k]_q + [c]_q} z^k \tag{4}$$

be a convex function. If $r, s \in \mathcal{A}$ with $r(0) = s(0) = 0$ and

$$r(z) + \frac{q^c z}{[c]_q} D_q r(z) \prec h(z) \tag{5}$$

then

$$s(z) = \frac{[c]_q}{z^c} \int_0^z t^{c-1} h(t) d_q t, \tag{6}$$

and

$$f(z) \prec s(z). \tag{7}$$

THEOREM 3.6. *Let* $0 < q < 1$, $r, h \in S_n^q(\lambda, \delta)$ *with* $h(z) = s(z) + \frac{q^c z}{[c]_q} D_q s(z)$ *and* $c \in \mathbb{N}$ *such that* $\Lambda_q^c(z)$ *defined in* (4) *is a convex function. If* $z^c F(z) = [c]_q \int_0^z t^{c-1} f(t) d_q t$, $z \in U$, *the following differential subordination*

$$D_{q,z}^{-\lambda} \mathcal{I}_q^n f(z) \prec h(z), \ z \in U, \tag{8}$$

implies

$$D_{q,z}^{-\lambda} \mathcal{I}_q^n F(z) \prec s(z), \ z \in U. \tag{9}$$

This result is sharp.

THEOREM 3.7. *Let* $0 < q < 1, z \in U$, $\delta \in [0, 1)$ *and* $h(z) = \frac{1+(2\delta-1)z}{1+qz}$, $\delta \in [0, 1)$ *and* $c \in \mathbb{N}$ *such that* $\Lambda_q^c(z)$, *defined in* (4), *is a convex function. If* $I_q^c f(z) = \frac{[c]_q}{z^c} \int_0^z t^{c-1} f(t) d_q t$, *then we have the following inclusion*

$$I_q^c[S_n^q(\lambda, \delta)] \subseteq S_n^q(\lambda, \delta^*),$$

where $\delta^* = 1 + (2\delta - 1 - q)[c]_q \int_0^1 \frac{t^c}{1+qt} d_q t$.

COROLLARY 3.8. *Let* $\delta \in [0, 1)$ *and* $h(z) = \frac{1+(2\delta-1)z}{1+z}$, $\delta \in [0, 1)$. *If* $If(z) = \frac{1}{z} \int_0^z f(t) dt$, *then we have the following inclusion*

$$I[S_n(\lambda, \delta)] \subseteq S_n(\lambda, \delta^*),$$

where $\delta^* = 1 - 2(1 - \delta)(1 - \ln 2)$.

4. Conclusion

In this paper, new subclasses of analytic functions and Ruscheweyh q-derivative operator were studied by virtue of an idea of Ruscheweyh operators. A new class of convex functions, $S_n^q(\lambda, \delta)$, are defined and an inclusion for the class $S_n^q(\lambda, \delta)$ is obtained. Further, some reliable results for differential subordinations of the analytic functions are investigated.

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Fixed point theorems for Jleli–Samet–Kannan contractions in metric spaces

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Abstract. In this paper, we introduce a new type of contraction mapping, called a *Jleli–Samet–Kannan contraction*, which combines the ideas of Jleli and Samet's generalization of the Banach contraction principle with Kannan's type contraction. We establish the existence and uniqueness of fixed points for such mappings in complete metric spaces. Our results generalize and unify several well-known fixed point theorems in the literature.

Keywords: Fixed point, Banach contraction, Kannan contraction, Jleli–Samet contraction, metric space.

AMS Mathematics Subject Classification [2020]: 47B49; 47B15

1. Introduction

The Banach contraction principle is a cornerstone of fixed point theory, ensuring the existence and uniqueness of fixed points for contractive self-mappings in complete metric spaces [3]. Over the years, numerous generalizations have been proposed, such as those more recently by Jleli and Samet [2].

Kannan's contraction relaxes the classical Lipschitz-type condition by involving the distances between each point and its image under the mapping. On the other hand, Jleli and Samet introduced a generalization based on an auxiliary function θ that provides a more flexible contraction framework.

In this paper, we combine both approaches and define a new hybrid contraction condition, referred to as a *Jleli–Samet–Kannan contraction*, which naturally extends and unifies the two classical principles.

First, to make the material clearer for the reader, we need to recall some introductory definitions.

*Speaker.

Definition 1.1. Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. Then T is said to be a Kannan contraction [1] if there exists $\gamma \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma \left[d(x, Tx) + d(y, Ty)\right], \quad \forall x, y \in X.$$

Within the setting of Branciari (rectangular) metric spaces, Jleli and Samet proposed a broader version of the Banach fixed-point theorem.

Definition 1.2. Let Φ denote the family of all functions $\nu : (0, \infty) \to (1, \infty)$ that satisfy the following conditions:

(θ1) θ is monotone and increasing;
(θ2) For every sequence $\{x_n\} \subset (0, \infty)$, one has

$$\lim_{n\to\infty} \theta(x_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} x_n = 0;$$

(θ3) There exist $\chi \in (0, 1)$ and $\tau \in (0, \infty]$ such that

$$\lim_{u\to 0^+} \frac{\theta(u) - 1}{u^\chi} = \tau.$$

As an extension of the Banach fixed-point theorem, Jleli and Samet [2] established the following result:

Theorem 1.3. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Suppose there exist $\theta \in \Phi$ and $\mu \in (0, 1)$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \quad \Rightarrow \quad \theta\big(d(Tx, Ty)\big) \leq \big(\theta(d(x, y))\big)^\mu. \tag{1}$$

Then, the mapping T has a unique fixed point in X.

Now we change the conditions applied to the function θ. In fact, we remove condition three and use the continuity assumption instead.

Let (X, d) be a metric space. Denote by Θ the class of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying:

(i) θ is non-decreasing,
(ii) θ is continuous,
(iii) $\theta(t_n) \to 1$ if and only if $t_n \to 0$.

The following definition will certainly be a generalization of the Kannan contraction, for which we will now examine the existence and uniqueness of a fixed point in a complete metric space.

Definition 1.4. Let (X, d) be a metric space. A self-mapping $T : X \to X$ is said to be a *Jleli–Samet–Kannan contraction* if there exist $\theta \in \Theta$ and $\gamma \in (0, 1)$ such that

$$\theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(\alpha d(x, Tx) + \beta d(y, Ty)\big)\big]^\gamma \tag{2}$$

for all $x, y \in X$, where $\alpha, \beta \geq 0$ so that $\alpha + \beta = 1$.

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ a Jleli–Samet–Kannan contraction. Then T has a unique fixed point $x^* \in X$.*

Proof. Choose an arbitrary fixed element x_0 of X and construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. So, we may assume $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Then, from (2), for any $n \in \mathbb{N}$, we have

$$\theta(d(x_{n+1}, x_{n+2})) \quad = \theta(d(Tx_n, Tx_{n+1})) \leq \theta(M(x_n, x_{n+1}))^\gamma \tag{3}$$

where

$$M(x_n, x_{n+1}) = \alpha d(x_n, Tx_n) + \beta d(x_{n+1}, Tx_{n+1}).$$

If

$$\max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\Big\} = d(x_{n+1}, x_{n+2}),$$

then

$$\theta(d(x_{n+1}, x_{n+2})) \leq (\theta(d(x_{n+1}, x_{n+2}))^\gamma < \theta(d(x_{n+1}, x_{n+2}))$$

which is a contradiction. Thus

$$\max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\Big\} = d(x_n, x_{n+1}).$$

Therefore,

$$1 \leq \theta(d(x_{n+1}, x_{n+2})) \leq (\theta(d(x_n, x_{n+1})))^\gamma \leq ... \leq (\theta(d(x_0, x_1)))^{\gamma^{n+1}} \tag{4}$$

for each $n \geq 0$. Taking limit on both sides of the above inequality, we get $\theta(d(x_{n+1}, x_{n+2})) \to 1$ as $n \to \infty$. Thus $d(x_{n+1}, x_{n+2}) \to 0$ as $n \to \infty$.

We claim that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ so that

$$n_i > m_i > i, \; d(x_{m_i}, x_{n_i}) \geq \varepsilon \tag{5}$$

and

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon.$$

From triangle inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq d(x_{m_i}, x_{n_i-1}) + d(x_{n_i-1}, x_{n_i}) < \varepsilon + d(x_{n_i-1}, x_{n_i}).$$

As $i \to \infty$, we find

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) = \varepsilon.$$

Also, we have

$$\begin{aligned} &d(x_{m_i}, x_{n_i}) - d(x_{m_i}, x_{m_i+1}) - d(x_{n_i}, x_{n_i+1}) \\ &\leq d(x_{m_i+1}, x_{n_i+1}) \\ &\leq d(x_{m_i}, x_{m_i+1}) + d(x_{m_i}, x_{n_i}) + d(x_{n_i}, x_{n_i+1}). \end{aligned}$$

As $i \to \infty$, we find

$$\limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i+1}) = \varepsilon.$$

By (2), we find

$$\theta(d(x_{m_i+1}, x_{n_i+1})) \leq (\theta(\alpha d(x_{m_i}, fx_{m_i}) + \beta d(x_{n_i}, fx_{n_i})))^\gamma. \tag{6}$$

Taking limit in both sides of (6), we obtain that

$$\theta(\limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i+1})) \leq 1,$$

a contradiction.

Thus, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), hence there is $x^* \in X$ so that

$$\lim_{n\longrightarrow\infty} x_n = x^*. \tag{7}$$

Finally, we claim that $d(x^*, Tx^*) = 0$. Suppose to the contrary, $d(x^*, Tx^*) > 0$. We have:

$$\theta(d(x^*, Tx^*)) = \lim_{n\to\infty} \theta(d(x_{n+1}, Tx^*)) = \lim_{n\to\infty} \theta(d(Tx_n, Tx^*)) \leq \lim_{n\to\infty} (\theta(M(x_n, x^*)))^\gamma. \tag{8}$$

Also

$$\lim_{n\longrightarrow\infty} M(x_{n_k}, x^*) = \lim_{n\longrightarrow\infty} \Big[\alpha d(x_n, x_{n+1}) + \beta d(x^*, Tx^*)\Big] < \beta d(x^*, Tx^*). \tag{9}$$

Now, (8) and (9) conclude that $\theta(d(x^*, Tx^*)) \leq (\theta(d(x^*, Tx^*)))^\gamma \leq \theta(d(x^*, Tx^*))$, which is a contradiction. Thus $d(x^*, Tx^*) = 0$ and so $x^* = Tx^*$. To show the uniqueness of the fixed point, suppose that $\hbar, \hbar'$ are two distinct fixed points of T. Using (2), we have

$$\begin{aligned}\theta(d(\hbar, \hbar')) &= \theta(d(T\hbar, T\hbar')) \\ &\leq (\theta(d(\hbar, \hbar')))^\gamma < \theta(d(\hbar, \hbar')),\end{aligned}$$

a contradiction. Thus, $\hbar = \hbar'$ and the proof is complete. □

3. Conclusion

We have presented a new generalization of the Kannan contraction principle based on the Jleli–Samet approach. The introduced concept unifies both classical results and opens further directions for fixed point theory in various generalized metric structures.

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On $\mathcal{MT}$-cyclic Ćirić quasicontractions

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Abstract. Following our previous work [**2**] on the existence and uniqueness of best proximity points for $\mathcal{MT}$-cyclic Ćirić *quasicontractions* of type I and type II, we now introduce the concepts of $\mathcal{MT}$-cyclic Ćirić *quasicontractions* of type III and type IV. Using these new notions, we investigate the existence of best proximity points for such mappings in metric spaces. The uniqueness of these points will be established by imposing an additional condition known as "property UC."

Keywords: Best proximity point, property UC, $\mathcal{MT}$-function, $\mathcal{MT}$-cyclic Ćirić quasicontractions.

AMS Mathematics Subject Classification [2020]: 47H10; 54H25

1. Introduction and Preliminaries

One of the generalizations of the well-known Banach theorem from 1922 is the best proximity point theory. The importance of fixed point theory emerges from the fact that it furnishes a unified approach and constitutes an important tool in solving equations which are not necessarily linear. A large number of problems can be formulated as nonlinear equations of the form $T(x) = x$, where T is a self-mapping in some framework. Nevertheless, an equation of the type $T(x) = x$ does not necessarily possess a solution if T happens to be a non-self-mapping. In this case, one seeks an appropriate solution that is optimal in the sense that $d(x, T(x))$ is minimum. That is, we resolve a problem of finding an element x such that x is in best proximity to $T(x)$ in some sense. A best proximity point problem is a problem of achieving the minimum distance between two sets through a function defined on one of the sets to the other. In the last decades, best proximity point theory have been appreciated by several authors, see e.g. [**5**] and [**6**].

Let A and B be nonempty subsets of metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a *cyclic* if $T(A) \subset B$ and $T(B) \subset A$. For any nonempty subsets A and B of X, we let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A \cup B$ is called to be a best proximity point for T if $d(x, Tx) = dist(A, B)$. Note that if $A = B$ then the best proximity point of T turns into fixed point of T.

*Speaker.

Definition 1.1. [**5**] Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a cyclic contraction if the following conditions hold:

(1) $T(A) \subset B$ and $T(B) \subset A$;
(2) There exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$ for all $x \in A, y \in B$.

Remark 1.1. Let A and B be nonempty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic contraction. If $A \cap B \neq \emptyset$, then $dist(A, B) = 0$ and T is a contraction on the complete metric space $(A \cap B, d)$. Hence, applying the Banach contraction principle, we know that T has a unique fixed point in $A \cap B$.

In [**5**], Eldred and Veeramani first proved the following interesting best proximity point theorem.

Theorem EV. [**[5], Proposition 3.2**] Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.

The concept of *property UC* was introduced by Suzuki *et al.* [**6**] as follows:
A pair (A, B) is said to satisfy the property UC if the following holds:

(UC) If $\{(x_n)_{n=1}^{\infty}\}$ and $\{(x'_n)_{n=1}^{\infty}\}$ are sequences in A and $\{y_n\}_{n=1}^{\infty}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = dist(A, B)$ and $\lim_{n\to\infty} d(x'_n, y_n) = dist(A, B)$ then $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

The concept of $\mathcal{MT}$-*function* was introduced by Du in [**3**].

Definition 1.1. [**3**] A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-*function* if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$).

Remark 1.2. [**3**] It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example 1.3. [**3**] Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t} & \text{, if } t \in (0, \frac{\pi}{2}] \\ 0 & \text{, otherwise.} \end{cases}$$

Since $\limsup_{s\to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

For convenience of the reader, we recall some of $\mathcal{MT}$-cyclic contractions in the framework of complete metric spaces which are defined in some papers; in these contractions the authors obtained the existence of best proximity point using by the axillary function $\mathcal{MT}$-function φ for such mappings in metric spaces. The uniqueness of this point will be obtain by imposing an additional condition, so called "property UC":
For mapping $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$; T is called
[**4**] [$\mathcal{MT}$-*cyclic contraction*] if

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))dist(A, B);$$

[$\mathcal{MT}$-*cyclic Kannan contraction*] if

$$d(Tx,Ty) \leq \frac{1}{2}\varphi(d(x,y))\Big(d(x,Tx)+d(y,Ty)\Big)+(1-\varphi(d(x,y)))dist(A,B);$$

[$\mathcal{MT}$-*cyclic Reich contraction*] if

$$d(Tx,Ty) \leq \frac{1}{3}\varphi(d(x,y))\Big(d(x,y)+d(x,Tx)+d(y,Ty)\Big)+(1-\varphi(d(x,y)))dist(A,B);$$

[**1**][*generalized* $\mathcal{MT}$-*cyclic contraction*] if

$$d(Tx,Ty) \leq \varphi(d(x,y))\max\{d(x,y),d(x,Tx),d(y,Ty)\}+(1-\varphi(d(x,y)))dist(A,B);$$

$Hardy-Rogers$ $\mathcal{MT}$-*cyclic contraction* if

$$d(Tx,Ty) \leq \frac{\varphi(d(x,y))}{5}\Big(d(x,y)+d(x,Tx)+d(Ty,y)+d(x,Ty)+d(Tx,y)\Big)+\big(1-\varphi(d(x,y))\big)dist(A,B),;$$

[**2**]$\mathcal{MT}$-*cyclic* Ćirić *quasicontraction* of type I if

$$d(Tx,Ty) \leq \varphi(d(x,y))M(x,y)+(1-\varphi(d(x,y)))dist(A,B);$$

and

[**2**]$\mathcal{MT}$-*cyclic* Ćirić *quasicontraction* of type II if

$$d(Tx,Ty) \leq \varphi(d(x,y))m(x,y)+(1-\varphi(d(x,y)))dist(A,B);$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(Tx,y)}{2}\};$$

and

$$m(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(Tx,y)\}.$$

It is showed there exists an example give a map T which is a $\mathcal{MT}$-cyclic contraction but not a cyclic contraction; see Example A in [**4**].

The following theorem was proved by Ćirić.

THEOREM 1.4. *Let (X,d) be a complete metric space. Let $T:X\to X$ be a Ćirić quasicontraction mapping; i.e., there exists a constant $\lambda\in[0,1)$ such that*

$$d(Tx,Ty) \leq \lambda M_{III}(x,y), \tag{1}$$

or

$$d(Tx,Ty) \leq \lambda M_{IV}(x,y), \tag{2}$$

for all $x,y\in X$; where

$$M_{III}(x,y)=\max\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(Tx,y)}{2}\};$$

and

$$M_{IV}(x,y)=\max\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},d(x,Ty),d(Tx,y)\}.$$

Then T has a unique fixed point.

The aim of this paper is generalization of by Ćirić's Theorems of type III or IV by applying the axillary function $\mathcal{MT}$-function φ.

2. Best Proximity point for $\mathcal{MT}$-cyclic Ćirić quasicontraction of type III and type IV

In this section, we present our main results. We, first, introduce the *generalized $\mathcal{MT}$-cyclic contraction* with respect to auxiliary $\mathcal{MT}$-function φ.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). For mapping $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$; T is called *$\mathcal{MT}$-cyclic* Ćirić *quasicontraction* of type III if

there exists a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y))M_{III}(x, y) + (1 - \varphi(d(x, y)))dist(A, B);$$

and it is called *$\mathcal{MT}$-cyclic* Ćirić *quasicontraction* of type IV if

$$d(Tx, Ty) \leq \varphi(d(x, y))M_{IV}(x, y) + (1 - \varphi(d(x, y)))dist(A, B).$$

In what follows that we establish the following theorem for best proximity point which is one of the main results in this paper.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X, d) and (A, B) satisfies the property UC. Let $T : A \cup B \to A \cup B$ be a cyclic map and let φ be a $\mathcal{MT}$-function. Suppose that A is complete and T is a Ćirić quasicontraction of type III or type IV with respect to φ. Then the following hold:*

(i) *T has a best proximity point z in A.*
(ii) *z is a unique fixed point of T^2 in A.*
(iii) *$\{T^{2n}x\}$ converges to z for every $x \in A$.*
(iv) *T has at least one best proximity point in B.*
(v) *If (B, A) satisfies the property UC, then Tz is unique best proximity point in B and $\{T^{2n}y\}$ converges to Tz for every $y \in B$.*

3. Conclusion

In this paper, we have introduced the concepts of $\mathcal{MT}$-cyclic Ćirić *quasicontractions* of type III and type IV. Using these new notions, we investigated the existence of best proximity points for such mappings in metric spaces. The uniqueness of these points established by imposing an additional condition known as "property UC."

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An application of fixed point theorems to dynamic programming in a generalized distance space

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Abstract. This study introduces a novel concept of a three-variable distance space, which extends several existing metric structures, including S-metric space, S_b-metric, S_θ-metric space, and S_p-metric space. Various topological properties of these spaces are examined. A fixed point theorem for generalized weak contraction mappings in this framework is established, accompanied by several related corollaries. To validate the space and the derived fixed-point theorem, illustrative examples are provided. Finally, the fixed point theorem is applied to analyze the solution of a functional equation arising in dynamic programming.

Keywords: Fixed point, S-metric space, S_Δ-metric space, Weak contraction mapping, Dynamic programming.

AMS Mathematics Subject Classification [2020]: 47H10, 54H25.

1. Introduction

S-metric spaces were proposed as a broader framework of standard metric spaces, allowing the notion of distance to depend on three points instead of the usual two. Historically, several three-variable metric structures have been studied, including 2-metric spaces, D-metric spaces, G-metric spaces, and D^*-metric spaces [4, 5, 10]. Among these, the S-metric provides a natural extension that relates a measurement involving three points to the sum of pairwise distances.

S-metric spaces have been further generalized into S_b-metric spaces, S_θ-metric spaces and S_p-metric spaces which relax the symmetry requirement. These spaces retain many properties of metric spaces, allowing fixed point results to be extended from classical metrics to the S-metric framework.

*Speaker.

2. S_Δ-metric space

Definition 2.1. Let $\aleph$ be assumed to be a nonempty set. and

$$S_\Delta : \aleph \times \aleph \times \aleph \to \mathbb{R}_+$$

be a mapping. Then S_Δ is said to be a S_Δ-*metric* if there exists a function $\Delta : \aleph \times \aleph \times \aleph \to \mathbb{R}_+$ such that for all $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'', a \in \aleph$, the following conditions hold:

($S_\Delta 1$) $0 \leq \Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'') \leq S_\Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'')$;
($S_\Delta 2$) $S_\Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'') = \Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'')$ if and only if $\mathfrak{q} = \mathfrak{q}' = \mathfrak{q}''$;
($S_\Delta 3$) $S_\Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'') \leq S_\Delta(\mathfrak{q}, \mathfrak{q}, a) + S_\Delta(\mathfrak{q}', \mathfrak{q}', a) + S_\Delta(\mathfrak{q}'', \mathfrak{q}'', a) + \Delta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'')$.

The pair $(\aleph, S_\Delta)$ is called a S_Δ-*m.s.*.

3. Fixed point results in Extended S_b-metric spaces

Mlaiki, in [9], introduced the concept of extended S_b-metric spaces as a broader framework of S_b-metric spaces and proved several fixed point theorems accompanied by notable applications.

Theorem 3.1. *Let $(\aleph, S_\theta)$ be a complete extended S_b-metric space such that S_θ is continuous, and let $\digamma : \aleph \to \aleph$ be a self-mapping satisfying*

$$S_\theta(\digamma\mathfrak{q}, \digamma\mathfrak{q}', \digamma\mathfrak{q}'') \leq k\, S_\theta(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'') \quad \text{for all } \mathfrak{q}, \mathfrak{q}', \mathfrak{q}'' \in \aleph,$$

where $0 \leq k < \frac{1}{2}$. Moreover, assume that for every $\mathfrak{q} \in \aleph$,

$$\lim_{n,m\to\infty} \theta(\digamma^n\mathfrak{q}, \digamma^n\mathfrak{q}, \digamma^m\mathfrak{q}) < \frac{1}{2k}. \tag{1}$$

It follows that $\digamma$ has a single fixed point in $\aleph$.

4. Fixed point results in S_b-metric spaces

The following results in S_b-metric spaces follow easily from the fixed point theorems proved in $\Delta - S$-metric spaces.

Corollary 4.1 (Banach-type contraction principle). *Let $(\aleph, S_b)$ be a complete S_b-metric space with parameter $s \geq 1$. Suppose that $\digamma : \aleph \to \aleph$ is a self-mapping such that*

$$S_b(\digamma\mathfrak{q}, \digamma\mathfrak{q}', \digamma\mathfrak{q}'') \leq \frac{a}{s}\, S_b(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}''), \tag{2}$$

for all $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'' \in \aleph$ and some constant $a \in [0, 1)$. Then $\digamma$ admits a unique fixed point in $\aleph$.

5. Fixed point results in S_p-metric spaces

Corollary 5.1 (Banach contraction type mappings). *Let $(\aleph, S_\Omega)$ be a complete S_p-metric space, and let $\digamma : \aleph \to \aleph$ be a mapping such that*

$$S_p(\digamma\mathfrak{q}, \digamma\mathfrak{q}', \digamma\mathfrak{q}'') \leq a\, \Omega^{-1}\big[S_p(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'')\big] \tag{3}$$

for all $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'' \in \aleph$ and some $a \in [0, 1)$. Then $\digamma$ admits a unique fixed point in $\aleph$.

Corollary 5.2 (Banach contraction type mappings). *Let $(\aleph, S_\Omega)$ be a complete S_p-metric space and $\digamma : \aleph \to \aleph$ satisfy*

$$\Omega^2\big[2S_p(\digamma\mathfrak{q}, \digamma\mathfrak{q}', \digamma\mathfrak{q}'')\big] \leq a\, S_p(\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'') \tag{4}$$

for all $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}'' \in \aleph$ and some $a \in [0, 1)$. Hence, $\digamma$ has exactly one fixed point in $\aleph$.

6. An application to dynamic programming

The pioneering work of Bellman [3] focused on the existence and iterative methods for approximating solutions of certain functional equations in dynamic programming. One representative equation among them is

$$\mathcal{F}(\mathfrak{q}) = \sup_{\mathfrak{q}'\in\Sigma}\Big\{\mathcal{P}(\mathfrak{q},\mathfrak{q}') + \mathcal{G}\big(\mathfrak{q},\mathfrak{q}',\mathcal{F}(\mathcal{Q}(\mathfrak{q},\mathfrak{q}'))\big)\Big\}, \tag{5}$$

where $\mathfrak{q} \in \Omega \subset \aleph$ represents the state space (namely, the set of initial states, admissible actions, and transitions of the process), while $\Sigma \subset \aleph'$ denotes the decision space (the collection of all possible actions). Here, $\mathfrak{q}$ and $\mathfrak{q}'$ stand for the state and decision vectors, respectively, and $\aleph$, $\aleph'$ are Banach spaces over $\mathbb{R}$.

The function $\mathcal{F}(\mathfrak{q})$ corresponds to the optimal return function associated with the initial state $\mathfrak{q}$. Moreover, the mappings are given as $\mathcal{P} : \Omega \times \Sigma \to \mathbb{R}$, $\mathcal{Q} : \Omega \times \Sigma \to \Omega$, and $\mathcal{G} : \Omega \times \Sigma \times \mathbb{R} \to \mathbb{R}$. Subsequently, numerous authors have examined the properties of solutions to such functional equations in dynamic programming and addressed them using different fixed point techniques (see, for instance, [8], [12], [16]).

Let $B_d(\Omega)$ denote the space of all bounded real-valued functions defined on Ω. For $\nu \in B_d(\Omega)$, introduce the norm $\|\nu\| = \sup_{\mathfrak{q}\in\Omega} |\nu(\mathfrak{q})|$. With this norm, the pair $(B_d(\Omega), \|\cdot\|)$ forms a Banach space.

We now define $S_\Delta : B_d(\Omega) \times B_d(\Omega) \times B_d(\Omega) \to [0,\infty)$ by

$$S_\Delta(\rho,\tau,\sigma) = \max\{\|\rho-\sigma\|^2, \|\tau-\sigma\|^2\}, \quad \forall\, \rho,\tau,\sigma \in B_d(\Omega).$$

This mapping S_Δ provides a S_Δ-metric on $B_d(\Omega)$, where

$$\Delta(\rho,\tau,\sigma) = \tfrac{1}{2}\max\{\|\rho-\sigma\|^2, \|\tau-\sigma\|^2\}, \quad \forall\, \rho,\tau,\sigma \in B_d(\Omega).$$

We now turn to the question of the existence of a solution to the functional equation (5), making use of the fixed point theorem obtained earlier. To this end, define the operator $\Psi : B_d(\Omega) \to B_d(\Omega)$ as

$$(\Psi\mathcal{F})(\mathfrak{q}) = \sup_{\mathfrak{q}'\in\Sigma}\Big\{\mathcal{P}(\mathfrak{q},\mathfrak{q}') + \mathcal{G}\big(\mathfrak{q},\mathfrak{q}',\mathcal{F}(\mathcal{Q}(\mathfrak{q},\mathfrak{q}'))\big)\Big\}, \tag{6}$$

for each $\mathcal{F} \in B_d(\Omega)$. It is immediate that whenever $\mathcal{P}$ and $\mathcal{G}$ are bounded functions, the operator Ψ is well-defined.

Theorem 6.1. *Consider the mapping $\Psi : B_d(\Omega) \to B_d(\Omega)$ introduced in (6). Suppose the following assumptions are satisfied.*

$(D1)$ $\mathcal{P} : \Omega \times \Sigma \to \mathbb{R}$ and $\mathcal{G} : \Omega \times \Sigma \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions.

$(D2)$ For any $\theta, \phi \in B_d(\Omega)$, the function $\mathcal{G}$ fulfills

$$\begin{aligned} &\big|\mathcal{G}(\mathfrak{q},\mathfrak{q}',\theta(\mathfrak{q}'')) - \mathcal{G}(\mathfrak{q},\mathfrak{q}',\phi(\mathfrak{q}''))\big|^2 \\ &\le \frac{\beta}{2}\max\left\{\sup_{\mathfrak{q}\in\Sigma}|\theta(\mathfrak{q})-\phi(\mathfrak{q})|^2, \sup_{\mathfrak{q}\in\Sigma}|\theta(\mathfrak{q})-\Psi(\theta)(\mathfrak{q})|^2, \sup_{\mathfrak{q}\in\Sigma}|\phi(\mathfrak{q})-\Psi(\phi)(\mathfrak{q})|^2\right\}, \end{aligned} \tag{7}$$

for all $(\mathfrak{q},\mathfrak{q}',\theta(\mathfrak{q}''))$ and $(\mathfrak{q},\mathfrak{q}',\phi(\mathfrak{q}'')) \in \Omega \times \Sigma \times \mathbb{R}$, where $\beta \in [0,1)$ and $L \ge 0$.

Then the functional equation (5) admits a unique bounded solution.

7. Conclusions

The family of all S-metric spaces, S_b metric spaces, $S-p$-metric spaces and S_θ metric spaces form proper subsets within the class of S_Δ-m.s.s. Because the triangle inequality in S_Δ-metric relies on an auxiliary function Δ, the topology τ_{S_Δ} introduced here is rather uncommon. Consequently, proving fixed point results in the framework of S_Δ-m.s.s becomes difficult. Because of the presence of control functions in the definition of S_Δ-metric structures, especially in the definition of S_θ metric spaces, the proof of fixed point theorems in these structures requires many conditions on the control functions. The idea of proving theorems in S_Δ-m.s.s and then obtaining similar results in S_θ metric spaces leads to results with fewer and weaker conditions. These conditions will be of great help in finding illustrative examples and applications.

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Conditional Expectation Operators and Kato Property Between L^p-spaces

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Abstract. This paper investigates the relationship between Lambert multipliers, defined via conditional expectation operators, and Kato operators acting between two L^p-spaces. The results establish new connections that clarify the structural and boundedness properties of Kato operators.

Keywords: conditional expectation, Lambert multipliers, Kato operators, composition operators.

AMS Mathematics Subject Classification [2020]: 58E11, 53B30, 53C50.

1. Introduction

Let (X, Σ, μ) be a complete σ-finite measure space and $\varphi : X \longrightarrow X$ be a measurable transformation with the following properties:

- the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ,
- $\mu \circ \varphi^{-1}$ is finite.

For convenience we set $h := \frac{d\mu\circ\varphi^{-1}}{d\mu}$, the Radon-Nikodym derivation of the measure $\mu \circ \varphi^{-1}$ with respect to μ. By $\varphi^{-1}(\Sigma)$ we mean the relative completion of the σ-algebra generated by $\{\varphi^{-1}(F) : F \in \Sigma\}$.

For any complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$ where $\mu_{|\mathcal{A}}$ is the restriction of μ to $\mathcal{A}$. Also its norm is denoted by $\|.\|_p$ on which $L^p(\mathcal{A})$ is a Banach subspace of $L^p(\Sigma)$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. (Similarly, $L^0(\mathcal{A})$ denotes the linear space of all complex-valued $\mathcal{A}$-measurable functions on X). The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Equalities and inequalities between measurable functions and also equality between sets can be interpreted as the almost everywhere sense, and the set of measure zero, respectively. For each nonnegative $f \in L^0(\Sigma)$ consider a measure

$$\rho_f(F) = \int_F f d\mu_{|\mathcal{A}}, \qquad \forall F \in \mathcal{A}.$$

*Speaker.

Since the measure ρ_f is absolutely continuous with respect to $\mu_{\mathcal{A}}$, then by the Radon-Nikodym theorem there exists a unique nonnegative function $E(f)$ with the following conditions:

(i) $E(f)$ is $\mathcal{A}$-measurable and integrable,

(ii) If F is any $\mathcal{A}$-measurable set for which $\int_F f d\mu$ exists, we have the functional relation

$$\int_F f d\mu = \int_F E(f) d\mu.$$

We should write something like a $E_\mu^{\mathcal{A}}(f)$ to reflect the dependence of $E(f)$ to the choice of subalgebra and measure.

Now associated with every complete σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$, the mapping $E : L^p(\Sigma) \longrightarrow L^p(\mathcal{A})$, $1 \leq p \leq \infty$, uniquely defined by the assignment $f \longmapsto E(f)$, is called the *conditional expectation operator with respect to* $\mathcal{A}$.

The mapping E is a linear operator and, in particular, it is a contraction map. In case $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. The role of this operator is important in this note and we list here some of its useful properties:

- If f is an $\mathcal{A}$-measurable function, then $E(fg) = fE(g)$.
- $|E(f)|^p \leq E(|f|^p)$.
- $\|E(f)\|_p \leq \|f\|_p$.
- If $f \geq 0$ then $E(f) \geq 0$; if $f > 0$ then $E(f) > 0$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^0(\mathcal{A})$.

A real-valued $f \in L^0(\Sigma)$ is said to be *conditionable* with respect to $\mathcal{A}$ if the measure of the set $\{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$ is zero, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{0, -f\}$ are nonnegative $\Sigma-$measurable functions.

In the next section, the Lambert composition multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator.

The algebra of all bounded linear operators from Banach space X into a Banach space Y is denoted by $\mathcal{B}(X, Y)$. Let $T \in \mathcal{B}(X, Y)$, the injectivity and surjectivity modulus of T are defined by

$$j(T) := \inf\{\|T(x)\| : x \in X, \|x\| = 1\}$$

and

$$\kappa(T) := \sup\{r \geq 0 : TU_X \supset r \cdot U_Y\},$$

respectively, where U_X and U_Y denotes the closed unit balls in X and Y.

It is said to be that T is *bounded below* if $j(T) > 0$.

By $\mathrm{R}^\infty(T)$ and $\mathrm{N}^\infty(T)$, we mean the linear subspaces of X, $\bigcap_{n=1}^\infty RanT^n$ and $\bigcup_{n=1}^\infty KerT^n$, respectively for $T \in \mathcal{B}(X)$.

2. Main results

In this section we bring some facts and definitions, which will be used later.

Definition 2.1. Let $T_u : L^2(\Sigma) \longrightarrow L^2(\Sigma)$. Define

$$W := \left\{u \in L^0(\Sigma) : T_u \textit{ is bounded on } L^2(\Sigma)\right\}.$$

We already know one important property of function in W, namely that $E(|u|^2)$ is bounded. However, Since

$$|E(u)|^2 \leq E|u|^2,$$

we see that $u \in W$ implies that $E(u)$ is bounded. Therefore, if a function is both $\mathcal{A}-$measurable and in W, then it must be bounded. Our next Lemma states that the converse also holds.

Lemma 2.2. $W \bigcap L^0(\mathcal{A}) = L^\infty(\mathcal{A})$.

Proof. Let $s \in L^\infty(\mathcal{A})$. Since s is $\mathcal{A}-$measurable, then $T_s f = sf$ for $f \in L^2(\Sigma)$. Also, we know $L^2(\mathcal{A}) \subset L^2(\Sigma)$, thus we get

$$\|T_s f\|_2^2 = \int_X |T_s f|^2 d\mu = \int_X |sf|^2 d\mu \leq \|s\|_\infty^2 \int_X |f|^2 d\mu = \|s\|_\infty^2 \|f\|_2^\infty$$

so that $T_s f \in L^2(\sigma)$ for all f. Hence, $W \bigcap L^0(\mathcal{A})$. The converse we proved in the remarks leading up to the lemma. □

Theorem 2.3. *Suppose $T_u : L^p(\Sigma) \to L^p(\Sigma)$ is bounded linear operator. Then T_u is bounded below linear operator if and only if $|E(u)| \geq \delta$ a.e on $\sigma(E(u))$ for some $\delta > 0$.*

By the following theorem, a bounded operator $T_u : L^p(\Sigma) \to L^p(\Sigma)$ is one-to-one and has closed range if and only if $|E(u)| \geq \delta$ on $\sigma(E(u))$ for some $\delta > 0$.

Theorem 2.4. *An operator $T \in \mathcal{B}(X, Y)$ is bounded below if and only if it is one-to-one and has closed range. T is onto if and only if $\kappa(T) > 0$.*

Proof. See [1]. □

Theorem 2.5. *Let $T \in \mathcal{B}(X, Y)$. Then $j(T) = \kappa(T^*)$ and $\kappa(T) = j(T^*)$.*

Proof. See [1]. □

Definition 2.6. Let $u \in L^0(\Sigma)$ be conditionable. For $1 \leq p \leq \infty$, the mapping $R_u : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $R_u f = E(uf)$ is called a *weighted conditional expectation operator with respect to* $\mathcal{A}$(or WCE operator), provided that for every $f \in L^p(\Sigma), uf$ is conditionable and $E(uf) \in L^p(\mathcal{A})$.

Theorem 2.7. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $R_u : L^p(\Sigma) \to L^p(\mathcal{A})$ is a bounded WCE operator, then $E(|u|^q) \in L^\infty(\mathcal{A})$ and $\|R_u\| = \|E(|u|^q)\|_\infty^{\frac{1}{q}}$.*

Proof. See [2, Theorem 2.1]. □

Theorem 2.8. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $T_u : L^p(\Sigma) \to L^p(\Sigma)$ and R_u are bounded linear operators. If $|E(u)| \geq \delta$ on $\sigma(E(u))$ for some $\delta > 0$, then for every $n \in \mathbb{N}$, $T_u^{*^{(n)}}$ is one-to-one and has closed range.*

Corollary 2.9. *T_u^n is onto if and only if $|E(u)| \geq \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$.*

Proof. By Theorems [2.5] and [2.4], it is trivial. □

Theorem 2.10. *For $T \in \mathcal{B}(X)$ with closed range, the following conditions are equivalent:*

(i) *$Ker(T) \subset R^\infty(T)$;*

(ii) $N^{\infty}(T) \subset Ran(T)$;
(iii) $N^{\infty}(T) \subset R^{\infty}(T)$;
(iv) $Ker(T) \subset \overline{R^{\infty}(T)}$.

Proof. See [1]. □

Lemma 2.11. *Let $1 < p < \infty$ and $T_u : L^p(\Sigma) \to L^p(\Sigma)$ be a bounded linear operator. Then $Ker(T_u) = L^p(X \setminus \sigma(E(u))$.*

Definition 2.12. Let $T \in \mathcal{B}(X)$. T is said to be Kato if $Ran(T)$ is closed and T satisfies any of the conditions of Theorem [2.10].

Theorem 2.13. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For bounded linear operators, R_u and $T_u : L^p(\Sigma) \to L^p(\Sigma)$, T_u is Kato if $|E(u)| \geq \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$.*

Proof. By Corollary [2.9], $R^{\infty}(T_u^n) = L^p(X, \Sigma, \mu)$ and by Lemma [2.11], $Ker(T_u) = L^p(X \setminus \sigma(E(u))$. Hence by (i) in Theorem [2.10], T_u is Kato.

□

Let $1 \leq p, q \leq \infty$ and $u \in L^0(\Sigma)$. We define Lambert weighted composition operator $K_u^{\varphi} : L^p(\Sigma) \longrightarrow L^q(\Sigma)$, as :

$$K_u^{\varphi} f := uE(f \circ \varphi) + (f \circ \varphi)E(u) - E(u)E(f \circ \varphi).$$

Theorem 2.14. *Let $1 \leq p < \infty$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ and $u \in L^0(\Sigma)$ and $h \in L^0(\mathcal{A})$. Suppose $E^{\mathcal{A}} E^{\varphi^{-1}\Sigma}$ is an orthogonal projection. Then K_u^{φ} is normal if and only if*

(i) $h\Big\{E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(|u^2|) + E^{\varphi^{-1}\Sigma}(uE^{\mathcal{A}}(\overline{u}) - E^{\mathcal{A}}(\overline{u})E^{\mathcal{A}}(u))\Big\} \circ \varphi^{-1} = uh \circ \varphi E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})$, *and*
(ii) $\mathcal{A} \cap \varphi^{-1}\Sigma \cap \sigma(u) = \Sigma \cap \sigma(u)$.

3. Conclusion

By framing Lambert multipliers through conditional expectation operators, new structural insights into Kato operators emerge. The established connections highlight how boundedness and operator behavior can be characterized more precisely within L^p-spaces. These findings contribute to the broader theory of operator analysis by unifying two previously distinct frameworks. Moreover, the results open potential avenues for extending Kato operator theory to more general functional settings. Such developments may influence future research in harmonic analysis and operator semigroups.

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Weak fixed point property for Kannan type mappings with respect to orbits in Banach spaces

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Abstract. This study focuses on extending the established theory of nonexpansive mappings. Specifically, we introduce and formally define the class of Kannan type mappings relative to orbits within the setting of a general Banach space. Following this definition, we establish a fundamental characterization: we prove that a Banach space $\mathcal{A}$ is endowed with weak normal structure if and only if it guarantees the existence of a weakly fixed point for every mapping belonging to this newly introduced class of Kannan type mappings with respect to orbits.

Keywords: Fixed point property, normal structure, Kannan type mappings.

AMS Mathematics Subject Classification [2020]: 46A40, 46B05, 46B20, 47H10.

1. Introduction

The study of fixed point theory for nonexpansive mappings is a cornerstone of nonlinear functional analysis. Given a nonempty subset $\mathcal{B}$ of a Banach space $\mathcal{A}$, and a mapping $\mathcal{F} : \mathcal{B} \to \mathcal{A}$, $\mathcal{F}$ is classically termed nonexpansive if for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$,

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leq \| \mathfrak{a} - \mathfrak{b} \| .$$

This area has been extensively investigated by numerous authors (see [2–4] and the references therein), yielding fundamental results on the existence of fixed points.

We denote the orbit of a point $\mathfrak{a} \in \mathcal{B}$ under a self-map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by $O_{\mathcal{F}}(\mathfrak{a})$, which is the set of all iterates $\{\mathcal{F}^n\mathfrak{a} : n = 0, 1, \ldots\}$, where $\mathcal{F}^0\mathfrak{a} = \mathfrak{a}$. The notation $r_{\mathfrak{a}}(\mathcal{C})$ is used for the radius of a nonempty subset $\mathcal{C} \subset \mathcal{A}$ with respect to $\mathfrak{a} \in \mathcal{A}$, defined as $r_{\mathfrak{a}}(\mathcal{C}) = \sup_{\mathfrak{b}\in\mathcal{C}} \| \mathfrak{a} - \mathfrak{b} \|$.

Motivated by the classical theory, Amini et al. [1] introduced the notion of nonexpansiveness with respect to orbits:

Definition 1.1 (Amini et al. [1]). Let $\mathcal{B}$ be a nonempty subset of a Banach space $\mathcal{A}$. A self-mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is nonexpansive with respect to (wrt) orbits if for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$,

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leq r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big).$$

A Banach space $\mathcal{A}$ is said to possess the weak fixed point property for nonexpansive mappings wrt orbits if every such mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ admits a fixed point, whenever $\mathcal{B}$ is an arbitrary weakly compact, convex subset of $\mathcal{A}$. Amini et al. [1] established a significant link between this property and the geometric structure of the space:

To contextualize this result, we recall that a point $\mathfrak{a} \in \mathcal{C}$ is nondiametral if $r_{\mathfrak{a}}(\mathcal{C}) < \operatorname{diam}(\mathcal{C})$. A convex subset $\mathcal{B}$ of $\mathcal{A}$ is said to have normal structure if every convex, bounded subset $\mathcal{C} \subset \mathcal{B}$ with $\operatorname{diam}(\mathcal{C}) > 0$ contains a nondiametral point. The space $\mathcal{A}$ itself has weak normal structure if every weakly compact, convex subset $\mathcal{B}$ of $\mathcal{A}$ with $\operatorname{diam}(\mathcal{B}) > 0$ contains a nondiametral point.

THEOREM 1.2 (Amini et al. [1]). *A Banach space $\mathcal{A}$ has weak normal structure if and only if every nonexpansive mapping wrt orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, where $\mathcal{B}$ is an arbitrary weakly compact, convex subset of $\mathcal{A}$.*

It is also pertinent to mention that a Banach space $\mathcal{A}$ has the fixed point property (fpp) if every standard nonexpansive mapping $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ has a fixed point, where $\mathcal{C}$ is a nonempty weakly compact, convex subset of $\mathcal{A}$. Kirk [6] famously proved in 1965 that weak normal structure implies the w-fpp. Notably, the spaces c_0 and the specially-normed l_2 space, $(l_2, \| a \|_{\sqrt{2}} = \max\{\| a \|_{l_2}, \sqrt{2} \| a \|_{\infty}\})$, possess the fpp but lack weak normal structure [5].

Inspired by the Kannan type contraction definition, we present the new concept:

DEFINITION 1.3. Let $\mathcal{B}$ be a nonempty subset of a Banach space $\mathcal{A}$. A self-mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is said to be Kannan type mapping with respect to orbits if for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$,

$$\| \mathcal{F}\mathfrak{a} - \mathcal{F}\mathfrak{b} \| \leq \alpha\Big(r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big) + r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big)\Big). \tag{1}$$

where $\alpha \in (0, \frac{1}{2})$.

It is clear that any nonexpansive mapping wrt orbits is a Kannan type mapping wrt orbits. Furthermore, every standard nonexpansive mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ satisfies the condition (1). However, the class of Kannan type mapping mappings wrt orbits strictly contains the nonexpansive class, as illustrated by the following example, which is similar to a construction found in [7, Example 5.2].

EXAMPLE 1.4. Consider the mapping $\mathcal{F} : [0, 1] \to [0, 1]$ defined by

$$\mathcal{F}(\mathfrak{a}) = \begin{cases} \frac{\mathfrak{a}}{2}, & \mathfrak{a} \geq \frac{1}{2} \\ \frac{\mathfrak{a}}{4}, & \mathfrak{a} < \frac{1}{2}. \end{cases}$$

Since $\mathcal{F}$ is discontinuous at $\mathfrak{a} = 1/2$, it cannot be a nonexpansive map. We assert that $\mathcal{F}$ is, nonetheless, a Kannan type mappinge mapping wrt orbits.

2. Main results

Following the foundational work of Amini et al. in [1], we now formally define the concept of the weak fixed point property for Kannan type mapping mappings with respect to their orbits in a Banach space setting.

DEFINITION 2.1. We say that a Banach space $\mathcal{A}$ possesses the weak fixed point property for Kannan type mapping mappings with respect to orbits if every mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ that is Kannan type mapping with respect to orbits admits a fixed point, where $\mathcal{B}$ denotes any arbitrary weakly compact and convex subset of $\mathcal{A}$

With this property formally established, we are now prepared to state our main result within this section.

THEOREM 2.2. *Let $\mathcal{A}$ be a Banach space. Then, $\mathcal{A}$ has weak normal structure if and only if $\mathcal{A}$ has the weak fixed point property for Kannan type mapping mappings wrt orbits.*

A Banach space $\mathcal{A}$ has the weak normal structure if and only if $\mathcal{A}$ possesses the weak fixed point property for Kannan type mapping mappings with respect to their orbits.

It is crucial to note that Kannan type mappinge mappings with respect to orbits are not necessarily continuous (as illustrated in Example 1.4). Consequently, Schauder's fixed point theorem cannot be directly invoked to guarantee the existence of a fixed point, even when $\mathcal{B} \subseteq \mathcal{A}$ is a non-empty, compact, convex set and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is a Kannan type mapping mapping with respect to its orbits. Nevertheless, since any non-empty, compact set possesses normal structure, we can leverage Theorem 2.2 to deduce the following corollary.

COROLLARY 2.3. *Let $\mathcal{B}$ be a nonempty, compact convex subset of a Banach space $\mathcal{A}$ and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be a Kannan type mapping mapping wrt orbits. Then, $\mathcal{F}$ has a fixed point.*

Let $\mathcal{A}$ be a Banach space, $\mathcal{B}$ be a non-empty, compact, and convex subset of $\mathcal{A}$, and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be a Kannan type mapping mapping with respect to its orbits. Then, $\mathcal{F}$ must have a fixed point.

For many years, the problem of the failure of the fixed point property for nonexpansive mappings defined on unbounded sets has attracted the interest of many researchers (see, for instance, [3, 8, 9]). This line of inquiry can also be extended to mappings which are Kannan type mapping with respect to orbits. For instance, we have the following result:

THEOREM 2.4. *Assume that $\mathcal{A}$ is a reflexive Banach space and $\mathcal{B}$ is a closed convex subset of $\mathcal{A}$. If every Kannan type mapping mapping with respect to orbits $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ has a fixed point, then $\mathcal{B}$ is bounded.*

PROOF. Assume that $\mathcal{B}$ is unbounded. Choose a sequence $\{\mathfrak{a}_n\}$ formed by distinct points such that $\| \mathfrak{a}_n \| \to \infty$. . Define $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by $\mathcal{F}\mathfrak{a} = \mathfrak{a}1$ if $a \notin \{\mathfrak{a}_n : \ n \in \mathbb{B}\}$ and $\mathcal{F}\mathfrak{a}_n = \mathfrak{a}n + 1$. It is clear that $\mathcal{F}$ is a fixed-pointfree nonexpansive mapping wrt orbits because $\max\left\{r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathfrak{b})\big), r_{\mathfrak{a}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{b})\big), r_{\mathfrak{b}}\big(O_{\mathcal{F}}(\mathcal{F}\mathfrak{a})\big)\right\}$ for every $\mathfrak{b} \in \mathcal{B}$.

□

3. Conclusion

This work established a framework for analyzing Kannan type mappings relative to orbits in Banach spaces, yielding several important fixed point theorems. The principal contribution lies in proving the equivalence between the weak fixed point property for these mappings and the existence of weak normal structure within the Banach space $\mathcal{A}$. These findings provide powerful analytical tools. Further research is warranted to explore the implications of these results for stability analysis or to adapt this framework to other geometric structures beyond the scope of Banach spaces.

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On the Loewner differential equations in geometric functions theory

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Abstract. In this article, two versions of the Loewner differential equations are introduced, which are mathematically equivalent but used in different contexts.
We introduce some properties of the versions Loewner equations.

Keywords: Analytic functions, Univalent functions, Loewner chains, Loewner differential equations

AMS Mathematics Subject Classification [2020]: 30C45, 30C20

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.
A function $f \in \mathcal{S}$ is called a slit mapping if f maps unit disc conformally onto the complex plane minus a set of Jordan arcs. These arcs must tend to ∞ because $f(\mathbb{U})$ is a simply connected domain [2, 3].

Theorem 1.1. *(Rieman Mapping Theorem) Let D be an open, simply connected subset of $\mathbb{C}$ and suppose that D is not equal to $\mathbb{C}$ itself. Let $z_0 \in D$ then there is an analytic map ϕ taking D onto the unit disk $\mathbb{U}$ in one-to-one function and satisfying $\phi(z_0) = 0$. Moreover, if we normalize so that $\phi^{'}(z_0) > 0$, then the map is unique. [3]*

In the lemma and theory of Loewner chains, if f is a function which depends holomorphically on $z \in \mathbb{U}$ and is also a function of other real variables, it is customary to write $f'(z, .)$ instead of $\frac{\partial f}{\partial z}(z, .)$ [3, 5].

Lemma 1.2. *The function $f : \mathbb{U} \times [0, \infty) \to \mathbb{C}$ with $f(0, t) = 0, f'(0, t) = e^t$, is a Löwner chain if and only if the following conditions hold:*
(i) There exist $r \in (0, 1)$ and a constant $M \geq 0$ such that $f(., t)$ is holomorphic on $\mathbb{U}_r$

*Speaker.

for each $t \geq 0$, where $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$, locally absolutely continuous in $t \geq 0$ locally uniformly with respect to $z \in \mathbb{U}_r$, and

$$|f(z,t)| \leq Me^t, \qquad |z| \leq r, \quad t \geq 0.$$

(ii) There exists a function $p(z,t)$ such that $p(.,t) \in \mathcal{P}$ for each $t \geq 0, p(z,.)$ is measurable on $[0,\infty)$ for each $z \in \mathbb{U}$, and for all $z \in \mathbb{U}_r$,

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \qquad a.e. \quad t \geq 0.$$

(iii) For each $t \geq 0, f(.,t)$ is the analytic continuation of $f(.,t)|_{\mathbb{U}_r}$ to $\mathbb{U}$, and furthermore this analytic continuation exists under the assumptions (i) and (ii).

Lemma 1.3. *Let $f(z,t)$ be a Loewner chain. Then there exists a function $p(z,t)$ such that $p(.,t) \in \mathcal{P}$, $t \geq 0$, $p(z,t)$ is measurable in $t \in [0,\infty)$ for each $z \in \mathbb{U}$, and*

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \quad z \in \mathbb{U}, t \geq 0. \tag{2}$$

2. Main results

In this section, we introduce some properties of the Loewner equations. conformal mappings that transform a domain with a slit of arbitrary geometry into the upper half-plane or the unit circle are governed by special differential equations. These equations, which describe how the mapping evolves with respect to the parameter defining the slit, are known as Loewner-type equations [1, 4].

2.1. Chordal Loewner equation. The most basic equation of this type, now called the chordal loewner equation, was first formulated in [4].
Consider a smooth curve Υ starting from a point on the real axis of $\mathbb{R}$ and extending to the upper half-plane $\mathbb{H}$, in other words,

$$\Upsilon : [0,\infty) \to \mathbb{H}, \qquad \Upsilon(0) \in \mathbb{R},$$

and Υ_t be its arc, $\Upsilon_t := \Upsilon([0,t])$.
By the theorem 1.1, there exists a unique univalent mapping $g(z,t)$ from the slit domain $\mathbb{H} - \Upsilon_t$ to $\mathbb{H}$. This mapping is normalized to show asymptotic behavior:

$$g(z,t) = z + a(t)z^{-1} + \bigcirc(z^{-2}), \qquad z \to \infty, \quad t > 0.$$

The coefficient $a(t)$, known as the capacity of the domain $\mathbb{H} - \Upsilon_t$ is a continuous increasing function of t, this allows the parameter to be varied in such a way that $a(t) = t$, and gives the following normalization:

$$g(z,t) = z + tz^{-1} + \bigcirc(z^{-2}), \qquad z \to \infty, \quad t > 0.$$

It seems, that there is a continuous real-valued driving function $\kappa(t)$, for which $g(z,t)$ satisfies the following differential equation:

$$\frac{\partial g(z,t)}{\partial t} = \frac{1}{g(z,t) - \kappa(t)}, \quad g(z,0) = z,$$

this is the chordal loewner equation.

2.2. Radial Loewner equation. The radial form of Loewner equation differs mainly in its normalization. Consider a Υ_t be an arc of the smooth curve in the exterior of the unit disk $\mathbb{U}$. The unique univalent conformal map $g(z,t)$ from $\mathbb{U}^c - \Upsilon_t$ to $\mathbb{U}^c$ is normalized as:

$$g(z,t) = e^{-\rho(t)}z + \bigcirc(1), \qquad z \to \infty, \quad t \in \mathbb{R}. \tag{3}$$

According the theorem 1.1, such a map exists and is unique. In 3, the coefficient $e^{-\rho(t)}$ is related to the radius of the domain $\mathbb{U}^c - \Upsilon_t$. Under a suitable parameterization $\rho(t) = t$, the map $g(z,t)$ satisfies the below normalization:

$$g(z,t) = e^{-t}z + \bigcirc(1), \qquad z \to \infty, \quad t \in \mathbb{R}.$$

Difference in normalization is the essential difference between the Chordal and radial types of the loewner equations. It seems, that there is a continuous real-valued driving function $\kappa(t)$, for which $g(z,t)$ satisfies the following differential equation:

$$\frac{\partial g(z,t)}{\partial t} = -g(z,t)\frac{g(z,t)+\psi(t)}{g(z,t)-\psi(t)}, \quad g(z,0) = z, \quad \psi(t) = e^{i\kappa(t)}.$$

This equation, formulated by Loewner in 1923 has served as a powerful tool in geometric function theory, particularly for determining bounds on the coefficients of univalent functions, including its application in proving the Bieberbach conjecture.

The Loewner differential equation (2) holds for arbitrary Loewner chains. Note that the function $f(z,t)$ satisfies Lipschitz condition in t as follow:

$$\exists M > 0; \qquad |f(z,t) - f(z,s)| < M|t-s|,$$

and equation (2) holds for almost all t. Conversely, every partial differential equation (2) determines a Loewner chain of conformal maps.

Example 2.1. The Loewner chain $g(z,t) = \frac{e^t z}{(1-z)^2}$ satisfies radial Loewner equation

$$\frac{\partial g}{\partial t}(z,t) = zg'(z,t)\frac{1+(-1)z}{1-(-1)z} = zg'(z,t)\frac{1-z}{1+z}, \quad z \in \mathbb{U}, t \geq 0. \tag{4}$$

So the driving function is $k(t) = -1$ for all t.

3. Conclusion

The Loewner differential equations provides a powerful and elegant model in complex analysis that illuminating various aspects of the geometric functions theory. The radial and chordal Loewner equations also provide deep insights into concepts about the geometry of slit maps and the behavior of conformal maps under special conditions.

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Fixed point theorems with α-admissible mapping in metric spaces

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Abstract. The aim of this paper is to establish some fixed point theorems for α-admissible contractive mappings defined on a metric space which generalize the results of N. Hussain and et al. In the end, we shall provide an example to support our result.

Keywords: fixed point, α-admissible mapping, complete metric space.

AMS Mathematics Subject Classification [2020]: Primary: 47H10, 54H25; Secondary: 46A80

1. Introduction

Fixed point theory has gained very large impetus due its wide range of applications in various fields such as engineering, economics, computer science, and many others. It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory and Banach's fixed point theorem for contraction mappings is one of the pivotal results in analysis. In 2012, Samet et al. [6] introduced the concepts of α-contractive and α-admissible mappings and established various fixed point theorems for such class of mappings defined on complete metric spaces and generalized several fixed point results from metric spaces. Thereafter, the existence of fixed points of α-admissible contractive-type mappings in complete metric spaces has been studied by several researchers [2].

In this paper we give some generalized versions of the fixed point theorems of [5].

To start with we give some notations and introduce some definitions which will be used in the sequel.

Definition 1.1. ([6]) Let $T : X \to X$ and $\alpha : X \times X \longrightarrow [0, \infty)$. T is said to be α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Obviously, $\alpha(., .)$ may or may not be symmetric and $\alpha(x, y) = \alpha(y, x)$ (i.e., symmetric) if and only if $x = y$.

Denote with Ψ the family of functions $\psi : [0, \infty) \to [0, 1]$ such that for all $t > 0$, $\psi(t) \in [0, 1]$ and $\psi(t) = 1$ iff $t = 0$ and for any bounded sequence $\{t_n\}$ of positive reals, $\psi(t_n) \to 1$ implies $t_n \to 0$.

*Speaker.

The function α can be chosen in several ways, so that one can obtain several useful forms of a mapping defined on a space. Further, such functions can be used to weaken and generalize the contractive constraints on mappings which are utilized to establish fixed point results for the mappings under consideration. For generalization and extension of α-admissible mappings and its application, we refer to [2–4] and the references therein.

THEOREM 1.2. *(see [1]) Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Assume that there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y),$$

for all $x, y \in X$. Then f has a unique fixed point.

EXAMPLE 1.3. ([6]) Let $X = \mathbb{R}$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$f(x) = \begin{cases} ln|x| & if \text{ x} \neq 0 \\ 3 & \text{otherwise} \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 3 & if \text{ x} \geq y \\ 0 & \text{otherwise.} \end{cases}$$

Then f is α-admissible.

2. Main results

Let (X, d) be a metric space and $T : X \to X$. In the next of paper we define for all $x, y \in X$,

$$M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right], \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right]\right\}.$$

THEOREM 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume there exists $\psi \in \Psi$ and*

$$(d(Tx, Ty) + l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \psi(M_T(x, y)).d(x, y) + l \tag{1}$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either

(a) *T is continuous or*
(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, then T has a fixed point.

THEOREM 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume there exists $\psi \in \Psi$ and*

$$(\alpha(x, Tx)\alpha(y, Ty) + l)^{d(Tx,Ty)} \leq 2^{\psi(M_T(x,y))M_T(x,y)} \tag{2}$$

for all $x, y \in X$. Suppose that either

(a) *T is continuous or*
(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, then T has a fixed point.

THEOREM 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume there exists $\psi \in \Psi$ and*

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq \psi(M_T(x, y))M_T(x, y) \tag{3}$$

for all $x, y \in X$. Suppose that either

(a) *T is continuous or*
(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, then T has a fixed point.

EXAMPLE 2.4. Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $f : X \to X$ defined by

$$f(x) = \begin{cases} \frac{1}{4}(1 - x^2) & \text{if x} \in [0, 1], \\ 3x & \text{if x} \in (1, \infty). \end{cases}$$

Define also $\alpha : X \times X \to [0, \infty)$ and $\beta : [0, \infty) \to [0, 1]$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if x,y} \in [0, 1], \\ 0 & \text{if otherwise} \end{cases} \quad \text{and} \quad \beta(t) = \frac{1}{2}.$$

We prove that Theorem 2.3 can be applied to f (here, a fixed point is $u = \sqrt{5}2$), but Theorem 1.2 can not be applied to f.

THEOREM 2.5. *Assume that all the hypotheses of Theorems 2.1, 2.2 and 2.3 holds. Adding the following condition:*

(c) *if $x = Tx$, then $\alpha(x, Tx) \geq 1$.*

We obtain the uniqueness of the fixed point of T.

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A Study of Composition Operators on Hardy-Type Spaces

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Abstract. In this paper, we study composition operators on general weighted Hardy spaces $H^2(\beta)$. We discuss when these operators are bounded or compact and how certain analytic functions affect this behavior. We also show that disc automorphisms of the unit disc produce cyclic composition operators.

Keywords: composition operators, Hardy space, Hilbert space, boundedness.

AMS Mathematics Subject Classification [2020]: 58E11, 53B30, 53C50.

1. Introduction

For a Hilbert space H of functions analytic in the unit disc $D = \{z : |z| < 1\}$ and φ an analytic function that maps D into itself the composition operator C_φ is defined by $C_\varphi f = f o \varphi$, for $f \in H$. Composition operators on the classical Hardy space H^2 have been studied extensively.

The purpose of this paper is to study the behavior of composition operators on genaral weighted Hardy space $H^2(\beta)$. In such a genral context even some basic questions, like boundedness, are sometimse hard to answer. There are some nice classes of functionsthat induce bounded composition operators on all weighted Hardy spaces. On some spaces for example, the ones that satisfy the relation $H^\infty(\beta) = H^2(\beta)$ and that contain the induction function, having supremum strictly smaller than one is sufficient for the function to induce not only a bounded operator but even one that is in the trace class. It is not known if this is true in the case of general weighted Hardy spaces.

Disc automorphisms of the unit disc play an importantrole in the question of compactness of composition operators. Disc automorphisms are also very intresting in the case of the classical Hardy space $H^2(\beta)$. We prove that they induce cyclic composition operators.

Corresponding to each sequence $\{s_n\}_{n=0}^\infty$ in ℓ^∞ there is the weighted unilateral shift S, defined on ℓ^2 by $S(x_0, x_1, x_2, ...) = (0, s_0x_0, s_1x_1, ...)$. One approach to the study of weighted unilateral shifts is through the representation of S as multiplication by z on certain weighted Hardy spaces. Given a positive sequence $\{s_n\}_{n=0}^\infty$, let $\beta_0 = 1$ and $\beta_n = s_0s_1...s_{n-1}$, for $n \geq 1$. Then S is unitary equivalent to multiplication by z on a certain space $H^2(\beta)$ of formal power series defined by the sequence $\beta = \{\beta_n\}_{n=0}^\infty$. For special β, the functions in $H^2(\beta)$ are analytic in the unit disc.

*Speaker.

In the case when $\beta_n = 1$ for all n, the space $H^2(\beta)$ is Hardy space H^2, i.e., the space of functions analytic in the unit disc whose power series coefficients at the origen are square summable.

Our main interest in this paper is composition operators on $H^2(\beta)$ spaces. In general, if H is a Hilbert space of complex valued functions on a set X, and if φ maps X into X, we define the operator C_φ on H by $C_\varphi f = fo\varphi$, for all f in H. We say that C_φ is composition operator induced by φ. The boundedness operators on H^2 follows from the Littlewood's subordination theorem.

2. Main results

For every composition operator on $H^2(\beta)$, the set $\mathbb{C}$ of all constant functions is an invariant subspace, for the constant function s_0 defined by $s_0(z) = 1$ for $z \in D$ belongs to $H^2(\beta)$ and $C_\varphi s_0 = s_0$ for all φ.

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$. The set $H^2(\beta)$ is defined as the set of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$||f||_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty.$$

Then $H^2(\beta)$ is a Hilbert space with the inner product

$$< f, g >_\beta = \sum_{n=0}^{\infty} a_n \bar{b}_n {\beta_n}^2$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Example 2.1. Let $\beta_n = 1$ for all n. Then $H^2(\beta)$ is the classical l Hardy space H^2.

Example 2.2. If $\beta_n = (1+n)^{\frac{-1}{2}}$, then $H^2(\beta)$ is the Begman space B, and $f \in B$ if and only if f is analytic in the unit disk D and $\int_D |f|^2 dm < \infty$, where m is the normalized Lebesgue area measure in D.

Example 2.3. If $\beta_n = (1+n)^{\frac{1}{2}}$, then $H^2(\beta)$ is the Dirichlet space D, and $f \in D$ if and only if f is analytic in the unit disk D and $\int_D |f^{`}|^2 dm < \infty$, where m is the same previous example.

Example 2.4. For a real number a, let $\beta_n = (1+n)^a$. Then we call those $H^2(\beta)$ spaces S_a spaces. All the above examples

Remark 2.5. Given the previous examples, we can easily observe that: $H^2(\beta) = S_0$, $B = S_{\frac{-1}{2}}$ and $D = S_{\frac{1}{2}}$.

By the closed graph theorem, a composition operator C_φ on a Hilbert space H is bounded if and only if $fo\varphi$ belongs to H for all $f \in H$. The function s_1 defined by $s_1(z) = z$, $z \in D$, belongs to every $H^2(\beta)$ space and $C_\varphi s_1 = \varphi$. So a necessary condition for C_φ to be bounded on $H^2(\beta)$ is that φ belongs to $H^2(\beta)$.

We have already that every composition operator on $H^2(\beta)$ is bounded. If the sequence β is given by $\beta_n = (1+n)^a$ and $a < 0$, then we also that every composition operator on $H^2(\beta)$ is bounded(see [1]).

In the following, we present the necessary and sufficient for boundedness of composition operator on $H^2(\beta)$. A necessery condition is that $H^\infty \subset H^2(\beta)$, where $H^\infty = \{f : ||f||_\infty <$

$\infty\}$. The reason for this is that we need to have all functions that map D into D in $H^2(\beta)$, and that H^∞ is closed under scaler multiplication.

PROPOSITION 2.6. *$H^\infty \subset H^2(\beta)$ if and only if the sequence β is bounded.*

Note that in the case when the sequence β is bounded and 0 is not an accumulation point for β, then β-norm is actually equivalent to the H^2-norm and so $H^2(\beta) = H^2$.

PROPOSITION 2.7. *If $H^\infty \subset H^2(\beta)$ and $||\varphi||_\infty < 1$, then C_φ is in the trace class of $H^2(\beta)$.*

THEOREM 2.8. *Let the space $H^2(\beta)$ be such that every function in $H^2(\beta)$ with sup norm stricly smaller than 1 induces a bounded composition operator on $H^2(\beta)$. Then , if $\varphi \in H^2(\beta)$ and $||\varphi||_\infty < 1$, we have that C_φ is in the trace class of $H^2(\beta)$.*

The attampt to characterize all functions that induce bounded composition operators on any $H^2(\beta)$ space seems to be too general. So we are going to consider a few special types off functions.

A necessary condition for φ to induce a bounded composition operator on every $H^2(\beta)$ space is that φ belongs to every $H^2(\beta)$. The following theorem gives a class of functions that are in every $H^2(\beta)$.

THEOREM 2.9. *Let $A_{\bar{D}} = \{f : f$ is analytic in some neighborhood of $\bar{D}\}$. Then $A_{\bar{D}} \subset \bigcap_\beta H^2(\beta)$.*

It is not true that every function φ in $A_{\bar{D}}$ induces a bounded composition operators on all $H^2(\beta)$ spaces.

Another interesting class of functions that belong to φ in $A_{\bar{D}}$ are disc automorphisms. The general from of a disc automorphism φ is

$$\varphi(z) = \lambda \frac{z - z_0}{1 - \bar{z_0} z},$$

where $|\lambda| = 1$ and $z_0 \in D$. If $z_0 = 0$, then $\varphi(z) = \lambda z$ and for $f(z) = \sum_{n=0}^\infty a_n z^n$, we have

$$f(\varphi(z)) = \sum_{n=0}^\infty a_n \lambda^n z^n$$

and

$$||fo\varphi||_\infty = ||f||_\beta;$$

i. e., C_φ is an isometry on all $H^2(\beta)$ spaces.

The set of all disc automorphisms is a group under the composition of functions and henc the boundedness composition operators induced by different disc automorphisms is not independent, as we can see in the following theorem.

THEOREM 2.10. *If the space $H^2(\beta)$ is such that there exisits a disc automorphism φ which induces a bounded composition operators on $H^2(\beta)$, then every disc automorphism induces a bounded composition operators on $H^2(\beta)$.*

DEFINITION 2.11. If disc automorphisms induce bounded composition operators on the space $H^2(\beta)$ we say that $H^2(\beta)$ is disc-automorphism invariant.

Beside rotations, another class of functions in $A_{\bar{D}}$ that induce bounded composition operators on all $H^2(\beta)$ spaces is the class of the form $\varphi(z) = \lambda z$, $|\lambda| < 1$. In this case $||\varphi||_\infty < 1$.

THEOREM 2.12. *Let φ in $A_{\bar{D}}$ and $||\varphi||_\infty < 1$. Then the operator C_φ is bounded on all $H^2(\beta)$ spaces.*

COROLLARY 2.13. *If φ in $A_{\bar{D}}$ and $||\varphi||_\infty < 1$, then C_φ is in the trace class of all $H^2(\beta)$ spaces.*

J. H. Shapiro has given examples of a $H^2(\beta)$ spaces which are not disc-automorphism invariant (see [6]). His result folows as a corollary of two statements about compact composition operators on certain Hilbert space H of analytic functions.

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Fixed Point Results for Graphical $(\psi - \varphi)$-Asymptotic Contractions in Metric Spaces with a Graph

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Abstract. This paper is dedicated to establishing the existence and uniqueness of fixed points for $G - (\psi - \varphi)$-asymptotic contractions within the setting of metric spaces endowed with a graph. The main result presented herein constitutes a novel extension and generalization of Kirk's fixed point theorem for asymptotic contractions. Specifically, we extend the theorem from classical metric spaces to the more complex framework of metric spaces equipped with a graph structure, thereby highlighting the significance and broad applicability of this class of contractions.

Keywords: Fixed point, $G - (\psi - \varphi)$-asymptotic contraction, orbitally G-continuous.

AMS Mathematics Subject Classification [2020]: Primary 47H10; Secondary 05C20

1. Introduction

"The study of fixed points in metric spaces has seen significant evolution, particularly concerning contractions. Kirk [8] initiated crucial work in 2003 on asymptotic contractions; however, this was later challenged by a counterexample from Jachymski and J'o'zwik [7], which underscored the necessity of uniform continuity for self-maps in Kirk's original theorem. Jachymski and J'o'zwik subsequently developed an alternative result for uniformly continuous asymptotic φ-contractions [7, Theorem 2]. Further refinement arrived with Aghanians et al. [1], who extended the fixed point theorem for p-continuous E-asymptotic contractions to separated uniform spaces using E-distances.

A pivotal shift in the field occurred in 2008 when Jachymski [6] integrated graph theory into the metric fixed point landscape by imposing a directed graph structure on the underlying metric space, effectively translating the Banach contraction principle into a graphical framework. This seminal approach has since been widely adopted and explored by numerous researchers (e.g., [2, 5]).

Building upon this foundation, this manuscript focuses on extending the fixed point results for Kirk's asymptotic contractions to the domain of metric spaces endowed with a directed graph E . Our central contribution is the formulation of a new iteration of [8, Theorem 2.1], specifically tailored to this graph-metric context, addressing both existence and uniqueness.

Let (X, d) be a metric space. We introduce a directed graph G, where the vertex set is $V(G) = X$ the edge set $E(G)$ mandates the inclusion of all loops, i.e., $(x, x) \in E(G)$ for all $x \in X$. Assume that G has no parallel edges, we denote G by $(V(G), E(G))$, signifying metric space (X, d) is endowed with G.

The metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

For the ensuing development, we adopt the necessary graph-theoretic terminology, consistent with Bondy and Murthy, and Jachymski. A comprehensive treatment of the underlying graph theory can be found in [3, 6].

2. Main Results

We begin by considering a metric space (X, d) that is further equipped with an underlying graph structure, denoted by G. For a self-map $f : X \to X$, we define $\mathrm{Fix}(f)$ as the set of its fixed points. Furthermore, we introduce the set C_f as the collection of points $x \in X$ that maintain adjacency within the extended graph $\widetilde{G}$ under iterated applications of f:

$$C_f = \{x \in X : (f^m x, f^n x) \in E(\widetilde{G}) \quad m, n = 0, 1, \cdots\}.$$

Also, assume that Ψ is the class of all continuous functions $\psi, \varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ are two such that

- ψ is nondecreasing and continuous;
- φ is lower semi-continuous (in short, lsc) on $\mathrm{R}^{\geq 0} and \varphi(t) = 0$ implies $t = 0$.)

Now, we are ready to define $G - (\psi - \varphi)$-asymptotic contractions in metric spaces endowed with a graph. This definition is motivated from [7, Definition 2.1] and [8, Definition 2.1].

Definition 2.1. Let (X, d) be metric space endowed with garph G. We say that a self-map f on X is a $G - (\psi - \varphi)$-asymptotic contraction if

($\mathcal{K}_1$) f preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;

($\mathcal{K}_2$) there exists a sequences $\varphi_n : [0, +\infty) \to [0, +\infty)$ converging uniformly to a $\varphi \in \Psi$ on the range of d such that

$$\psi\big(d(f^n x, f^n y)\big) \leq \psi\big(d(x, y)\big) - \varphi_n\big(d(x, y)\big) \tag{1}$$

for all $n \geq 1$ and all $x, y \in X$ with $(x, y) \in E(G)$.

To establish our main theorem, we first require the following lemma.

Lemma 2.2. *Let (X, d) be a metric space endowed with a graph G and let $f : X \to X$ be a $G - (\psi - \varphi)$-asymptotic contraction. Assume that the sequence of functions ψ_n appearing in (1) is continuous on $[0, +\infty)$ for all sufficiently large indices n. Then, for every $x \in C_f$, the sequence $\{f^n x\}$ is Cauchy in (X, d).*

Now we are prepared to establish the main theorem concerning the existence of fixed points for $G - (\psi - \varphi)$-asymptotic contractions.

Theorem 2.3. *Let (X, d) be a complete metric space endowed with a graph G, and let $f : X \to X$ be an orbitally G-continuous $G - (\psi - \varphi)$-asymptotic contraction. Assume that the sequence of functions ψ_n appearing in (1) is continuous on $[0, +\infty)$ for all sufficiently large indices n.*

Then f admits a fixed point in X if and only if $C_f \neq \emptyset$. Moreover, if the subgraph of G with the vertex set $\mathrm{Fix}(f)$ *is connected, then the restriction of f to C_f is a Picard operator.*

We next present several direct consequences of Theorem 2.3.

By setting $G = G_0$ in Theorem 2.3, where G_0 is the complete graph with the vertex set X, that is, $V(G_0) = X$ and $E(G_0) = X \times X$, it follows immediately that G_0-orbital continuity with the notion of orbital continuity introduced by Ćirić [4] and the set $C_f$associated with any self-map f on X reduces to the entire space X. Accordingly, we obtain the following counterpart of [8, Theorem 2.1].

Corollary 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ be an orbitally continuous $(\psi - \varphi)$-asymptotic contraction such that the functions φ_n in (1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f is a Picard operator.*

Secondly, consider a partially ordered set $(X, \preceq)$ and poset graph G_1 with $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. Put $G = G_1$ in Theorem 2.3. This yields a partially ordered counterpart of [8, Theorem 2.1] for complete metric spaces equipped with a partial order.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an orbitally G_1-continuous $G_1 - (\psi - \varphi)$-asymptotic contraction such that the functions ψ_n in (1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.*

Smilarly, by considering a partially ordered set $(X, \preceq)$ and poset graph G_2 defined by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \preceq y \ \vee \ y \preceq x\}$. Taking $G = G_2$ in Theorem 2.3, we obtain another partially ordered extension of [8, Theorem 2.1] in complete metric spaces endowed with a partial order.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an orbitally G_2-continuous $G_2 - (\psi - \varphi)$–asymptotic contraction such that the functions ψ_n in (1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are comparable elements of $(X, \preceq)$ for all $m, n \in \mathbb{N} \cup \{0\}$.*

In particular, if every two elements of $\mathrm{Fix}(f)$ *are comparable, then the restriction of f to the set of all $x \in X$ whose every two iterates under f are comparable elements of $(X, \preceq)$ is a Picard operator.*

Finally, suppose that (X, d) is a metric space and ε is a fixed positive real number. Two elements $x, y \in X$ are called ε-close whenever $d(x, y) < \varepsilon$. Define a graph G_3 by $V(G_3) = X$ and $E(G_3) = \{(x, y) \in X \times X : \ d(x, y) < \varepsilon\}$. Then $(\mathcal{K}_1)$ holds if and only if f maps ε-close elements of X onto ε-close elements, and $(\mathcal{K}_2)$ means that

$$\psi\big(d(f^n x, f^n y) \leq \psi\big(d(x, y)\big) - \varphi_n\big(d(x, y)\big)$$

for all ε-close elements $x, y \in X$, where $\psi_n : [0, +\infty) \to [0, +\infty)$ converges uniformly to a $\psi \in \Psi$ on the range of d. By setting $G = G_3$ in Theorem 2.3 we obtain the following metric version of [8, Theorem 2.1]:

Corollary 2.7. *Let (X, d) be a complete metric space and ε be a fixed positive real number. Let $f : X \to X$ be an orbitally G_3-continuous $G_3-(\psi-\varphi)$–asymptotic contraction such that the functions ψ_n in (1) are continuous on $[0, +\infty)$ for sufficiently large indices n. Then f has a fixed point in X if and only if there exixts an $x_0 \in X$ such than $f^m x$ and $f^n x$ are ε-close elements of (X, d) for all $m, n \in \mathbb{N} \cup \{0\}$.*

Moreover, if every two elements of $\mathrm{Fix}(f)$ *are ε-close, then the restriction of f to the set of all $x \in X$ whose every two iterates under f are ε-close elements of (X, d) is a Picard operator.*

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