



# دوازدهمین همایش ملی ریاضی دانشگاه پیام نور



۱۵ الی ۱۶ آبان ماه ۱۴۰۴

دانشگاه پیام نور استان مازندران - مرکز ساری

کد اختصاصی:  
۰۴۲۵۰-۲۴۴۱۸

چاپ مقالات برتر  
در مجلات علمی  
پژوهشی معتبر



محورهای همایش  
گرایش های علوم ریاضیات  
و کاربردهای مرتبط با آن

# Proceeding of 12<sup>th</sup> National Mathematics Conference of Payame Noor University

<https://12nma.conf.pnu.ac.ir>

<https://eita.com/mazmath>

12mazmath@gmail.com

۰۱۱۳۳۲۰۸۹۴۴



In the name of  
Allah, the most  
Gracious and  
the most  
Merciful



کد اختصاصی:

۰۴۲۵۰-۲۴۴۱۸



دانشگاه پیام نور استان مازندران

# 12<sup>th</sup> National Mathematics Conference of Payame Noor University

6-7-November 2025  
Sari – Iran

# Proceeding of 12<sup>th</sup> National Mathematics Conference of Payame Noor University



**Authored by:**  
**Mostafa Nouri Jouybari and**  
**Davood Darvishi Salookolayi**

**Adress:**  
**Payame Noor University of Mazandaran**  
**Sari, Mazandaran, Iran**

**Website: <https://12nma.conf.pnu.ac.ir>**  
**Email: [12mazmath@gmail.com](mailto:12mazmath@gmail.com)**



# Table of Content

- **Preface**
  - **Poster of Conference**
  - **Organizers and Sponsors**
  - **Scope of Conference**
  - **Strategic Committee of Conference**
  - **Scientific Committee of Conference**
  - **Executive Committee of Conference**
  - **Program of Conference**
  - **Table of Content of Articles**
  - **Articles**
- 

# Preface

Payame Noor University, in cooperation and partnership with the Islamic World Science Citation Center, universities, institutions and organizations, educational and research centers, and respected members of scientific and executive committees, and with the unprecedented presence and participation of researchers, academics, scholars, and national and international practitioners in the field of mathematical sciences, The 12th National Mathematics Conference of Payame Noor University was held on November 6 and 7, 2025, simultaneously with the celebration of Mazandaran National Day, with the participation of more than 300 researchers from all over the country, virtually and in person, with the support of reputable scientific associations, at Payame Noor University, Mazandaran Province, Sari Center, with the aim of exchanging knowledge and examining the latest achievements in the field of mathematical sciences and its applications.

Certainly, the warm and well-deserved welcome of researchers and scholars in this field to present papers and participate in this conference, in addition to indicating the growth and advancement of this dynamic knowledge in the country in recent years, It indicates the existence of potential and actual capabilities among the country's researchers and scholars in innovation and conducting original research. In particular, the presence and participation of prominent researchers indicates the appropriate position and importance of this science in society.

The conference secretariat received 403 papers, and after reviewing the status of the papers and considering the time constraints for presentation, it was decided to accept 235 papers for presentation at the conference in the form of lectures and 124 papers for poster presentation. On the sidelines of this conference, 6 specialized workshops were also held in the field of artificial intelligence, mathematical laboratory, and challenges of financial markets.

Here, I consider it necessary to thank Dr. Mohammad Hadi Amin Naji, the esteemed President of Payame Noor University and his esteemed deputies, the Dean of the Faculty of Basic Sciences, the Secretary of the Department's Specialized Council, the esteemed President of Payame Noor University of Mazandaran Province, Dr. Saber Ghasempoor, Dr. Mostafa Nouri Jouybari, the esteemed Secretary of the Executive Committee, The esteemed members of the scientific and executive committees, and I would like to express my deepest gratitude and appreciation to all the esteemed sponsors who supported and encouraged us in holding this conference. Finally, I would like to express my deepest gratitude and appreciation to all the students, provincial headquarters managers, and respected colleagues who helped us in holding this conference.

**Davood Darvishi Salokolaei**

**Chair of the scientific committee**

# Poster of Conference

**دوازدهمین همایش ملی ریاضی دانشگاه پیام نور**

کد اختصاصی: ۴۲۵۰-۲۴۴۱۸  
ISC

۱۵ الی ۱۶ آبان ماه ۱۴۰۴  
دانشگاه پیام نور استان مازندران - مرکز ساری

موضوعات همایش  
گرایش های علوم ریاضیات  
و کاربردهای مرتبط با آن

چاپ مقالات برتر  
در مجلات علمی  
پژوهشی معتبر

$y = ax^2 + bx + c$   
 $x = \frac{L}{2}$   
 $y = \frac{4xh(L-x)}{L^2}$

دانشگاه پیام نور استان مازندران

هدیه کفیه فرک  
جهان هستی است،  
بدون دانش هدیه  
زرود  
عسوق  
است.

<https://12nma.conf.pnu.ac.ir>  
<https://eita.com/mazmath>

12mazmath@gmail.com  
۰۱۱۳۳۲۰۸۹۴۴

CIVILICA

QR Code

# Organizers and Sponsors

► **Organizer:**

**Payame Noor university of Mazandaran**

► **Head of Conference:**

**Saber Ghasempour, Ph.D.**

► **Scientific Secretary of Conference:**

**Davood Darvishi Salookolayi, Ph.D.**

► **Executive Secretary of Conference:**

**Mostafa Nouri Jouybari, Ph.D.**

► **Sponsors:**

**University of Mazandaran, Civilica, ISC,  
University of Technology of Noshirvani,  
Journal The of Mahani Mathematical  
Research (JMMR), International Journal of  
Applied Operational Research, Grey Systems:  
Theory and Application, Control and  
Optimization in Applied Mathematics,**

# Scope of Conference

- ▶ **-Algebra**
- ▶ **-Application of Mathematical Science**
- ▶ **-Differential Equation**
- ▶ **-Functional Analysis**
- ▶ **-Fuzzy Mathematics**
- ▶ **-Geometry**
- ▶ **-Graph and Combinatorics**
- ▶ **-Mathematical Analysis**
- ▶ **-Mathematical Finance**
- ▶ **-Mathematical Learning**
- ▶ **-Numerical Analysis**
- ▶ **-Operation Research**
- ▶ **-Optimization and Control**
- ▶ **-Statistics**
- ▶ **-Topology**



# Strategic Committee of Conference

- ▶ **Shahriar Farahmand Rad, Ph.D.**  
Head of Strategic Committee
  - ▶ **Jahangir Cheshmavar, Ph.D.**  
Member of Strategic Committee
  - ▶ **Saber Ghasempour, Ph.D.**  
Secretary of Strategic Committee
  - ▶ **Davood Darvishi Salookolayi, Ph.D.**  
Member of Strategic Committee
  - ▶ **Mostafa Nouri Jouybari, Ph.D.**  
Member of Strategic Committee
- 

# Scientific Committee of Conference

Esmael Babolian	Kharazmi University	Seyed Mehdi Mirhosseini Alizamini	Payame Noor University
Amir Jafari	Sharif University of Technology	Seyed Hadi Naseri	University of Mazandaran
Arsham Broomand Saeed	Shaid Bahonar Kerman University	Aliasghar Talebi Rostami	University of Mazandaran
Hossein Jafari	University of Mazandaran	Mohammad Hamidi	Payame Noor University
Doostali Mojdeh	University of Mazandaran	Mohammad Ali Hossein Zadeh	Science and Technology Amol University
Rahim Zare Nahandi	Tehran University	Maryam Shams Solary	Payame Noor University
Saman Babayi	Semnan University	Mehdi Jafari	Payame Noor University
Ahmad Erfanian	University of Mashhad	Nader Kanzi	Payame Noor University
Taber Ghaemi Honari	Kharazmi University	Hadi Rezazadeh	Science and Technology Amol University
Aghileh Heidari	Payame Noor University	Akbar Hashemi Borzabadi	Science and Technology Mazandaran University
Ghasem Alizadeh Afroози	University of Mazandaran	Amihesam Zaeem	Payame Noor University
Kazem Khashayarmanesh	University of Mashhad	Akbar Rezayi	Payame Noor University
Maziar Salahi	University of Guilan	Peyman Ghiasvand	Payame Noor University
Majid Eshaghi Gorji	Semnan University	Farkhoneh Farzalipour	Payame Noor University
Majid Mirzavaziri	University of Mashhad	Fahimeh Soltanian	Payame Noor University
Mohsen Alimohammadi	University of Mazandaran	Esmael Nikoofar	Payame Noor University
Yahya Talebi Rostami	University of Mazandaran	Saeed Mirvakili	Yazd University
Javad Vahidi	Science and Technology of Tehran	Saeed Rajayi	Payame Noor University
Saber Ghasempour	Payame Noor University	Mostafa Amini	Payame Noor University
Allahbakhsh Yazdani Charati	University of Mazandaran	Mohammadreza Forootan	Payame Noor University
Jafar Poormahmood	Madani Azarbaygan University	Alireza Vahidi	Payame Noor University
Jahangir Cheshmavar	Payame Noor University	Shahram Rezayi	Payame Noor University
Habibeh Nazif	Payame Noor University	Reza Ameri	Tehran University
Davood Darvishi Salookolayi	Payame Noor University	Kamal Fallahi	Kordesatan University
Mehdi Rafiee Rad	University of Mazandaran	Seyed Ahmad Edalatpanah	Ayandegan University
Seyed Hashem Rasooli	Noshishvani Thecnology University	Hamidreza Yoosefzadeh	Payame Noor University

# Executive Committee of Conference

<b>Saber Ghasempour</b>	<b>Mostafa Ramzani</b>	<b>Nazli Besharati</b>
<b>Davood Darvishi Salookolayi</b>	<b>Aliasghar Shaabanzadeh</b>	<b>Mehdi Jafari</b>
<b>Mostafa Nouri Jouybari</b>	<b>Mohammad Reza Mohammadian</b>	<b>Seyed Hosein Jafari Petroodi</b>
<b>Seyed Mehdi Mirhosseini</b>	<b>Rahim Montazeri</b>	<b>Mohammad Hamidi</b>
<b>Masoomeh Hezarjaribi</b>	<b>Elmira Moosavi</b>	<b>Mohammad Dehghandar</b>
<b>Parisa Hemmatian</b>	<b>Fatemeh Mohajervatan</b>	<b>Hadi Razaghian</b>
<b>Mohammad Reza Najafi</b>	<b>Reza Hashaem adeh</b>	<b>Akbar Rezayi</b>
<b>Hamid Reza Mardani</b>	<b>Seyed Hasan Hashemi</b>	<b>Hadi Shokoohi Amiri</b>
<b>Mojtaba Kakooyi</b>	<b>Mohtaram Yoosefi</b>	<b>Maryam Shams Sloary</b>
<b>Zeinab Kazemi</b>	<b>Seyed Mohammad Hasan Mehrjoo</b>	<b>Fereshteh Samadi</b>
<b>Elham Noroozi</b>	<b>Javad Shafinejad</b>	<b>Aram Azizi</b>
<b>Amir Nabipoor</b>	<b>Roya Royintan</b>	<b>Maryam Alizadeh Afrouzi</b>
<b>Alireza Arefi</b>	<b>Kosar Bidel</b>	<b>Peyman Ghiasvand</b>
<b>Mehdi Valizadeh</b>	<b>Sahar Badakhshan</b>	<b>Farkhondeh Farzalipoor</b>
<b>Reza Shakerian</b>	<b>Rasool Ebrahimi</b>	<b>Adeleh Fallah Talooki</b>
<b>Maryam Akbari</b>	<b>Zahra Amini</b>	<b>Kamal Fallahi</b>
<b>Tahereh Borzoyi</b>	<b>Seyed Zahra Kazem Nejad</b>	<b>Nade Kanzi</b>
<b>Sirus Jahanpanah</b>	<b>Ramtin Farivar</b>	<b>Ehsan Golzadeh</b>
<b>Iman Hosseinpoor</b>	<b>Gohar Shakoori</b>	<b>Elahe Mohammadzadeh</b>
<b>Ami Abbas Heidari</b>	<b>Bentolhoda Ravanbakhsh</b>	<b>Arash Kakoolarimi</b>
<b>Leila Khanmohammadi</b>	<b>Mehdi Rahmani</b>	<b>Akbar Rezayi</b>
<b>Ehsan Dejkam</b>	<b>Parastoo Niksefat</b>	<b>Maryam Malvandi</b>
<b>Ahmad Rezayi</b>	<b>Zahra Yazdani Charati</b>	<b>Khadijeh Ghaziani</b>
<b>Seyed Mostafa Razavi</b>	<b>Zahrasadat Khalili</b>	<b>Seyed Ali Kazemnejad</b>

# Program of Conference

ROOM 13	ROOM 12	ROOM 11	ROOM 10	ROOM 9	ROOM 8	ROOM 7	ROOM 6	ROOM 5	ROOM 4	ROOM 3	ROOM 2	ROOM 1	روز	ساعت
کاربرد ریاضی در سایر علوم	آمار ریاضی مالی معادلات	بهینه سازی، کنترل، کاربرد ریاضی در سایر علوم	تحقیق در عملیات	هندسه توپولوژی	آنالیز عددی	آنالیز ریاضی، آنالیز تابعی، کاربرد ریاضی در سایر علوم	آنالیز ریاضی	آنالیز ریاضی	گراف آموزش ریاضی	جبر ریاضیات فازی	جبر	جبر		
<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>	<a href="#">لینک</a>		پنج شنبه
۸۰۶۹	۷۸۷۴	۷۹۹۵	۷۹۵۷	۷۸۸۲	۷۸۸۹	۷۲۷۶	۷۵۵۰	۷۸۹۳	۷۹۲۵	۷۳۸۸	۷۲۶۲	۷۹۶۲	پنج شنبه	۱۴:۰۰-۱۴:۲۰
۷۹۰۰	۷۷۲۴	۷۹۳۲	۷۹۵۶	۷۸۴۰	۷۸۸۷	۷۲۶۸	۷۵۴۲	۷۸۸۳	۷۸۷۸	۷۳۸۷	۷۳۴۰	۷۸۴۹	پنج شنبه	۱۴:۲۰-۱۴:۴۰
۷۸۸۸	۷۵۳۷	۷۹۳۱	۷۹۵۴	۷۸۱۹	۷۸۸۱	۷۱۴۰	۷۵۲۳	۷۸۷۷	۷۷۴۷	۷۳۹۷	۷۳۳۰	۷۷۸۱	پنج شنبه	۱۴:۴۰-۱۵:۰۰
۷۸۷۰	۷۵۳۲	۷۸۰۳	۷۹۳۹	۷۸۱۷	۷۸۷۹	۷۸۵۰	۷۵۱۹	۷۸۶۵	۷۵۵۷	۷۲۹۱	۷۳۰۲	۷۷۷۳	پنج شنبه	۱۵:۰۰-۱۵:۲۰
۷۸۶۷	۷۴۵۲	۷۷۹۳	۷۹۲۶	۷۸۰۴	۷۸۴۲	۷۷۸۷	۷۵۰۱	۷۸۶۲	۷۵۵۴	۷۲۴۵	۷۲۹۸	۷۷۵۲	پنج شنبه	۱۵:۲۰-۱۵:۴۰
													پنج شنبه	استراحت ۱۵:۴۰-۱۶:۰۰
۷۸۴۵	۷۲۱۸	۷۶۸۲	۷۸۹۰	۷۶۶۴	۷۷۵۳	۷۵۳۴	۷۴۶۰	۷۸۴۸	۷۵۵۳	۷۲۴۴	۷۲۹۰	۷۶۳۹	پنج شنبه	۱۶:۰۰-۱۶:۲۰
۷۸۳۱	۷۲۷۵	۷۶۸۱	۷۸۸۶	۷۶۴۹	۷۶۴۳	۷۳۹۵	۷۴۴۶	۷۸۴۷	۷۵۲۴	۷۲۰۲	۷۲۷۸	۷۵۸۸	پنج شنبه	۱۶:۲۰-۱۶:۴۰
۷۸۳۰	۷۲۶۶	۷۶۷۴	۷۸۵۴	۷۶۴۲	۷۶۳۰	۷۳۹۴	۷۴۱۳	۷۸۳۲	۷۵۲۲	۷۱۳۸	۷۲۴۶	۷۵۶۶	پنج شنبه	۱۶:۴۰-۱۷:۰۰
۷۸۲۵	۷۲۵۳	۷۶۷۳	۷۸۴۴	۷۵۹۹	۷۵۷۸	۷۲۶۲	۷۴۱۲	۷۸۲۱	۷۵۱۸	۸۰۸۳	۷۲۰۵	۷۵۳۶	پنج شنبه	۱۷:۰۰-۱۷:۲۰
۷۸۲۴	۷۲۲۸	۷۵۱۵	۷۷۱۶	۷۵۸۳	۷۵۷۲	۷۶۰۹	۷۲۹۱	۷۷۲۰	۷۵۰۰	۷۷۰۲	۷۷۰۵	۷۵۲۵	پنج شنبه	۱۷:۲۰-۱۷:۴۰
													جمعه	لینک جلسات
۷۸۱۶	۷۲۰۰	۷۴۹۱	۷۶۷۹	۷۵۱۷	۷۵۷۰	۷۵۷۶	۷۲۴۸	۷۷۰۶	۷۴۸۵	۷۶۲۸	۷۷۷۹	۷۵۲۰	جمعه	۰۸:۰۰-۰۸:۵۰
۷۸۰۷	۷۶۹۳	۷۴۰۹	۷۶۳۵	۷۴۹۴	۷۵۶۸	۷۵۴۹	۷۳۴۲	۷۶۶۳	۷۴۷۰	۷۶۳۷	۷۷۷۰	۷۵۰۹	جمعه	۰۸:۵۰-۰۹:۱۰
۷۷۸۰	۷۶۲۶	۷۳۹۶	۷۵۹۱	۷۴۶۸	۷۵۴۵	۷۴۲۱	۷۳۳۹	۷۶۵۲	۷۲۵۱	۷۶۲۹	۷۵۶۳	۷۴۷۷	جمعه	۰۹:۱۰-۰۹:۳۰
۷۷۶۷	۷۸۹۹	۷۳۹۳	۷۵۳۰	۷۸۵۷	۷۵۴۲	۷۳۳۷	۷۳۲۲	۷۶۲۲	۷۶۲۰	۷۴۸۳	۷۵۳۹	۷۴۷۳	جمعه	۰۹:۳۰-۰۹:۵۰
													جمعه	استراحت ۱۰:۱۰-۰۹:۵۰
۷۷۰۳	۷۸۹۶	۷۲۲۴	۷۵۱۳	۷۸۵۶	۷۵۳۳	۷۳۱۰	۷۳۱۲	۷۶۱۹	۷۳۵۱	۷۴۴۳	۷۵۱۱	۷۴۳۲	جمعه	۱۰:۱۰-۱۰:۳۰
۷۶۶۶	۷۸۴۱	۷۱۸۹	۷۴۳۹	۷۵۴۴	۷۵۰۲	۷۲۳۳	۷۲۸۴	۷۶۰۳	۷۲۱۳	۷۴۳۱	۷۴۷۵	۷۴۱۴	جمعه	۱۰:۳۰-۱۰:۵۰
۷۶۴۵	۷۱۴۹	۷۴۰۲	۷۳۰۱	۷۴۹۶	۷۲۱۸	۷۲۷۷	۷۵۸۴			۷۱۴۶	۷۴۶۳	۷۳۸۹	جمعه	۱۰:۵۰-۱۱:۱۰

## Table of Content

Number	Scope	Code	Title	First Name	Last Name	Page
1	Algebra	7173	On the Generalized Essential and Superfluous Submodules	Alireza	Najafizadeh	1
2	Algebra	7174	On the Generalized Principally Injective Modules	Alireza	Najafizadeh	5
3	Algebra	7202	On the right exactness of Cousin functors	Alireza	Vahidi	9
4	Algebra	7203	Finiteness properties of extension functors of generalized local cohomology modules	Alireza	Vahidi	13
5	Algebra	7204	On the finiteness of extension functors of generalized local cohomology modules	Alireza	Vahidi	17
6	Algebra	7205	Finiteness dimensions of generalized local cohomology modules	Alireza	Vahidi	21
7	Algebra	7244	Isomorphism Theorems for Grey S-Acts	Masoomeh	Hezarjaribi	25
8	Algebra	7245	New Results on Injective Measurable S-Acts	Masoomeh	Hezarjaribi	29
9	Algebra	7246	Some Properties of Flatness and Stability in S-Posets over Pomonoids	Masoomeh	Hezarjaribi	33
10	Algebra	7254	On the d-Dimension of Finitely Generated Modules	Sayed Hamid	Masoudi	37
11	Algebra	7256	p-Solvable of Finite Groups under Special Prime $p = 7$	Sara	Pouyandeh	42
12	Algebra	7278	A note on commuting mappings	Amir Hosein	Mokhtari	45
13	Algebra	7290	Some results on the top general local cohomology modules	seadat ollah	Faramarzi	48
14	Algebra	7291	Cohomological dimension filtration of an R-module	seadat ollah	Faramarzi	53
15	Algebra	7297	Topics in 1-Absorbing Prime Substructures	Mohammad Ali	Dehghanizadeh	58
16	Algebra	7298	The index of the n-center subgroup of an n-abelian group	Azam	Hokmabadi	61
17	Algebra	7302	On Weakly 1-Absorbing Prime Subsemimodules: Triple-Zero Phenomena	Mohammad Ali	Dehghanizadeh	64
18	Algebra	7309	1-Absorbing Prime Subsemimodules: Extended Triple-Zero Structures and Localization Effects	Mohammad Ali	Dehghanizadeh	68
19	Algebra	7340	(n,m)-cotorsion modules respect to a semidualizing module	Mostafa	Amini	72
20	Algebra	7363	Some results on mj-clean rings and strongly mj-clean rings	Mansoorah	Maani-Shirazi	76
21	Algebra	7387	An approach to graded fuzzy primary hyperideals	Peyman	Ghiasvand	79
22	Algebra	7388	On pseudo 2-prime hyperideals in multiplicative hyperrings	Peyman	Ghiasvand	83
23	Algebra	7389	Some results on graded S-prime hyperideals	Peyman	Ghiasvand	87
24	Algebra	7414	Foxby Equivalence relative to $FP_n$ -flat (n,m)-cotorsion modules	Mostafa	Amini	91
25	Algebra	7432	ON C-EXTENDING MODULES	Amirmohammad	Momeni Kohestani	95
26	Algebra	7463	Graded S-superfluous and Graded S-essential submodules	Saeed	Rajae	99
27	Algebra	7473	A subspace of homogeneous preserving star operations	Parviz	Sahandi	103
28	Algebra	7475	Embeddings of finite maximal parabolic subgroups of symplectic group	Maryam	Ghorbani	107
29	Algebra	7477	The rational character table of the group $K22(2n)$	Maryam	Ghorbani	112
30	Algebra	7509	Some quotients of the Rees Algebra	Nematollah	Shirmohammadi	116
31	Algebra	7510	Vector space generated by the multiplicative commutators of a division ring	Asghar	Madadi	120
32	Algebra	7511	Homological properties of some quotients of the Rees Algebra	Nematollah	Shirmohammadi	123
33	Algebra	7520	Derived Length Bounds for Soluble-by-Finite Subgroups in Valued Division Algebras	Mehran	Motiee	127
34	Algebra	7521	Conditions on Lie ideals in rings	Asghar	Madadi	131
35	Algebra	7525	Algorithmic Framework for Symbolic-Numeric Sign Analysis of Mathematical Expressions	Mahdi	Dehghani Darmian	134
36	Algebra	7536	Nakayama Lemma in the Category Act-S	Kamal	Ahmadi	138
37	Algebra	7539	Dedekind-finite Acts over Monoids	Kamal	Ahmadi	142
38	Algebra	7563	Formal Concept for EQ-algebras	Sirus	Jahanpanah	145

39	Algebra	7566	Formal Concept for BL-algebras	Sirus	Jahanpanah	<a href="#">129</a>
40	Algebra	7588	Some Results In Fuzzy Groupoid-Poset Structure	Karim	Ghadimi	<a href="#">153</a>
41	Algebra	7639	Some upper bounds for the index of the second $n$ -center subgroup of an $n$ -abelian group	Azam	Hokmabadi	<a href="#">157</a>
42	Algebra	7705	Various Types of Injectivity in the Categories of Krasner $(m, n)$ -ary Hypermodules	Najmeh	Jafarzadeh	<a href="#">161</a>
43	Algebra	7752	Classification of generalized Heisenberg 3-Lie algebras of dimension 10 and rank 2, 3	Zari	Ahmadieh	<a href="#">166</a>
44	Algebra	7770	The structures of the prime graphs of almost simple groups related to $PSL_{\{3\}}(p^2)$ -Part(1)	Masoumeh	Sajjadi	<a href="#">170</a>
45	Algebra	7773	The structures of the prime graphs of almost simple groups related to $PSL_{\{3\}}(p^2)$ -Part(2)	Masoumeh	Sajjadi	<a href="#">174</a>
46	Algebra	7779	Constructing Graphs from EQ-algebras Using Operation Inverters	Marzieh	Rahmati	<a href="#">178</a>
47	Algebra	7781	On Dihypergraphs and Applications	Marzieh	Rahmati	<a href="#">182</a>
48	Algebra	7839	A generalization of a converse of Schur's theorem in the variety of $n$ -abelian groups	Azam	Hokmabadi	<a href="#">186</a>
49	Algebra	7849	Some aspects of certain automorphism groups of finite $p$ -groups	Rasoul	Soleimani	<a href="#">189</a>
50	Algebra	7962	Grey Congruences on S-Acts: A Structural and Categorical Approach	Zohreh	Habibi	<a href="#">193</a>
51	Algebra	7964	A note on central reversible group ring	Maryam	Masoudi Arani	<a href="#">197</a>
52	Algebra	8174	Some Objects Of Category Of Grey Sets	Zohreh	Habibi	<a href="#">200</a>
53	Algebra	8177	Morphic S-Acts over Commutative Monoids with Unique Zero	Zohreh	Habibi	<a href="#">202</a>
54	Application	7149	On Asymmetric Structures of FGM Copulas: Measures of Association and Properties	Hakim	Bekrizadeh	<a href="#">207</a>
55	Application	7310	Computational Algebraic Analysis of 3R Manipulators	Mahdi	Dehghani Darmian	<a href="#">211</a>
56	Application	7337	Minimization of Total Mechanical Rotary Energy required for High-Pressure Natural Gas Transmission Using Thermodynamic Optimization	Seyed Hossain	Ebrahimi	<a href="#">215</a>
57	Application	7549	Mathematical SIV epidemic model with standard incidence rate and temporary-perfect vaccination	Mahmood	Parsamanesh	<a href="#">219</a>
58	Application	7576	An Adaptive Perona-Malik Model with Coupled Gradient Weight and Gradient-Dependent Fidelity	Neda	Mohamadi	<a href="#">223</a>
59	Application	7609	Performance analysis of Mamdani and Takagi-Sugeno fuzzy systems for estimating the systolic blood pressure gradient in women using Persian medicine pulsology	Mohammad	Dehghandar	<a href="#">227</a>
60	Application	7632	New perspectives on differential and integral calculus	Mohammad	Jafari	<a href="#">236</a>
61	Application	7767	Discrete-time SIV epidemic model with constant recruitment rate and population size	Mahmood	Parsamanesh	<a href="#">239</a>
62	Application	7807	Optimizing One-Stage Crowd Detection: An Enhanced SSD with Residual and Trainable Downsampling Blocks	Seyedsaeid	Mirkamali	<a href="#">243</a>
63	Application	7824	Equation of Motion in $f(R)$ Modified Gravity with Cosmological Constant for imperfect fluid	Mojtaba	Safdarian	<a href="#">247</a>
64	Application	7825	Equation of Motion and Extra Force and Generalized Raychaudhuri equation in $f(R, T)$ Modified Gravity	Mojtaba	Safdarian	<a href="#">250</a>
65	Application	7888	ANFIS-Based Modeling of Pulse Parameters for Objective Determination of Warmness, Coldness, Wetness, Dryness, and Heart Strength in Persian Medicine	Mohammad	Dehghandar	<a href="#">253</a>
66	Application	7891	Gradient-Weighted Diffusion PDE for Image Denoising	Neda	Mohamadi	<a href="#">262</a>
67	Application	7900	Three-Manifolds and the Global Geometry of the Universe: A Mathematical Approach to Cosmic Topology	Parviz	Asefi Nazarlou	<a href="#">266</a>
68	Differential Equation	7841	Exact finite difference and nonstandard finite difference schemes for solving the Fitzhugh-Nagumo equation	Mehran	Namjoo	<a href="#">270</a>
69	Differential Equation	7859	Dynamics analysis of a novel 3D chaotic system	Pooneh	Omidiniya	<a href="#">275</a>
70	Differential Equation	7896	Spectrum of a Sturm-liouville operator with discontinuities at two points	Isa	Dehghani	<a href="#">279</a>

71	Differential Equation	7899	Completeness of eigenfunctions of a discontinuous Sturm-Liouville problem with eigenparameter-dependent boundary conditions	Isa	Dehghani	<a href="#">283</a>
72	Differential Equation	8664	On solutions of nonlinear Langevin equation featuring three fractional orders with anti-periodic boundary conditions	Abdol Hadi	Dabbaghian	<a href="#">287</a>
73	Functional Analysis	7262	Polynomial convexity of the spectrum of generators and its extensions on linearly complete regular algebras	Ali	Morovatpoor	<a href="#">291</a>
74	Functional Analysis	7300	Babenko's attitude based on new sequential Game-Caputo differential equations via measures of noncompactness	Bahram	Agheli	<a href="#">296</a>
75	Functional Analysis	7383	Generalization of Darbo's fixed point theorem via the measure of noncompactness for solving a system of implicit integral equations	Mozhgan	Amirsoleimani	<a href="#">300</a>
76	Functional Analysis	7394	Majorization and Berezin symbol for bounded linear operators in Hilbert space	Mohammadr eza	Foroutan	<a href="#">302</a>
77	Functional Analysis	7395	Analysis via A-valued reproducing kernel Hilbert spaces and applications	Mohammadr eza	Foroutan	<a href="#">308</a>
78	Functional Analysis	7482	Fixed point Results for $\Phi$ -set contractions in Menger spaces	Mahboobe	Tatari	<a href="#">312</a>
79	Functional Analysis	7534	New set contractions in Menger spaces via KKM principle	Mahboobe	Tatari	<a href="#">316</a>
80	Functional Analysis	7787	Iterative Methods for Equilibrium Problems in Banach Spaces	Mostafa	Ghadampour	<a href="#">320</a>
81	Functional Analysis	7850	Structure and Operators in a Class of Weighted Sequence Spaces	Hadi	Khodabakhshian	<a href="#">322</a>
82	Fuzzy Mathematics	7407	Some properties of BCH-algebras	Amir	Shomali	<a href="#">328</a>
83	Fuzzy Mathematics	7431	On Fuzzy Small Submodules and Fuzzy Direct Sums of Modules	Behnam	Talae	<a href="#">332</a>
84	Fuzzy Mathematics	7443	Triangular Fuzzy Solution of the Fuzzy Heat Equation	Hesamoddin	Saadati	<a href="#">338</a>
85	Fuzzy Mathematics	7483	Unitary semi-ring on finite nexuses	Vajiheh	Nazemi Niya	<a href="#">340</a>
86	Fuzzy Mathematics	7637	Modeling Uncertainty in Logistics: Application of Interval-Valued Trapezoidal Fuzzy Numbers in Transportation Optimization	Madineh	Farnam	<a href="#">342</a>
87	Fuzzy Mathematics	7638	A Theoretical and Computational Study of a Defuzzification Function for the Generalized Fuzzy mTSP	Madineh	Farnam	<a href="#">348</a>
88	Fuzzy Mathematics	7702	Interior Anti-Ideals in BE-Algebras	Somayeh	Borhani Nejad Raveni	<a href="#">352</a>
89	Fuzzy Mathematics	8659	Pattern Recognition in Student Career Selection	Amal	Kumar Adak	<a href="#">356</a>
90	Geometry	7494	Riemann solitons properties on Morris-Thorne wormhole spacetime	Sajiyeh	Safaeinejad	<a href="#">361</a>
91	Geometry	7517	Riemann Solitons on the Peres Metric	Adele	Alemahdi	<a href="#">368</a>
92	Geometry	7580	On characterizations of hypersurfaces in a Kenmotsu space form	Maryam	Jedali Bahram	<a href="#">369</a>
93	Geometry	7581	On characterizations of hypersurfaces in a Sasakian space form	Maryam	Jedali Bahram	<a href="#">373</a>
94	Geometry	7583	On characterizations of Contact CR Submanifolds of maximal Contact CR dimension of Sasakian Space Form	Maryam	Jedali Bahram	<a href="#">377</a>
95	Geometry	7599	G- Manifolds with Negative Curvature	Hamed	Soroush	<a href="#">381</a>
96	Geometry	7642	Organic 3D Forms with Local Learning in Parametric Design	Parisa	Hemmatian Dehkordi	<a href="#">388</a>
97	Geometry	7649	On 1-maximal hypersurfaces in Lorentzian hyperbolic 5-space	Sara	Hoseinpour	<a href="#">389</a>
98	Geometry	7664	A Reinforcement Learning Approach for Generating Fractal Curves and Surfaces	Parisa	Hemmatian Dehkordi	<a href="#">393</a>
99	Geometry	7804	A Parabolic Analysis of Arches under Uniform Vertical load	Parisa	Hemmatian Dehkordi	<a href="#">397</a>
100	Geometry	7817	On the Expected Dimension of Brill–Noether Loci of Stable Rank-2 Vector Bundles with Fixed Determinant	Parviz	Asefi Nazarlou	<a href="#">401</a>
101	Geometry	7819	Non-Abelian Brill–Noether Loci of Stable Rank-2 Bundles with Fixed Determinant: The Case $k = 2$	Parviz	Asefi Nazarlou	<a href="#">403</a>
102	Geometry	7882	3D Cup Modeling Using Bessel–Gamma Functions and Octopus Machine Learning	Parisa	Hemmatian Dehkordi	<a href="#">407</a>
103	Graph and Combinatorics	7251	Edge Coalitions in Graphs	Nazli	Besharati	<a href="#">411</a>

104	Graph and Combinatorics	7382	Perfect Star Packings in One Type of (2, 6)-Fullerene Graphs	Meysam	Taheri-Dehkordi	<a href="#">415</a>
105	Graph and Combinatorics	7403	Degree-based topological indices of sedative drugs with M-polynomial	Hamideh	Aram	<a href="#">419</a>
106	Graph and Combinatorics	7485	NL-coloring of Vertex-Deleted Subgraphs	Ali	Ghanbari	<a href="#">423</a>
107	Graph and Combinatorics	7500	Fuzzy E-Banhatti Index and Its Application to Autism Drugs	Mahsa	Sadeghi	<a href="#">427</a>
108	Graph and Combinatorics	7505	K-distance degree index for Dendrimer	Hamideh	Aram	<a href="#">431</a>
109	Graph and Combinatorics	7518	Doubly resolving set of a class of graphs on dihedral group	Ali	Zafari	<a href="#">434</a>
110	Graph and Combinatorics	7522	Existence and fundamental properties of dominating induced matchings	Akram	Mahmoodi	<a href="#">437</a>
111	Graph and Combinatorics	7524	Some relations between dominating induced matchings and other graph parameters	Akram	Mahmoodi	<a href="#">441</a>
112	Graph and Combinatorics	7554	The relationship between the Sombor index and other topological indices	Fatemeh	Heidari	<a href="#">444</a>
113	Graph and Combinatorics	7557	A GENERALIZATION OF INJECTIVE COLORING	Shahrzad Sadat	Mirdamad	<a href="#">448</a>
114	Graph and Combinatorics	7747	Some more properties of the graph $SC^*(M)$	Saeed	Rajaei	<a href="#">452</a>
115	Graph and Combinatorics	7925	A Generalization of Lifting Properties for Directed Graphs	Majid	Kowkabi	<a href="#">457</a>
116	Graph and Combinatorics	8597	Some properties of Scalar Graphs Product Graphs on Modules	Mostafa	Nouri Jouybari	<a href="#">461</a>
117	Graph and Combinatorics	8598	Automorphism Groups of Scalar Product graph of Modules	Mostafa	Nouri Jouybari	<a href="#">465</a>
118	Mathematical Analysis	7140	Inequalities for sector matrices and matrix monotone functions	Hamideh	Mohammadzadehkan	<a href="#">468</a>
119	Mathematical Analysis	7144	Inequalities involving interpolation paths for sectorial matrices	Hamideh	Mohammadzadehkan	<a href="#">472</a>
120	Mathematical Analysis	7268	A Theoretical Study of Multipliers in Semisimple Banach Algebras	Marjan	Adib	<a href="#">476</a>
121	Mathematical Analysis	7276	Lie mappings on some matrix rings	AmirHossein	Mokhtari	<a href="#">480</a>
122	Mathematical Analysis	7277	Linear mappings through commutative zero products	AmirHossein	Mokhtari	<a href="#">483</a>
123	Mathematical Analysis	7284	Topological Structures on the Space of Module Actions in Banach Algebras	Marjan	Adib	<a href="#">486</a>
124	Mathematical Analysis	7311	Investigation Hyers-Ulam stability of hyper 3-homomorphisms and hyper 3-derivations in complex ternary algebras	Parastoo	Heiatian Naeini	<a href="#">490</a>
125	Mathematical Analysis	7312	Generalized Hyers-Ulam Stability of an Additive Quadratic Functional Equation in 2-Banach Space	Parastoo	Heiatian Naeini	<a href="#">494</a>
126	Mathematical Analysis	7332	Solvability of an infinite system of the fractional order nonlinear fractional Hadamard functional integral equations in the Banach algebra $(C(J))^\infty$	Nayereh	Gholamian	<a href="#">498</a>
127	Mathematical Analysis	7339	Existence Results of a Nonlinear Fractional Integral Equation in the Space of Regulated Functions	Nayereh	Gholamian	<a href="#">502</a>
128	Mathematical Analysis	7342	Solvability of an Infinite System of Differential Equations of Order n in the space $bvq$	Mahnaz	Khanehgir	<a href="#">506</a>
129	Mathematical Analysis	7348	Continuity Results of the Boundary Spectrum in p-Banach Algebras	Ali	Naziri-Kordkandi	<a href="#">510</a>
130	Mathematical Analysis	7349	A note on CQ-algebras	Ali	Naziri-Kordkandi	<a href="#">514</a>
131	Mathematical Analysis	7391	Boundedness of composition operators on weighted variable exponent Bergman spaces on polydisk	Ali	Morovatpoor	<a href="#">517</a>
132	Mathematical Analysis	7412	On weakly interpolative Kannan contractions	Hosein	Lakzian	<a href="#">521</a>

133	Mathematical Analysis	7460	A subclass of p-valent meromorphic functions associated by defined q differential operator	Aboalghasem	Alishahi	<a href="#">526</a>
134	Mathematical Analysis	7462	Certaion subclass of p-valent meromorphic multivalent q -starlike and q-convex sunctions with q -differential operator	Aboalghasem	Alishahi	<a href="#">530</a>
135	Mathematical Analysis	7464	On the Ma-Minda Starlikeness of the Polynomial $(1+xz)^3$	Vali	Soltani Masih	<a href="#">534</a>
136	Mathematical Analysis	7488	Fixed Point Approximations for Mappings in CAT(0) Spaces	Hossein	Soleimani	<a href="#">538</a>
137	Mathematical Analysis	7519	Hybrid Algorithm for Generalized Mixed Equilibrium Problems and Fixed Points of W-Mappings in Banach Spaces	Mostafa	Ghadampour	<a href="#">542</a>
138	Mathematical Analysis	7540	The Stable Fixed Point Property for Continuous Mappings in Metric Trees	Hossein	Soleimani	<a href="#">546</a>
139	Mathematical Analysis	7543	Existence of coupled best approximation point for almost cyclic contractive mappings	Khadije	Amiri	<a href="#">550</a>
140	Mathematical Analysis	7550	The class of meromorphic $\{\boldsymbol{\alpha};\boldsymbol{\varphi}\}\text{mbox{-}}\{\text{starlike}\}$ functions for $\{\boldsymbol{p}\}\text{hspace{1pt}}\{\text{-valency}$	Vali	Soltani Masih	<a href="#">554</a>
141	Mathematical Analysis	7584	Remarks on the paper: "Best proximity point theorems for some classes of contractions"	Khadije	Amiri	<a href="#">558</a>
142	Mathematical Analysis	7603	Some fixed point theorems for $(\zeta, \eta)$ -weakly cyclic contraction mappings in b-metric spaces	Nayereh	Gholamian	<a href="#">562</a>
143	Mathematical Analysis	7611	Independece property of Parseval continuous K-g-frames	Narjes sadat	Banitaba	<a href="#">566</a>
144	Mathematical Analysis	7619	Resulted Sharp for a New Subclass of Ma-Minda Starlike Functions	Seyed Hadi	Sayedain Boroujeni	<a href="#">570</a>
145	Mathematical Analysis	7622	Bionomial Distribution and its Associated New Ma-Minda Type Class	Seyed Hadi	Sayedain Boroujeni	<a href="#">574</a>
146	Mathematical Analysis	7640	SOME FIXED POINT THEOREMS IN SEQUENTIAL G-METRIC SPACES AND THEIR APPLICATION TO THE SOLVABILITY OF A SYSTEM OF INTEGRAL EQUATIONS	Hasan	Hosseinzadeh	<a href="#">578</a>
147	Mathematical Analysis	7646	On Ruscheweyh q-derivative operator associated with fractional integral operator	Ebrahim	Amini	<a href="#">582</a>
148	Mathematical Analysis	7652	Fixed point theorems for Jleli--Samet--Kannan contractions in metric spaces	Zeinab	Izadi	<a href="#">586</a>
149	Mathematical Analysis	7663	On MT -cyclic Ćirić quasicontractions	Hosein	Lakzian	<a href="#">590</a>
150	Mathematical Analysis	7665	AN APPLICATION OF FIXED POINT THEOREMS TO DYNAMIC PROGRAMMING IN A GENERALIZED DISTANCE SPACE	Hasan	Hosseinzadeh	<a href="#">594</a>
151	Mathematical Analysis	7821	Conditional Expectation Operators and Kato Property Between $L_p$ -spaces	Sannar	Khalil Sarbaz	<a href="#">598</a>
152	Mathematical Analysis	7832	Weak fixed point property for Kannan type mappings with respect to orbits in Banach spaces	Kamal	Fallahi	<a href="#">602</a>
153	Mathematical Analysis	7862	On the Loewner differential equations in geometric functions theory	Hormoz	Rahmatan	<a href="#">606</a>
154	Mathematical Analysis	7877	Fixed point theorems with $\alpha$ -admissible mapping in metric spaces	Hossein	Rahimpoor	<a href="#">610</a>
155	Mathematical Analysis	7883	A Study of Composition Operators on Hardy-Type Spaces	Sannar	Khalil Sarbaz	<a href="#">614</a>
156	Mathematical Analysis	7893	Fixed Point Results for Graphical $(\psi\text{-}\varphi)$ -Asymptotic Contractions in Metric Spaces with a Graph	Kamal	Fallahi	<a href="#">617</a>
157	Mathematical finance	7626	Heston Model Calibration: Implementation and Empirical Evaluation	Somayeh	Fallah	<a href="#">621</a>
158	Mathematical finance	7693	Portfolio Optimization Interdiction: A Novel Risk-Averse Vision for Risk Management	Javad	Tayyebi	<a href="#">625</a>
159	Mathematics Learning	7243	The Impact of AI in Teaching Undergraduate Mathematics	Seyed Amjad	Samareh Hashemi	<a href="#">629</a>

160	Numerical Analysis	7198	Determinants for certain band matrices by the theory of matrix polynomials	Maryam	Shams Solary	<a href="#">633</a>
161	Numerical Analysis	7199	The primary value for matrix Lambert W function	Maryam	Shams Solary	<a href="#">637</a>
162	Numerical Analysis	7230	A two-dimensional Boubaker polynomial expansion scheme for the numerical solution of the nonlinear Schrödinger equation in (2+1) dimensions	Farshad	Mehdifar	<a href="#">641</a>
163	Numerical Analysis	7242	A Hybrid Fast Numerical Method for the Lane-Emden Differential Equation Using GHFs	Seyed Amjad	Samareh Hashemi	<a href="#">645</a>
164	Numerical Analysis	7329	Numerical solution of pantograph differential equations using the double Sinc method	Mansour	Shiralizadeh	<a href="#">649</a>
165	Numerical Analysis	7367	Numerical solution of singular Volterra-Fredholm integro-differential equations using the Sinc method	Mansour	Shiralizadeh	<a href="#">653</a>
166	Numerical Analysis	7496	Solutions of modified Fornberg--Whitham equation via the $\tan(\phi/2)$ -expansion method	Mehdi	Fazli Aghdaei	<a href="#">657</a>
167	Numerical Analysis	7502	Analytical treatment of the Black-Scholes equation for European option pricing by Saul'yev finite difference scheme	Mehdi	Fazli Aghdaei	<a href="#">661</a>
168	Numerical Analysis	7542	Construction of Operational Matrix for Solving Non-linear Fractional Differential Equations Via Genocchi Polynomials and Collocation Method	Azar Sadat	Shabani	<a href="#">665</a>
169	Numerical Analysis	7545	He's semi-inverse variational method for the fifth-order integrable equations	Mehdi	Fazli Aghdaei	<a href="#">669</a>
170	Numerical Analysis	7564	An iterative method for computing eigenpairs of symmetric matrices	Behzad	Kafash	<a href="#">673</a>
171	Numerical Analysis	7568	A Boundary Element Method for the Numerical Solution of the Variable-Order Time-Fractional Diffusion Equation Based on the Caputo Derivative	Leila	Hasani	<a href="#">677</a>
172	Numerical Analysis	7570	A Boundary Element Method Approach to Fractional Diffusion-Reaction Equations with Caputo Derivative: Applications to Mathematical Biology	Leila	Hasani	<a href="#">681</a>
173	Numerical Analysis	7572	A local RBF-FD scheme for the two-dimensional fractional integro-differential equation with weakly singular kernel	Samira	Eslami	<a href="#">685</a>
174	Numerical Analysis	7578	APPLICATION OF LOCAL MESHLESS MOVING KRIGING METHOD FOR SOLVING 2D STOCHASTIC ADVECTION-DIFFUSION EQUATIONS	Zahra	Jeihouni	<a href="#">689</a>
175	Numerical Analysis	7630	Rational Radial Basis Functions for Solving Ordinary Differential Equations	Mansour	Shiralizadeh	<a href="#">693</a>
176	Numerical Analysis	7643	A review of the various fields of wavelet applications	Aram	Azizi	<a href="#">697</a>
177	Numerical Analysis	7753	Fibonacci-Based Spectral Method for Caputo Fractional PDEs	Shahed	Mashhoodi	<a href="#">701</a>
178	Numerical Analysis	7842	A spectral framework for numerical solution of the time-fractional heat conduction problems	Younes	Talaei	<a href="#">705</a>
179	Numerical Analysis	7887	A Numerical method of Elliptic-Parabolic Integro-Differential Equations Arising in Applied Problems	Neda	Najafzadeh	<a href="#">710</a>
180	Numerical Analysis	7889	Analysis of Time-Dependent Fractional Integro-Differential Equations under the Maximum Principle	Neda	Najafzadeh	<a href="#">713</a>
181	Operation Resaerch	7402	Dynamic Starting Point Updating for Guaranteed Convergence in Non-Convex Optimization	Tajedin	Derikvand	<a href="#">717</a>
182	Operation Resaerch	7439	Optimizing Multi-Level Aspirations through Multi-Segment Goal Programming	Parastoo	Niksefat	<a href="#">720</a>
183	Operation Resaerch	7512	Integer reverse obnoxious median facility location models on tree graphs	Sepideh	Mohammadi	<a href="#">724</a>
184	Operation Resaerch	7513	A novel algorithm for general minimum cost inverse obnoxious median location optimization on graphs	Sepideh	Mohammadi	<a href="#">727</a>

185	Operation Resaerch	7635	Solving the flow-shop scheduling problem using a hybrid nature-inspired optimization algorithm	Habibeh	Nazif	<a href="#">739</a>
186	Operation Resaerch	7679	A novel most reliable path problem for data transmission	Javad	Tayyebi	<a href="#">736</a>
187	Operation Resaerch	7844	An Enhanced Feasible Value Constraint for Multiobjective Optimization Problems	Hossein	Salmei	<a href="#">740</a>
188	Operation Resaerch	7854	Solving initial value problems using artificial neural networks and collective intelligence	Fatemeh	Ahmadkhanpour	<a href="#">744</a>
189	Optimization And Control	7491	Theoretical Perspectives and Open Problems in Generalized Semi-Infinite (Multiobjective) Nonsmooth Optimization	Nader	Kanzi	<a href="#">748</a>
190	Optimization And Control	7673	Numerical method for time-invariant delay optimal control problems	Seyed Mehdi	Mirhosseini-Alizami	<a href="#">752</a>
191	Optimization And Control	7682	A Genetic Optimization Approach to Brachytherapy Dose Rate Planning with Fuzzy Constraints	Mohammad	Mohammadi Najafabadi	<a href="#">756</a>
192	Optimization And Control	7803	Piecewise orthogonal function neural network: A general framework for function approximation	Ghasem	Ahmadi	<a href="#">760</a>
193	Optimization And Control	7931	A Mathematical Model for cancer by using immunotherapy	Maryam	Nikbakht	<a href="#">764</a>
194	Optimization And Control	7932	A Mathematical Model for cancer by using chemotherapy	Maryam	Nikbakht	<a href="#">768</a>
195	Optimization And Control	7995	Research on Controllability and Observability of Discrete-time Linear System with Interval Coefficients	Hadi	Shokohi Amiri	<a href="#">772</a>
196	Statistics	7228	Bayesian and E-Bayesian Estimation for Rayleigh Distribution Using Progressive Type-II Censored	Kazem	Fayyaz Heidari	<a href="#">776</a>
197	Statistics	7265	Bayesian Inference for the Power Modified Lindley Distribution based on Symmetric and Asymmetric Balanced Loss Functions	Adeleh	Fallah	<a href="#">779</a>
198	Statistics	7266	Confidence Intervals for the Power Modified Lindley Distribution based on Progressive Type-II Censoring Samples	Adeleh	Fallah	<a href="#">783</a>
199	Statistics	7318	Hierarchical estimation of Rayleigh distribution	Kazem	Fayyaz Heidari	<a href="#">787</a>
200	Statistics	7451	The effect of the loss function on Bayes Estimator and the posterior risk for the exponential distribution	Elham	Basiri	<a href="#">791</a>
201	Statistics	7452	Comparing three different censoring schemes from the perspective of experimental design cost	Elham	Basiri	<a href="#">795</a>
202	Statistics	7486	A new distribution as a combination of Rayleigh and log-series distributions	Sajjad	Piradl	<a href="#">799</a>
203	Statistics	7498	A Case Study of Concomitants of Ordered Random via Generalized Exponential Distributions Variables from Bairamov Family	Reza	Akbari	<a href="#">803</a>
204	Statistics	7499	On introduction of Poisson-Pranav Distribution	Reza	Akbari	<a href="#">807</a>
205	Statistics	7532	A new generalization of the log-Rayleigh probability distribution	Sajjad	Piradl	<a href="#">811</a>
206	Statistics	7537	A new extension of the generalized beta probability distribution	Sajjad	Piradl	<a href="#">815</a>
207	Statistics	7724	A 15-year Research Profile of the Bulletin of the Iranian Mathematical Society (2008-2022): a Sci-entometric Approach	Heidar	Mokhtari	<a href="#">819</a>
208	Topology	7301	Existence of Solutions for Sequential Liouville-Caputo Fractional Differential Equations	Bahram	Agheli	<a href="#">829</a>
209	Topology	7544	Generalization of the Topological Interior Operator via Stacks	Ali	Shahrezaei	<a href="#">833</a>
210	Topology	7775	The equivalent conditions for QHC-subspace and QHC-subset	Fatemeh	Heidari	<a href="#">837</a>

211	Topology	7856	When $C_{\infty}(X)$ is an ideal of $C_c(X)$	Somayeh	Soltanpour	841
212	Topology	7857	$C_{\infty}(X)$ and Related Ideals in $C_c(X)$	Somayeh	Soltanpour	844







# Algebra



## On the Generalized Essential and Superfluous Submodules

Alireza Najafizadeh\*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: najafizadeh@pnu.ac.ir

**ABSTRACT.** In this talk the concepts of essential and superfluous submodules are recalled. Moreover, some generalizations of such notions concerning the results of the author and his colleagues in their later research are investigated.

**Keywords:** Essential, superfluous, projective

**AMS Mathematics Subject Classification [2020]:** 13C99

### 1. Introduction

Throughout this talk all ring are commutative with identity and all modules are unitary. Among principal classes of submodules in module theory one may cite essential and superfluous submodules with many important properties and applications. Moreover, such notions have been generalized from various points of view by several authors over the past decades. Let  $M$  be a module over a commutative ring  $R$  with  $S = \text{End}_R(M)$ . A submodule  $N$  of  $M$  is called small or superfluous, denoted by  $N \ll M$ , if for any submodule  $X$  of  $M$ ,  $X + N = M$  implies that  $X = M$ . Following Amouzegar-Kalati and Keskin-Tütüncü [1],  $N$  is called annihilator small (a-small), denoted by  $N \ll_a M$ , if for any submodule  $X$  of  $M$ ,  $X + N = M$  implies that  $(0 :_S M) = 0$ . Moreover,  $N$  is called essential, denoted by  $N \subseteq_e M$ , provided that for each submodule  $L$  of  $M$ ,  $N \cap L = 0$  implies that  $L = 0$ . Following Amouzegar [2],  $N$  is called  $S$ -essential, denoted by  $N \subseteq_{s-e} M$ , provided that for each submodule  $X$  of  $M$ ,  $N \cap X = 0$  implies that  $(X :_S M) = 0$ . An  $R$ -module  $M$  is called retractable if for any non-zero submodule  $N$  of  $M$  there exists a non-zero  $\alpha : M \rightarrow N$ . Moreover,  $M$  is called co-retractable if for any proper submodule  $N$  of  $M$ , there exists a non-zero  $\alpha : M \rightarrow M$  such that  $\alpha(N) = 0$ . Let  $U$  and  $M$  be  $R$ -modules. Then,  $U$  is said to be  $M$ -projective in case for each  $R$ -epimorphism  $g : M \rightarrow N$  and each  $R$ -homomorphism  $f : U \rightarrow N$  there exists an  $R$ -homomorphism  $h : U \rightarrow M$  such that  $f = gh$ . If  $U$  is  $U$ -projective, then  $U$  is said to be quasi-projective. If  $U$  is  $M$ -projective for every  $R$ -module  $M$ , then  $U$  is said to be projective. A projective cover of a module  $M$  is an epimorphism  $p : P \rightarrow M$  such that  $P$  is a projective module and  $\ker(p) \ll P$ . Following [5], a submodule  $N$  of an  $R$ -module  $M$  is called epi-superfluous in  $M$ , denoted by  $N \leq_{epi} M$ , if for any submodule  $L$  of  $M$ ,  $L + N = M$  implies that  $L$  is an epimorphic

\*Speaker.

image of  $M$ . An epimorphism  $g : M \rightarrow N$  is called epi-superfluous if  $\ker g \leq_{\text{epi}} M$ . An  $R$ -module  $M$  is said to be epi-Noetherian (epi-Artinian) if for every ascending (descending) chain  $M_1 \leq M_2 \leq M_3 \leq \dots$  ( $M_1 \geq M_2 \geq M_3 \geq \dots$ ) of submodules of  $M$ , there exists an index  $n \geq 1$  such that for every  $i \geq n$ , there exists an epimorphism  $\phi_i : M_{i+1} \rightarrow M_i$  ( $\phi_i : M_i \rightarrow M_{i+1}$ ). The trace of  $M$  in  $N$ , which is denoted by  $\text{Tr}_N(M)$ , is defined as  $\text{Tr}_N(M) = \sum \{\text{Im}(\varphi) : \varphi \in \text{Hom}_R(M, N)\}$ . In this talk we state some generalizations of the results in [3] related mainly to essential and superfluous submodules. The results we present here are parts of the author's works as well as his colleagues' in [4].

## 2. Main results

We begin with the following result which is in [4] as Theorem 3.26. We give the proof for the sake of completeness.

**Theorem 2.1.** *Let  $M$  and  $P$  be modules over a commutative ring  $R$  and  $P \xrightarrow{\theta} M$ . Let  $N$  be a submodule of  $M$  such that  $\theta(P) + N = M$ . Then,  $\theta^{-1}(N) \ll P$  exactly if any submodule  $P' \subseteq P$  satisfying  $\theta(P') + N = M$  implies that  $P' = P$ .*

PROOF. Let  $\theta^{-1}(N)$  be a superfluous submodule of  $P$ . Moreover, let  $P' \subseteq P$  be such that  $\theta(P') + N = M$ . We prove that  $P' = P$ . We claim that  $P' + \theta^{-1}(N) = P$ . To do this, suppose that  $y \in P$  be arbitrary. Therefore, we have  $\theta(y) \in M = \theta(P') + N$ . Hence, there exist elements  $z \in P'$  and  $n \in N$  such that  $\theta(y) = \theta(z) + n$ . This implies that  $y - z \in \theta^{-1}(N)$ . Therefore,  $y = z + (y - z) \in P' + \theta^{-1}(N)$ . Consequently,  $P \subseteq P' + \theta^{-1}(N)$ . We conclude that  $P' + \theta^{-1}(N) = P$ . Now the hypothesis that  $\theta^{-1}(N)$  is a superfluous submodule of  $P$ , yields  $P' = P$ . Conversely, we prove that  $\theta^{-1}(N)$  is a superfluous submodule of  $P$ . Let  $L$  be a submodule of  $P$  such that  $\theta^{-1}(N) + L = P$ . This implies that  $\theta(\theta^{-1}(N) + L) = \theta(P)$ . Hence  $\theta(\theta^{-1}(N)) + \theta(L) = \theta(P)$ . We get  $\theta(\theta^{-1}(N)) + \theta(L) + N = \theta(P) + N = M$ . The fact that  $\theta(\theta^{-1}(N)) \subseteq N$ , implies that  $\theta(L) + N = M$ . Consequently, the hypothesis implies that  $L = P$ .  $\square$

The following result is concluded from Theorem 2.1.

**Corollary 2.2.** *Let  $P \xrightarrow{\theta} M \rightarrow 0$  be an exact sequence of  $R$ -modules. Then,  $\ker(\theta)$  is a superfluous submodule of  $P$  exactly if any submodule  $P' \subseteq P$  satisfying  $\theta(P') = M$  implies that  $P' = P$ . In particular,  $P \xrightarrow{\theta} M \rightarrow 0$  is a projective cover for  $M$  exactly if any submodule  $P' \subseteq P$  satisfying  $\theta(P') = M$  implies that  $P' = P$ .*

PROOF. See [4, Corollary 3.27].  $\square$

The following result appears in [4] as Theorem 3.28.

**Theorem 2.3.** *Let  $M$  and  $P$  be modules over a commutative ring  $R$  and  $P \xrightarrow{\theta} M$ . Let  $N$  be a submodule of  $M$  such that  $\theta(P) + N = M$ . Then,  $\theta^{-1}(N) \ll P$  exactly if any module  $A$  and any  $R$ -homomorphism  $A \xrightarrow{h} P$  satisfying  $\theta h(A) + N = M$  imply that  $h(A) = P$ .*

PROOF. Let  $\theta^{-1}(N)$  be a superfluous submodule of  $P$ . In view of Theorem ??, we put  $P' = h(A)$ . Then, we have  $h(A) = P$ . Conversely, we prove that  $\theta^{-1}(N)$  is a superfluous submodule of  $P$ . In view of Theorem ??, let  $\theta(P') + N = M$  for some submodule  $P'$  of  $P$ . This implies that  $\theta i(P') + N = M$ , where  $P' \xrightarrow{i} P \xrightarrow{\theta} M$ . The hypothesis implies that  $i(P') = P$ . Consequently,  $P' = P$ .  $\square$

The following result is a consequence of Theorem 2.3.

**Corollary 2.4.** *An epimorphism  $g : M \rightarrow N$  is superfluous if and only if for all homomorphisms (equivalently, monomorphisms)  $h$ , if  $gh$  is epic, then  $h$  is epic.*

PROOF. See [3, Corollary 5.15]. □

The following result which is also an immediate consequence of Theorem 2.3, appears in [3] as Proposition 5.14.

**Proposition 2.5.** *For a submodule  $K$  of  $M$  the following statements are equivalent:*

- (1)  $K \ll M$ .
- (2) *The natural map  $p_K : M \rightarrow M/K$  is a superfluous epimorphism.*
- (3) *For every module  $N$  and for every  $h \in \text{Hom}(N, M)$  the relation  $\text{Im}h + K = M$  implies  $\text{Im}h = M$ .*

PROOF. Put  $N = 0$  in Theorem 2.3 and suppose that  $\theta = p_K$ . Then we have  $p_K^{-1}(0) \ll M$  exactly if any module  $A$  and any  $R$ -homomorphism  $A \xrightarrow{h} M$  satisfying  $p_K h(A) + 0 = M/K$  implies that  $h(A) = M$ . But we have  $M/K = p_K h(A) = (h(A) + K)/K$ . This implies that  $K \ll M$  exactly if any module  $A$  and any  $R$ -homomorphism  $A \xrightarrow{h} M$  satisfying  $h(A) + K = M$  we have  $h(A) = M$ . □

The results investigated above have some consequences in the generalizations of superfluous and essential submodules.

**Lemma 2.6.** *Let  $M$  be a module over a commutative ring  $R$  with  $S = \text{End}_R(M)$ . Let  $T$  be a submodule of  $M$  such that  $(T :_S M)$  is an essential ideal of  $S$ . Then,  $(T :_S M)M$  is an  $S$ -essential submodule of  $M$ . In particular,  $T$  is an  $S$ -essential submodule of  $M$ .*

PROOF. Let  $(T :_S M)M \cap X = 0$  for some submodule  $X$  of  $M$ . Clearly, we have  $(T :_S M) \cap (X :_S M) = 0$ . Now, the hypothesis that  $(T :_S M)$  is an essential ideal of  $S$ , implies that  $(X :_S M) = 0$ . We conclude that  $(T :_S M)M$  is an  $S$ -essential submodule of  $M$ .

Moreover, we observe that if  $N$  is a submodule of  $T$  such that  $N$  is an  $S$ -essential submodule of  $M$ , then  $T$  is an  $S$ -essential submodule of  $M$ . Now, the last assertion is clear. □

**Proposition 2.7.** *Let  $M$  be a module over a commutative ring  $R$  with  $S = \text{End}_R(M)$ . If  $\text{Tr}_S(M)$  is an essential (left) ideal of  $S$ , then we have the following.*

- (1)  $(0 :_M \text{Tr}_S(M))$  is an  $a$ -small submodule of  $M$ .
- (2) *If  $M$  is a co-retractable module, then  $(0 :_M \text{Tr}_S(M))$  is a small submodule of  $M$ .*

PROOF. Let  $(0 :_M \text{Tr}_S(M)) + X = M$  for some submodule  $X$  of  $M$ . Then, we have  $0 = (0 :_S M) = (0 :_S [(0 :_M \text{Tr}_S(M)) + X]) = [0 :_S (0 :_M (\text{Tr}_S(M)))] \cap (0 :_S X)$ . Now in view of the fact that  $\text{Tr}_S(M) \subseteq [0 :_S (0 :_M (\text{Tr}_S(M)))]$ , we have  $\text{Tr}_S(M) \cap (0 :_S X) \subseteq [0 :_S (0 :_M (\text{Tr}_S(M)))] \cap (0 :_S X) = 0$ . Therefore,  $\text{Tr}_S(M) \cap (0 :_S X) = 0$ . But,  $\text{Tr}_S(M)$  is an essential ideal of  $S$ , hence  $(0 :_S X) = 0$  as desired. For the second assertion, suppose that  $(0 :_M \text{Tr}_S(M)) + X = M$ . By the way of contradiction, suppose that  $X \neq M$ . Since  $M$  is co-retractable, there exists a non-zero  $\beta : M \rightarrow M$  such that  $\beta(M) = 0$ . Therefore,  $\beta \in (0 :_S X)$ . This means  $(0 :_S X) \neq 0$ , a contradiction. □

At this point, we give some results concerning epi-superfluous modules which are generalized superfluous submodules from another point of view.

**Lemma 2.8.** *Let  $M$  an  $R$ -module. Then  $M$  is epi-Artinian (epi-Noetherian) exactly if for every non-empty set  $\Sigma$  of submodules of  $M$ , there exists  $N \in \Sigma$  such that for every submodule  $K \leq N$  ( $N \leq K$ ) if  $K \in \Sigma$ , then  $K$  is a homomorphic image of  $N$  ( $N$  is a homomorphic image of  $K$ ).*

PROOF. See [5, Lemma 2.6]. □

**Corollary 2.9.** *Every epi-Artinian module contains an epi-superfluous submodule.*

PROOF. See [5, Corollary 2.7]. □

**Theorem 2.10.** *The following statements are equivalent for an  $R$ -module  $M$ .*

- (1) *Every epi-superfluous submodule of  $M$  is epi-Artinian (epi-Noetherian).*
- (2) *Let  $\Sigma$  be a non-empty set of epi-superfluous submodules of  $M$ . Then, there exists  $L \in \Sigma$  such that for every  $K \leq L$  ( $L \leq K$ ), if  $K \in \Sigma$ , then  $K$  ( $L$ ) is a homomorphic image of  $L$  ( $K$ ).*
- (3)  *$M$  is epi-Artinian (epi-Noetherian) on its epi-superfluous submodules.*

PROOF. See [5, Theorem 2.8]. □

**Proposition 2.11.** *Let  $R$  be an epi-Artinian domain and  $F$  be a finitely generated free  $R$ -module. Then there exists an epi-superfluous epimorphism  $\phi : F \rightarrow N$  for every finitely generated  $R$ -module  $N$ .*

PROOF. See [5, Proposition 2.9]. □

**Corollary 2.12.** *Let  $R$  be a ring and  $\{M_i\}_{i=1}^n$  be a finite family of  $R$ -modules. Then  $\bigoplus_{i=1}^n M_i$  is epi-Artinian (epi-Noetherian) on epi-superfluous submodules if and only if each  $M_i$  is epi-Artinian (epi-Noetherian) on epi-superfluous submodules.*

PROOF. See [5, Corollary 2.17]. □

### 3. Conclusion

In this talk some generalizations of the concepts of essential and superfluous submodules are investigated. It is observed that some results are true in the generalized forms provided that further conditions are imposed. Most of the results are both the author's and his colleagues works in their later research in [4].

### Acknowledgement

The author would like to thank Payame Noor University of I. R. of Iran (PNU) for the financial support during the preparation of this conference paper.

### References

1. Amouzegar, T. (2016) *S-Essential Submodules*, Gen. Math. Notes, **34** (1), 56-63.
2. Amouzegar-Kalati, T. and Keskin-Tütüncü, D. (2013) *Annihilator-small submodules*, Bulletin of the Iranian Math. Soc., **39** (6), 1053-1063.
3. Anderson, F. W. and Fuller, K. R. (1974), *Rings and categories of modules*, Springer-Verlag.
4. Khaksari, A., Najafzadeh, A. and Zafarkhah, M. (2022) *The dual notion of the square submodule of a module*, J. Math. Ext. **16** (11), 1-16.
5. Nikandish, R. and Amini, M. (2024) *Epi-superfluous submodules and epi-artinian modules*, Rend. Circ. Mat. Palermo, **73** (3), 1059-1072.



## On the Generalized Principally Injective Modules

Alireza Najafizadeh\*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: najafizadeh@pnu.ac.ir

---

**ABSTRACT.** In this talk we present some of the results related to principally injective modules over arbitrary rings. Moreover, it is proved that the results are valid to some other extended injectivity conditions which may be defined over modules.

**Keywords:** Principal, injective, projective

**AMS Mathematics Subject Classification [2020]:** 13C99

---

### 1. Introduction

Throughout this paper all rings  $R$  are associative with unity and all modules are unitary right modules unless otherwise stated. Recall that a ring  $R$  is called right principally injective (or right  $p$ -injective for short) if, every homomorphism from a principally right ideal of  $R$  to  $R$  can be extended to an endomorphism of  $R$ . This concept has been generalized to modules in various ways. Following [2], a module  $M$  is called principally quasi-injective (or PQ-injective for short) if, each homomorphism from a principal submodule of  $M_R$  to  $M_R$  can be extended to an endomorphism of  $M_R$ . Given the right  $R$ -modules  $M$  and  $N$ , we say that  $M$  is  $N$ -injective if every homomorphism from a submodule of  $N$  to  $M$  can be extended to a homomorphism from  $N$  to  $M$ . Moreover, we say that  $M$  is principally  $N$ -injective if every homomorphism from a cyclic submodule of  $N$  to  $M$  can be extended to a homomorphism of  $N$  to  $M$ . An  $R$ -module  $M$  is called quasi-injective if it is  $M$ -injective. Over the past decades, there have been several achievements in relation to  $N$ -injective and principally  $N$ -injective modules specially in the case of  $p$ -injective rings and PQ-injective modules. In this talk we present some of the results related to principally injective modules over arbitrary rings. Moreover, it is proved that the results are valid to some other extended injectivity conditions which may be defined over modules. The results are parts of the author's work in [1].

### 2. Main results

If  $M_R$  is a module, we write  $l_M(r) = \{m \in M : mr = 0\}$  for all  $r \in R$ , and  $r_R(m) = \{r \in R : mr = 0\}$  for all  $m \in M$ . Given the  $R$ -modules  $M$  and  $N$ , the trace of  $M$  in  $N$  denoted by  $Tr_N(M)$ , is defined as  $Tr_N(M) = \sum \{\text{Im}(\varphi) \mid \varphi \in \text{Hom}_R(M, N)\}$ . We say

---

\*Speaker.

that  $M$  is (semi-)  $N$ -injective if every homomorphism  $f : K \rightarrow M$  with  $K$  a ( $N$ -cyclic) submodule of  $N$ , can be extended to a homomorphism  $g : N \rightarrow M$ . The module  $M$  is called weakly  $N$ -injective if for every finitely generated submodule  $K \subset N^{(\mathbb{N})}$  all homomorphisms  $f : K \rightarrow M$  can be extended to  $g : N^{(\mathbb{N})} \rightarrow M$ . An  $R$ -module  $M$  is called a Quasi-Frobenius module or a QF module if  $M$  is weakly  $M$ -injective and a weak cogenerator in  $\sigma[M]$ . We recall that a right  $R$ -module  $M$  is a duo module if every submodule of  $M$  is fully invariant. Moreover, an  $R$ -module  $M_R$  is called strongly duo if  $Tr_M(N) = N$  for every submodule  $N$  of  $M_R$ . A proper submodule of a right  $R$ -module  $M$  is called completely prime if for each  $r \in R$  and every  $m \in M$  such that  $mr \in P$ , we have  $m \in P$  or  $Mr \subseteq P$ .

**Lemma 2.1.** *Let  $M$  and  $N$  be right modules over a ring  $R$ . Then the following statements are equivalent.*

- (1)  $M_R$  is principally  $N$ -injective.
- (2)  $l_M r_R(n) = \text{Hom}_R(N, M).n$  for all  $n \in N$ .

PROOF.  $1 \implies 2$ ) Clearly,  $\text{Hom}_R(N, M).n \subseteq l_M r_R(n)$ . Now, suppose  $m \in l_M r_R(n)$ . This implies that the map  $f : nR \rightarrow M$  defined as  $f(nt) = mt$  is correctly defined. Hence, the hypothesis gives a homomorphism  $g : N \rightarrow M$  such that  $g(n) = f(n)$ . Consequently,  $m = f(n) = g(n) = g.n \in \text{Hom}_R(N, M).n$ .

$2 \implies 1$ ) Let  $f : nR \rightarrow M$  be a homomorphism for some  $n \in N$ . We have  $f(n) \in l_M r_R(n) = \text{Hom}_R(N, M).n$ . Then, there exists a homomorphism  $g : N \rightarrow M$  such that  $f(n) = g(n)$ , that is  $f$  extends to  $g$ .  $\square$

**Lemma 2.2.** *Let  $M$  and  $N$  be right modules over a ring  $R$ . If  $M$  is a direct summand of  $N_R$ , then the following statements are equivalent.*

- (1)  $M_R$  is principally  $N$ -injective.
- (2)  $l_M r_R(n) = \text{Hom}_R(N, M).n$  for all  $n \in N$ .
- (3) If  $r_R(n_1) \subseteq r_R(n_2)$  where  $n_1, n_2 \in N$ , then we have

$$\text{Hom}_R(N, M).n_2 \subseteq \text{Hom}_R(N, M).n_1.$$

- (4)  $l_M(bR \cap r_R(n)) = l_M(b) + \text{Hom}_R(N, M).n$ , for all  $b \in R$  and  $n \in N$ .

PROOF. In view of Lemma 2.1, we need to prove the equivalences of (2) with (3) and (4).  $2 \implies 3$ ) We prove  $l_M r_R(n_2) \subseteq l_M r_R(n_1)$ , where  $n_1, n_2 \in N$ . This is done by some easy calculations.

$3 \implies 4$ ) Let  $m = m_1 + m_2$ , where  $m_1 \in l_M(b)$  and  $m_2 \in \text{Hom}_R(N, M).n$ . Then  $m = m_1 + f(n)$  for some homomorphism  $f : N \rightarrow M$  and  $n \in N$ . Now, if  $s = br \in bR \cap r_R(n)$  then,  $m.s = (m_1 b)r + f(ns) = 0$ . Conversely, suppose that  $m \in l_M(bR \cap r_R(n))$ . Then,  $m.(bR \cap r_R(n)) = 0$ . Now an easy calculation shows that  $r_R(nb) \subseteq r_R(mb)$ . Hence, the hypothesis implies that  $\text{Hom}_R(N, M).mb \subseteq \text{Hom}_R(N, M).nb$ . Now, in view of the fact that  $M$  is direct summand of  $N_R$ , we get  $\text{Hom}_R(N, N)mb \subseteq \text{Hom}_R(N, M)mb \subseteq \text{Hom}_R(N, M)nb$ . Since  $1_N \in \text{Hom}_R(N, N)$ , there exists a homomorphism  $\alpha : N \rightarrow M$  such that  $mb = 1_N(mb) = \alpha(nb) = \alpha(n)b$ . Therefore,  $m - \alpha(n) \in l_M(b)$ . Finally, we have  $m = m - \alpha(n) + \alpha(n) \in l_M(b) + \text{Hom}_R(N, M).n$ .

$4 \implies 2$ ): Set  $b = 1$ .  $\square$

**Corollary 2.3.** *Given a module  $M_R$  with  $S = \text{End}(M_R)$ , the following are equivalent:*

- (1)  $M_R$  is PQ-injective.
- (2)  $l_M r_R(m) = Sm$ .
- (3) If  $r_R(m) \subseteq r_R(n)$  where  $m, n \in M$ , then  $Sn \subseteq Sm$ .

(4)  $l_M(bR \cap r_R(m)) = l_M(b) + Sm$ , for all  $b \in R$  and  $m \in M$ .

PROOF. The equivalence of assertions (1), (2) and (3) follows in case we set we set  $N = M$  in Lemma 2.2. To show the equivalence of (3) and (4), let  $n = m_1 + m_2$ , where  $m_1 \in l_M(b)$  and  $m_2 \in Sm$ . Then  $m = m_1 + f(n)$  for some  $m \in M$  and  $f \in S$ . If  $s = br \in bR \cap r_R(m)$  then,  $ns = m_1(br) + f(m) = (m_1b)r + f(ms) = 0$ . Conversely, suppose that  $n \in l_M(bR \cap r_R(m))$ . Then,  $n(bR \cap r_R(m)) = 0$ . Now an easy calculation shows that  $r_R(mb) \subseteq r_R(nb)$ . Consequently, there exists an  $f \in S$  such that  $nb = f(mb) = f(m)b$ . This yields  $n - fm \in l_M(b)$ , hence  $n \in l_M(b) + Sm$ .

4  $\implies$  2): Set  $b = 1$ . □

**Proposition 2.4.** *Let  $M_R$  be a module over a ring  $R$  with  $S = \text{End}(M_R)$ . Then the following statements are equivalent:*

- (1) *If  $r_R(m) \subseteq r_R(n)$  where  $m, n \in M$ , then  $Sn \subseteq Sm$ .*
- (2) *Let  $N$  and  $K$  be submodules of  $M_R$  and let  $f : N \rightarrow K$  be a homomorphism. If  $y = f(x)$  for some  $x \in N$  and  $y \in K$ , then  $Sy \subseteq Sx$ .*
- (3) *Let  $N$  and  $K$  be submodules of  $M_R$  and let  $f : N \rightarrow K$  be a homomorphism. If  $y = f(x)$  for some  $x \in N$  and  $y \in K$ , then there exists some  $\beta : M_R \rightarrow M_R$  such that  $y = \beta(x)$ .*

PROOF. See [1, Proposition 3.9]. □

**Proposition 2.5.** *Let  $\{R_i\}_{i \in I}$  be a collection of rings. Suppose that  $\{M_i\}_{i \in I}$  and  $\{N_i\}_{i \in I}$  be collections of modules such that  $M_i$  and  $N_i$  are  $R_i$ -modules for every  $i \in I$ . Let  $R = \prod_{i \in I} R_i$ ,  $M = \prod_{i \in I} M_i$  and  $N = \prod_{i \in I} N_i$ . Then,  $M$  is principally  $N$ -injective as an  $R$ -module exactly if each  $M_i$  is principally  $N_i$ -injective as  $R_i$ -module for all  $i \in I$ .*

PROOF.  $\implies$ ) Clear.

$\impliedby$ ) We show that  $M = \prod_{i \in I} M_i$  is principally  $(\prod_{i \in I} N_i)$ -injective. In view of Lemma 2.2, we need to show that  $l_M(r_R(n)) = \text{Hom}_R(\prod_{i \in I} M_i, \prod_{i \in I} N_i)n$  for all  $n = \{n_i\}_{i \in I} \in N$ . Suppose that  $m = \{m_i\}_{i \in I} \in l_M(r_R(n))$ , then  $m \cdot r_R(n) = 0$ . Clearly,  $m_i \in l_{M_i}r_{R_i}(n_i)$  for all  $i \in I$ . Now in view of the fact that each  $M_i$  is principally  $N_i$ -injective as  $R_i$ -module, we have  $l_{M_i}r_{R_i}(n_i) = \text{Hom}_{R_i}(M_i, N_i)n_i$ . Therefore,  $m_i = \alpha_i n_i$  for some  $\alpha_i \in \text{Hom}_{R_i}(M_i, N_i)$  and  $i \in I$ . Clearly, for  $\alpha = \{\alpha_i\}_{i \in I}$  we have  $m = \{m_i\}_{i \in I} = \{\alpha_i n_i\}_{i \in I} = \alpha \{n_i\}_{i \in I} = \alpha(n)$ , which yields the result. □

**Corollary 2.6.** *Let  $\{R_i\}_{i \in I}$  be a collection of rings and  $\{M_i\}_{i \in I}$  be a collection of modules such that  $M_i$  is an  $R_i$ -module for all  $i \in I$ . Let  $R = \prod_{i \in I} R_i$  and  $M = \prod_{i \in I} M_i$ . Then,  $M$  is PQ-injective as an  $R$ -module exactly if each  $M_i$  is PQ-injective as  $R_i$ -module.*

PROOF. Set  $N = M$  in Proposition 2.5. □

**Proposition 2.7.** *Let  $M_R$  and  $N_R$  be modules over a ring  $R$ . Suppose that  $N_R$  is faithful and  $\{0\}$  is a completely prime submodule of  $N_R$ . Then,  $M_R$  is principally  $N$ -injective exactly if  $\text{Hom}_R(N, M)n = M$  for all non zero  $n \in N$ .*

PROOF. See [1, Proposition 3.15]. □

**Corollary 2.8.** *Let  $M_R$  be a faithful module over a ring  $R$  with  $S = \text{End}(M_R)$ . Suppose that  $\{0\}$  is a completely prime submodule of  $M$ . Then,  $M_R$  is PQ-injective exactly if  ${}_S M$  is simple.*

PROOF. Set  $N = M$  in Proposition 2.7. □

**Proposition 2.9.** *Let  $M_R$  be a module over a ring  $R$  with  $S = \text{End}(M_R)$ . Moreover, let  $N_R$  be a submodule of  $M_R$  such that  $M_R$  is a principally  $N$ -injective module. Let  $X$  be the set of all  $m \in M$  such that  $\text{Hom}_R(N_R, M_R).m$  is a superfluous  $I$ -submodule of  ${}_I M$ , where  $I = \text{Hom}(N_R, M_R)$ . Then  $X \subseteq Z(M_R)$ .*

PROOF. See [1, Corollary ]. □

**Proposition 2.10.** *Let  $M_R$  be a PQ-injective module with  $S = \text{End}(M_R)$ . Then the following are equivalent.*

- (1) *If  $r(m) \subseteq r(n)$  for some  $m$  and  $n$  in  $M$ , then  $nR \subseteq mR$ .*
- (2)  *$M_R$  is a duo module.*

PROOF.  $1 \Rightarrow 2$ ) Let  $N$  be a submodule of  $M_R$  and  $\alpha : M_R \rightarrow M_R$ . Given an element  $n \in N$ , we have  $r(n) \subseteq r(\alpha(n))$ . The hypothesis implies that  $\alpha(n) \in nR \subseteq N$ .

$2 \Rightarrow 1$ ) The hypothesis that  $r(m) \subseteq r(n)$ , implies that the map  $\alpha : mR \rightarrow nR$  defined as  $\alpha(mr) = nr$  for all  $r \in R$  is correctly defined. Therefore, there exists a homomorphism  $\bar{\alpha} : M \rightarrow M$  such that  $\bar{\alpha}i = j\alpha$ , where  $i : mR \rightarrow M$  and  $j : nR \rightarrow M$  are the inclusion maps, respectively. Hence,  $n = \alpha(m) = \bar{\alpha}(m)$ . But  $mR$  is fully invariant, so  $\bar{\alpha}(m) \in mR$  which yields  $n \in mR$ . □

**Proposition 2.11.** *Let  $M_R$  be a duo module with  $S = \text{End}(M_R)$ . If  $K$  is a direct summand of  $M_R$  such that  $M$  is principally  $K$ -injective, then  $K$  is strongly duo.*

PROOF. See [1, Corollary ]. □

**Corollary 2.12.** *Every PQ-injective duo module is strongly duo.*

PROOF. Set  $K = M$  in Proposition 2.11. □

### 3. Conclusion

The results presented in this talk are parts of the author's work in [1]. Indeed, some of the results related to principally injective modules over arbitrary rings are presented.

### Acknowledgement

This work was supported by a grant from Payame Noor University of I. R. of Iran. The author would like to thank Payame Noor University for the support during the preparation of this work.

### References

1. Gholami, F., Habibi, Z. and Najafzadeh, A. (2024) *On the generalized principally injective modules*, Bull. Korean Math. Soc., **61**(2), 301-315.
2. Nicholson, W. K., Park J. K. and Yousif M. F., *Principally quasi-injective modules*, Comm. Algebra **27** (4) (1999), 1683-1693.



## On the right exactness of Cousin functors

Alireza Vahidi\*

Department of Mathematics, Payame Noor University, Tehran, Iran

Email: vahidi.ar@pnu.ac.ir

**ABSTRACT.** Let  $R$  be a commutative Noetherian ring with non-zero identity and let  $\mathcal{F}$  be a filtration of  $\text{Spec}(R)$ . In this paper, we demonstrate that the Cousin functor with respect to  $\mathcal{F}$ , denoted by  $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \rightarrow \text{Comp}(R)$  where  $\mathcal{C}_{\mathcal{F}}(R)$  represents the category of  $R$ -modules which are admitted by  $\mathcal{F}$  and  $\text{Comp}(R)$  denotes the category of complexes of  $R$ -modules, is a right exact functor.

**Keywords:** Cousin complexes, Cousin functors, right exact functors

**AMS Mathematics Subject Classification [2020]:** 13D02, 18A22

### 1. Introduction

Let  $R$  be a commutative Noetherian ring with non-zero identity. The concept of Cousin complex was initially introduced in [2], with a corresponding algebraic analogue provided by Sharp in [3]. Subsequently, in [6], Sharp extended this framework to define the Cousin complex for an  $R$ -module  $X$  with respect to a filtration  $\mathcal{F}$  of  $\text{Spec}(R)$ , denoted by  $C_R(\mathcal{F}, X)$ . This construction has proven to be a significant analytical tool, enabling the characterization of various classes of rings and modules including Gorenstein rings, Cohen-Macaulay modules, local cohomology modules, and balanced big Cohen-Macaulay modules via properties of Cousin complexes (see, for example, [3–6]).

Let  $\mathcal{C}_{\mathcal{F}}(R)$  represent the category of  $R$ -modules which are admitted by  $\mathcal{F}$  and let  $\text{Comp}(R)$  denote the category of complexes of  $R$ -modules. In [1], Bamdad and the present author introduced the Cousin functor with respect to  $\mathcal{F}$ , defined as  $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \rightarrow \text{Comp}(R)$ . This functor was employed to construct Cousin spectral sequences with respect to  $\mathcal{F}$  and to investigate the extension functors of Cousin cohomologies (i.e., the cohomology modules of Cousin complexes). Furthermore, utilizing this functor, they established conditions equivalent to the vanishing of Cousin cohomologies and derived several results pertaining to modules with finitely generated Cousin cohomologies. In this paper, we investigate the property of right exactness of this functor.

### 2. Main results

A filtration of  $\text{Spec}(R)$  is defined as a descending sequence  $\mathcal{F} = (F_i)_{i \geq 0}$  of subsets of  $\text{Spec}(R)$ , so that  $\text{Spec}(R) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots$ , with the property that, for

\*Speaker.

all  $i \geq 0$ , each member of  $F_i \setminus F_{i+1}$  is a minimal member of  $F_i$  with respect to inclusion. We say that  $\mathcal{F}$  admits an  $R$ -module  $X$  if  $\text{Supp}_R(X) \subseteq F_0$ .

Suppose that  $\mathcal{F} = (F_i)_{i \geq 0}$  is a filtration of  $\text{Spec}(R)$  that admits an  $R$ -module  $X$ . The Cousin complex  $C_R(\mathcal{F}, X)$  for  $X$  with respect to  $\mathcal{F}$  has the following structure:

$$0 \xrightarrow{d_X^{-2}} X \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} \dots \xrightarrow{d_X^{i-3}} X^{i-2} \xrightarrow{d_X^{i-2}} X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} \dots$$

where, for all  $i \geq 0$ ,  $X^i = \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (\text{Coker } d_X^{i-2})_{\mathfrak{p}}$  and  $d_X^{i-1}(x) = \{(x + \text{Im } d_X^{i-2})/1\}_{\mathfrak{p} \in F_i \setminus F_{i+1}}$  for every element  $x$  of  $X^{i-1}$ . This complex satisfies the following properties (see [3, Proposition 2.2 and Corollary 2.3] and [6, Definitions 1.1, Definition 1.3, and Proposition 1.4]):

- (P1)  $\text{Supp}_R(X^i) \subseteq \text{Supp}_R(X) \cap F_i$ ;
- (P2)  $\text{Supp}_R(\text{Coker } d_X^{i-2}) \subseteq \text{Supp}_R(X) \cap F_i$ ;
- (P3)  $\text{Supp}_R(H^{i-1}(C_R(\mathcal{F}, X))) \subseteq \text{Supp}_R(X) \cap F_{i+1}$ ;
- (P4) The natural  $R$ -homomorphism  $\xi_{X^i} : X^i \longrightarrow \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (X^i)_{\mathfrak{p}}$ , defined by  $\xi_{X^i}(x) = \{x/1\}_{\mathfrak{p} \in F_i \setminus F_{i+1}}$ , is an isomorphism.

We adopt the convention that  $X^{-1} = X$ . Bamdad and the current author established the following lemma and used it to define the Cousin functor with respect to  $\mathcal{F}$ ,  $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \text{Comp}(R)$ , which is both  $R$ -linear and covariant (see [1, Theorem 2.2]).

LEMMA 2.1. (see [1, Lemma 2.1]) *Let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$  which admits  $R$ -modules  $X$  and  $Y$ . Then, for every  $R$ -homomorphism  $f : X \longrightarrow Y$ , there exists a unique morphism of complexes  $C_R(\mathcal{F}, f) = (f^i)_{i \geq -2} : C_R(\mathcal{F}, X) \longrightarrow C_R(\mathcal{F}, Y)$  such that  $f^{-1} = f$ . In particular, the following property holds:*

- (P5)  $\xi_{Y^i} f^i = (\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}}) \xi_{X^i}$  for all  $i \geq 0$ .

The following theorem establishes that the Cousin functor concerning  $\mathcal{F}$  is right exact.

THEOREM 2.2. *Let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$ . Then, the Cousin functor with respect to  $\mathcal{F}$ ,  $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \longrightarrow \text{Comp}(R)$ , is a right exact functor.*

PROOF. Let  $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$  be an exact sequence in  $\mathcal{C}_{\mathcal{F}}(R)$ . By Lemma 2.1, there exist morphisms of complexes  $C_R(\mathcal{F}, f) = (f^i)_{i \geq -2} : C_R(\mathcal{F}, X) \longrightarrow C_R(\mathcal{F}, Y)$  and  $C_R(\mathcal{F}, g) = (g^i)_{i \geq -2} : C_R(\mathcal{F}, Y) \longrightarrow C_R(\mathcal{F}, Z)$  such that  $f^{-1} = f$  and  $g^{-1} = g$ . Thus

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow d_X^{-2} & & \downarrow d_Y^{-2} & & \downarrow d_Z^{-2} & \\ X & \xrightarrow{f^{-1}} & Y & \xrightarrow{g^{-1}} & Z & \longrightarrow & 0 \\ & \downarrow d_X^{-1} & & \downarrow d_Y^{-1} & & \downarrow d_Z^{-1} & \\ X^0 & \xrightarrow{f^0} & Y^0 & \xrightarrow{g^0} & Z^0 & \longrightarrow & 0 \\ & \downarrow d_X^0 & & \downarrow d_Y^0 & & \downarrow d_Z^0 & \\ & \vdots & & \vdots & & \vdots & \\ & \downarrow d_X^{i-1} & & \downarrow d_Y^{i-1} & & \downarrow d_Z^{i-1} & \\ X^i & \xrightarrow{f^i} & Y^i & \xrightarrow{g^i} & Z^i & \longrightarrow & 0 \\ & \downarrow d_X^i & & \downarrow d_Y^i & & \downarrow d_Z^i & \\ & \vdots & & \vdots & & \vdots & \end{array}$$

is a commutative diagram of  $R$ -modules. Using induction on  $i$ , we demonstrate that  $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow 0$  forms an exact sequence of  $R$ -modules. The base case for  $i = -1$  is clear. Assume that  $i \geq 0$  and  $X^j \xrightarrow{f^j} Y^j \xrightarrow{g^j} Z^j \longrightarrow 0$  is exact for all  $-1 \leq j \leq i-1$ . For the inductive step, select  $\mathfrak{p} \in F_i \setminus F_{i+1}$ . By (P1) and (P3), the localized complexes

$$\begin{array}{ccccc}
 (X^{i-2})_{\mathfrak{p}} & \xrightarrow{(f^{i-2})_{\mathfrak{p}}} & (Y^{i-2})_{\mathfrak{p}} & \xrightarrow{(g^{i-2})_{\mathfrak{p}}} & (Z^{i-2})_{\mathfrak{p}} \longrightarrow 0 \\
 \downarrow (d_X^{i-2})_{\mathfrak{p}} & & \downarrow (d_Y^{i-2})_{\mathfrak{p}} & & \downarrow (d_Z^{i-2})_{\mathfrak{p}} \\
 (X^{i-1})_{\mathfrak{p}} & \xrightarrow{(f^{i-1})_{\mathfrak{p}}} & (Y^{i-1})_{\mathfrak{p}} & \xrightarrow{(g^{i-1})_{\mathfrak{p}}} & (Z^{i-1})_{\mathfrak{p}} \longrightarrow 0 \\
 \downarrow (d_X^{i-1})_{\mathfrak{p}} & & \downarrow (d_Y^{i-1})_{\mathfrak{p}} & & \downarrow (d_Z^{i-1})_{\mathfrak{p}} \\
 (X^i)_{\mathfrak{p}} & \xrightarrow{(f^i)_{\mathfrak{p}}} & (Y^i)_{\mathfrak{p}} & \xrightarrow{(g^i)_{\mathfrak{p}}} & (Z^i)_{\mathfrak{p}} \longrightarrow 0 \\
 \downarrow (d_X^i)_{\mathfrak{p}} & & \downarrow (d_Y^i)_{\mathfrak{p}} & & \downarrow (d_Z^i)_{\mathfrak{p}} \\
 0 & & 0 & & 0
 \end{array}$$

is a commutative diagram with exact columns. Since the induction hypothesis guarantees the exactness of the top two rows, it follows that

$$(X^i)_{\mathfrak{p}} \xrightarrow{(f^i)_{\mathfrak{p}}} (Y^i)_{\mathfrak{p}} \xrightarrow{(g^i)_{\mathfrak{p}}} (Z^i)_{\mathfrak{p}} \longrightarrow 0$$

is exact. Thus

$$\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (X^i)_{\mathfrak{p}} \xrightarrow{\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}}} \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (Y^i)_{\mathfrak{p}} \xrightarrow{\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (g^i)_{\mathfrak{p}}} \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (Z^i)_{\mathfrak{p}} \longrightarrow 0$$

is exact. Since, by (P5),

$$\begin{array}{ccccccc}
 X^i & \xrightarrow{f^i} & Y^i & \xrightarrow{g^i} & Z^i & \longrightarrow & 0 \\
 \downarrow \xi_{X^i} & & \downarrow \xi_{Y^i} & & \downarrow \xi_{Z^i} & & \\
 \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (X^i)_{\mathfrak{p}} & \xrightarrow{\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (f^i)_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (Y^i)_{\mathfrak{p}} & \xrightarrow{\bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (g^i)_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p} \in F_i \setminus F_{i+1}} (Z^i)_{\mathfrak{p}} & \longrightarrow & 0.
 \end{array}$$

is a commutative diagram and, by (P4),  $\xi_{X^i}$ ,  $\xi_{Y^i}$ , and  $\xi_{Z^i}$  are isomorphisms,  $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow 0$  is exact. Therefore

$$C_R(\mathcal{F}, X) \xrightarrow{C_R(\mathcal{F}, f)} C_R(\mathcal{F}, Y) \xrightarrow{C_R(\mathcal{F}, g)} C_R(\mathcal{F}, Z) \longrightarrow 0$$

is an exact sequence in  $\text{Comp}(R)$  as desired.  $\square$

**DEFINITION 2.3.** Let  $n$  be a non-negative integer and let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$  such that  $F_0 = \text{Spec}(R)$ . We define the  $n$ th Cousin functor with respect to  $\mathcal{F}$ , denoted by  $C_{n,R}(\mathcal{F}, -)$ , as the  $n$ th left derived functor of  $C_R(\mathcal{F}, -)$ . For an arbitrary  $R$ -module  $X$ , we refer to  $C_{n,R}(\mathcal{F}, X)$  as the  $n$ th Cousin complex for  $X$  with respect to  $\mathcal{F}$ .

**COROLLARY 2.4.** Let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$  such that  $F_0 = \text{Spec}(R)$ . Then,  $C_{0,R}(\mathcal{F}, -)$  is naturally equivalent to  $C_R(\mathcal{F}, -)$ .

COROLLARY 2.5. Let  $\mathcal{F} = (F_i)_{i \geq 0}$  be a filtration of  $\text{Spec}(R)$  such that  $F_0 = \text{Spec}(R)$  and let  $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$  be a short exact sequence of  $R$ -modules. Then, there exists a long exact sequence

$$\begin{array}{ccccccc} & & & & \dots & & \longrightarrow \\ & & & & & & \\ C_{n,R}(\mathcal{F}, X) & \xrightarrow{C_{n,R}(\mathcal{F},f)} & C_{n,R}(\mathcal{F}, Y) & \xrightarrow{C_{n,R}(\mathcal{F},g)} & C_{n,R}(\mathcal{F}, Z) & \longrightarrow & \\ & & & & \dots & & \longrightarrow \\ C_{1,R}(\mathcal{F}, X) & \xrightarrow{C_{1,R}(\mathcal{F},f)} & C_{1,R}(\mathcal{F}, Y) & \xrightarrow{C_{1,R}(\mathcal{F},g)} & C_{1,R}(\mathcal{F}, Z) & \longrightarrow & \\ C_R(\mathcal{F}, X) & \xrightarrow{C_R(\mathcal{F},f)} & C_R(\mathcal{F}, Y) & \xrightarrow{C_R(\mathcal{F},g)} & C_R(\mathcal{F}, Z) & \longrightarrow & 0 \end{array}$$

in  $\text{Comp}(R)$ .

The Cousin complex is a significant construct within the fields of commutative and homological algebra, serving to characterize various important notions such as Gorenstein rings, Cohen-Macaulay modules, local cohomology modules, and balanced big Cohen-Macaulay modules (see [3, Theorem 5.4], [4, Theorem 2.4], [5, Theorem], and [6, Corollary 3.7]). Since, by Corollary 2.4, the Cousin complex is isomorphic to the 0th Cousin complex, it is natural to consider the following question:

QUESTION. Let  $n$  be a positive integer. Does the  $n$ th Cousin complex serve to characterize any other concepts within commutative or homological algebra?

### 3. Conclusion

Our analysis indicates that the Cousin functor with respect to  $\mathcal{F}$ ,  $C_R(\mathcal{F}, -) : \mathcal{C}_{\mathcal{F}}(R) \rightarrow \text{Comp}(R)$  where  $\mathcal{C}_{\mathcal{F}}(R)$  is the category of  $R$ -modules which are admitted by  $\mathcal{F}$  and  $\text{Comp}(R)$  is the category of complexes of  $R$ -modules, is a right-exact functor.

### References

1. Bamdad, H. and Vahidi, A. (2018) *Extension functors of Cousin cohomology modules*, Bull. Iranian Math. Soc., **44**(2), 253–267.
2. Hartshorne, R. (1966) *Residues and Duality*, Springer-Verlag, Berlin–New York.
3. Sharp, R. Y. (1969) *The Cousin complex for a module over a commutative Noetherian ring*, Math. Z., **112**, 340–356.
4. Sharp, R. Y. (1970) *Gorenstein modules*, Math. Z., **115**, 117–139.
5. Sharp, R. Y. (1977) *Local cohomology and the Cousin complex for a commutative Noetherian ring*, Math. Z., **153**(1), 19–22.
6. Sharp, R. Y. (1982) *A Cousin complex characterization of balanced big Cohen-Macaulay modules*, Quart. J. Math. Oxford Ser. (2), **33**(132), 471–485.



# Finiteness properties of extension functors of generalized local cohomology modules

Alireza Vahidi\*

Department of Mathematics, Payame Noor University, Tehran, Iran

Email: vahidi.ar@pnu.ac.ir

**ABSTRACT.** Let  $R$  be a commutative Noetherian ring with non-zero identity,  $\mathfrak{a}$  an ideal of  $R$ ,  $M$  and  $X$  finitely generated  $R$ -modules, and  $t$  a non-negative integer. In this paper, we demonstrate that if the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ , then the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finitely generated.

**Keywords:** cofinite modules, generalized local cohomology modules, local cohomology modules

**AMS Mathematics Subject Classification [2020]:** 13D02, 13D45

## 1. Introduction

Throughout, let  $R$  denote a commutative Noetherian ring with non-zero identity,  $\mathfrak{a}$  an ideal of  $R$ ,  $M$  a finite (i.e., finitely generated)  $R$ -module,  $X$  an arbitrary  $R$ -module that may not be finite, and  $t$  a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [1].

Consider a finite  $R$ -module  $L$ . In the work referenced as [3], Hartshorne posed the following question:

**QUESTION.** *Under what conditions is the module  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(L))$  finite for all integers  $i$  and  $j$ ?*

Multiple studies have explored conditions under which the modules  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(X))$  are finite. For instance, in [2, Theorem A], Dibaei and Yassemi examined the finiteness of  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^j(X))$ . They demonstrated that  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite whenever  $\text{Ext}_R^{t+1}(R/\mathfrak{a}, X)$  is finite, and additionally, for all  $j$  and all  $i < t$ , the modules  $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  are finite.

The  $i$ th generalized local cohomology module

$$H_{\mathfrak{a}}^i(M, X) \cong \varinjlim_{n \in \mathbb{N}} \text{Ext}_R^i(M/\mathfrak{a}^n M, X)$$

\*Speaker.

of two  $R$ -modules  $M$  and  $X$  with respect to an ideal  $\mathfrak{a}$  was introduced by Herzog in [4]. Notably, when  $M = R$ ,  $H_{\mathfrak{a}}^i(R, X)$  coincides with the classical local cohomology module  $H_{\mathfrak{a}}^i(X)$ . This leads to a natural extension of Hartshorne's question:

QUESTION. *Under what conditions is the module  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M, L))$  finite for all integers  $i$  and  $j$ ?*

In this paper, we investigate this research question with particular emphasis on the finiteness properties of the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$ . Our findings seek to build upon and enhance previous work, including results such as [2, Theorem A], by extending these to the framework of generalized local cohomology modules.

## 2. Main results

In the following theorem, we establish sufficient conditions ensuring the finiteness of the  $R$ -module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$ .

THEOREM 2.1. *Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that the modules  $\text{Ext}_R^{t+1-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  are finite for all  $i \leq t+1$  and the modules  $\text{Ext}_R^{t+2-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  are finite for all  $i < t$ . Under these conditions, it follows that the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite.*

PROOF. Let

$$E^{\bullet} = 0 \longrightarrow X \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^i \longrightarrow \cdots$$

be an injective resolution of  $X$  and apply  $\text{Hom}_R(M, \Gamma_{\mathfrak{a}}(-))$  to its deletion  $E^{\bullet X}$  to get the complex

$$\text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X})) = 0 \longrightarrow \text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^0)) \longrightarrow \cdots \longrightarrow \text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^i)) \longrightarrow \cdots$$

Let

$$0 \longrightarrow \text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X})) \longrightarrow T^{\bullet, 0} \longrightarrow T^{\bullet, 1} \longrightarrow \cdots \longrightarrow T^{\bullet, i} \longrightarrow \cdots$$

be a Cartan–Eilenberg injective resolution of  $\text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X}))$  and consider the third quadrant bicomplex  $\mathcal{T} = \{\text{Hom}_R(R/\mathfrak{a}, T^{p,q})\}$ . We denote the total complex of  $\mathcal{T}$  by  $\text{Tot}(\mathcal{T})$ .

The first filtration has  ${}^I E_2$  term the iterated cohomology  $H^p H^{p,q}(\mathcal{T})$ . We have

$$\begin{aligned} H^{p,q}(\mathcal{T}) &\cong H^q(\text{Hom}_R(R/\mathfrak{a}, T^{p,\bullet})) \\ &\cong \text{Ext}_R^q(R/\mathfrak{a}, \text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^p))) \\ &\cong \text{Hom}_R(\text{Tor}_q^R(R/\mathfrak{a}, M), \Gamma_{\mathfrak{a}}(E^p)). \end{aligned}$$

Therefore

$$\begin{aligned} {}^I E_2^{p,q} &\cong H^p H^{p,q}(\mathcal{T}) \\ &\cong H^p(\text{Hom}_R(\text{Tor}_q^R(R/\mathfrak{a}, M), \Gamma_{\mathfrak{a}}(E^{\bullet X}))) \\ &\cong H_{\mathfrak{a}}^p(\text{Tor}_q^R(R/\mathfrak{a}, M), X) \end{aligned}$$

which yields the third quadrant spectral sequence

$${}^I E_2^{p,q} := \text{Ext}_R^p(\text{Tor}_q^R(R/\mathfrak{a}, M), X) \Longrightarrow_p H^{p+q}(\text{Tot}(\mathcal{T})).$$

For all  $i$ ,  $0 \leq i \leq t+1$ , we have  ${}^I E_{\infty}^{t+1-i,i} = {}^I E_{t+3}^{t+1-i,i}$  because  ${}^I E_j^{t+1-i-j, i-1+j} = 0 = {}^I E_j^{t+1-i+j, i+1-j}$  for all  $j \geq t+3$ ; so that  ${}^I E_{\infty}^{t+1-i,i}$  is finite from the fact that  ${}^I E_{t+3}^{t+1-i,i}$  is a subquotient of  ${}^I E_2^{t+1-i,i}$  which is finite by assumptions. There exists a finite filtration

$$0 = \phi^{t+2} H^{t+1} \subseteq \phi^{t+1} H^{t+1} \subseteq \cdots \subseteq \phi^1 H^{t+1} \subseteq \phi^0 H^{t+1} = H^{t+1}(\text{Tot}(\mathcal{T}))$$

such that  ${}^I E_{\infty}^{t+1-i,i} \cong \phi^{t+1-i} H^{t+1} / \phi^{t+2-i} H^{t+1}$  for all  $i$ ,  $0 \leq i \leq t+1$ . Now the exact sequences

$$0 \longrightarrow \phi^{t+2-i} H^{t+1} \longrightarrow \phi^{t+1-i} H^{t+1} \longrightarrow {}^I E_{\infty}^{t+1-i,i} \longrightarrow 0,$$

for all  $i$ ,  $0 \leq i \leq t+1$ , show that  $H^{t+1}(\text{Tot}(\mathcal{T}))$  is finite.

On the other hand, the second filtration has  ${}^{II} E_2$  term the iterated cohomology  $H''^p H^{q,p}(\mathcal{T})$ . Note that every short exact sequence of injective modules splits and so it remains split after applying the functor  $\text{Hom}_R(R/\mathfrak{a}, -)$ . By using this fact and the fact that  $T^{\bullet,\bullet}$  is a Cartan–Eilenberg injective resolution of  $\text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X}))$ , we get

$$\begin{aligned} H^{q,p}(\mathcal{T}) &\cong H^q(\text{Hom}_R(R/\mathfrak{a}, T^{\bullet,p})) \\ &\cong \text{Hom}_R(R/\mathfrak{a}, H^q(T^{\bullet,p})) \\ &\cong \text{Hom}_R(R/\mathfrak{a}, H^{q,p}). \end{aligned}$$

Therefore

$$\begin{aligned} {}^{II} E_2^{p,q} &\cong H''^p H^{q,p}(\mathcal{T}) \\ &\cong H^p(\text{Hom}_R(R/\mathfrak{a}, H^{q,\bullet})) \\ &\cong \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M, X)) \end{aligned}$$

which gives the third quadrant spectral sequence

$${}^{II} E_2^{p,q} := \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M, X)) \implies_p H^{p+q}(\text{Tot}(\mathcal{T})).$$

There exists a finite filtration

$$0 = \psi^{t+2} H^{t+1} \subseteq \psi^{t+1} H^{t+1} \subseteq \dots \subseteq \psi^1 H^{t+1} \subseteq \psi^0 H^{t+1} = H^{t+1}(\text{Tot}(\mathcal{T}))$$

such that  ${}^{II} E_{\infty}^{t+1-i,i} \cong \psi^{t+1-i} H^{t+1} / \psi^{t+2-i} H^{t+1}$  for all  $i$ ,  $0 \leq i \leq t+1$ . Since  $H^{t+1}(\text{Tot}(\mathcal{T}))$  is finite,  $\psi^1 H^{t+1}$  is finite. Hence  ${}^{II} E_{\infty}^{1,t} \cong \psi^1 H^{t+1} / \psi^2 H^{t+1}$  is finite. Therefore  ${}^{II} E_{t+3}^{1,t}$  is finite because  ${}^{II} E_{t+3}^{1,t} = {}^{II} E_{\infty}^{1,t}$  from the fact that  ${}^{II} E_j^{1-j,t-1+j} = 0 = {}^{II} E_j^{1+j,t+1-j}$  for all  $j \geq t+3$ . For all  $r \geq 2$ , let  ${}^{II} Z_r^{1,t} = \text{Ker}({}^{II} E_r^{1,t} \rightarrow {}^{II} E_r^{1+r,t+1-r})$  and  ${}^{II} B_r^{1,t} = \text{Im}({}^{II} E_r^{1-r,t-1+r} \rightarrow {}^{II} E_r^{1,t})$ . We have the short exact sequences

$$0 \longrightarrow {}^{II} Z_r^{1,t} \longrightarrow {}^{II} E_r^{1,t} \longrightarrow {}^{II} E_r^{1,t} / {}^{II} Z_r^{1,t} \longrightarrow 0$$

and

$$0 \longrightarrow {}^{II} B_r^{1,t} \longrightarrow {}^{II} Z_r^{1,t} \longrightarrow {}^{II} E_{r+1}^{1,t} \longrightarrow 0.$$

Since  ${}^{II} E_2^{1+r,t+1-r}$  is finite by assumptions and  ${}^{II} E_2^{1-r,t-1+r} = 0$ ,  ${}^{II} E_r^{1+r,t+1-r}$  is finite and  ${}^{II} E_r^{1-r,t-1+r} = 0$ , and so  ${}^{II} E_r^{1,t} / {}^{II} Z_r^{1,t}$  is finite and  ${}^{II} B_r^{1,t} = 0$ . Hence  ${}^{II} E_r^{1,t}$  is finite whenever  ${}^{II} E_{r+1}^{1,t}$  is finite. Therefore  ${}^{II} E_2^{1,t} = \text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^1(M, X))$  is finite.  $\square$

As applications of the aforementioned theorem, we present the following corollaries. Recall that, an  $R$ -module  $X$  is said to be  $\mathfrak{a}$ -cofinite if  $\text{Supp}_R(X) \subseteq \{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$  and  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is finite for all integers  $i$  [3].

**COROLLARY 2.2.** *Let  $M$  and  $X$  be finite  $R$ -modules and let  $t$  be a non-negative integer such that the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ . Under these assumptions, it can be concluded that the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite.*

**REMARK 2.3.** Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that  $\text{Ext}_R^{t-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  is finite for all  $i \leq t$  and  $\text{Ext}_R^{t+1-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  is finite for all  $i < t$ . Then, by employing an approach similar to the proof strategy used in Theorem 2.1,  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite. Moreover,

in the particular case where  $X$  is finite and  $H_{\mathfrak{a}}^i(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i < t$ , the finiteness of  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  holds.

**COROLLARY 2.4.** *Let  $R$  be a ring with Krull dimension less than or equal to 2 and let  $M$  and  $X$  be finite  $R$ -modules. Then, the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite over  $R$  for all integers  $i$ .*

**PROOF.** This outcome can be directly derived from Remark 2.3, Corollary 2.2, [6, Corollary 2.5], and by applying an induction on  $i$ .  $\square$

**COROLLARY 2.5.** *Let  $R$  be a ring such that the dimension of  $R/\mathfrak{a}$  is less than or equal to 1 and let  $M$  and  $X$  be finite  $R$ -modules. Under these conditions, the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite over  $R$  for all integers  $i$ .*

**PROOF.** This conclusion follows from Remark 2.3, Corollary 2.2, [5, Theorem 2.3], and by using an induction on  $i$ .  $\square$

By setting  $M = R$  in Theorem 2.1 and Corollaries 2.2, 2.4, and 2.5, we derive the corresponding results related to standard local cohomology modules.

**COROLLARY 2.6.** (see [2, Theorem A]) *Let  $X$  be an arbitrary  $R$ -module and let  $t$  be a non-negative integer such that the module  $\text{Ext}_R^{t+1}(R/\mathfrak{a}, X)$  is finite and the modules  $\text{Ext}_R^{t+2-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  are finite for all  $i < t$ . Under these conditions, it follows that the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite.*

**COROLLARY 2.7.** *Let  $X$  be a finite  $R$ -module and let  $t$  be a non-negative integer such that the local cohomology modules  $H_{\mathfrak{a}}^i(X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ . Under these assumptions, it can be concluded that the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite.*

**COROLLARY 2.8.** *Let  $R$  be a ring with Krull dimension less than or equal to 2 and let  $X$  be a finite  $R$ -module. Then, the local cohomology modules  $H_{\mathfrak{a}}^i(X)$  are  $\mathfrak{a}$ -cofinite over  $R$  for all integers  $i$ .*

**COROLLARY 2.9.** *Let  $R$  be a ring such that the dimension of  $R/\mathfrak{a}$  is less than or equal to 1 and let  $X$  be a finite  $R$ -module. Under these conditions, the local cohomology modules  $H_{\mathfrak{a}}^i(X)$  are  $\mathfrak{a}$ -cofinite over  $R$  for all integers  $i$ .*

### 3. Conclusion

Our analysis indicates that the module  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite, provided that both  $M$  and  $X$  are finite modules over  $R$  and that the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ .

### References

1. Brodmann, M. P. and Sharp, R. Y. (1998) *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge.
2. Dibaei, M. T. and Yassemi, S. (2006) *Finiteness of extension functors of local cohomology modules*, Comm. Algebra, **34**(8), 3097–3101.
3. Hartshorne, R. (1969/70) *Affine duality and cofiniteness*, Invent. Math., **9**, 145–164.
4. Herzog, J. (1970) *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Universität Regensburg, Habilitationsschrift.
5. Melkersson, L. (2012) *Cofiniteness with respect to ideals of dimension one*, J. Algebra, **372**, 459–462.
6. Vahidi, A. and Papari-Zarei, M. (2021) *Cofiniteness of local cohomology modules in the class of modules in dimension less than a fixed integer*, Rev. Un. Mat. Argentina, **62**(1), 191–198.



## On the finiteness of extension functors of generalized local cohomology modules

Alireza Vahidi\*

Department of Mathematics, Payame Noor University, Tehran, Iran  
 Email: vahidi.ar@pnu.ac.ir

**ABSTRACT.** Let  $R$  be a commutative Noetherian ring with non-zero identity,  $\mathfrak{a}$  an ideal of  $R$ ,  $M$  and  $X$  finitely generated  $R$ -modules, and  $t$  a non-negative integer. In this study, we establish that if the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M, X))$  is finitely generated and if the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ , then the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finitely generated.

**Keywords:** cofinite modules, generalized local cohomology modules, local cohomology modules

**AMS Mathematics Subject Classification [2020]:** 13D02, 13D45

### 1. Introduction

Throughout this document, let  $R$  denote a commutative Noetherian ring with non-zero identity;  $\mathfrak{a}$  an ideal of  $R$ ;  $M$  a finite (i.e., finitely generated)  $R$ -module;  $X$  an arbitrary (not necessarily finite)  $R$ -module; and  $t$  a non-negative integer. For fundamental results, definitions, and terminology not explicitly included here, readers are encouraged to consult [2]. Consider a finite  $R$ -module  $L$ . In [4], Hartshorne posed the following question:

**QUESTION.** *Under what circumstances is  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(L))$  finite for all  $i$  and  $j$ ?*

Several studies have explored criteria ensuring the finiteness of modules of the form  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(X))$ . For example, in [3, Theorem B], Dibaei and Yassemi investigated the finiteness of  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^j(X))$ . They established that  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite when the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(X))$  is finite, the modules  $\text{Ext}_R^j(R/\mathfrak{a}, X)$  are finite for all  $j$ , and additionally, for all  $j$  and all  $i < t$ , the modules  $\text{Ext}_R^j(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  are finite.

The concept of the  $i$ th generalized local cohomology module was introduced by Herzog in [5], defined for  $R$ -modules  $M$  and  $X$  with respect to an ideal  $\mathfrak{a}$  as:

$$H_{\mathfrak{a}}^i(M, X) \cong \varinjlim_{n \in \mathbb{N}} \text{Ext}_R^i(M/\mathfrak{a}^n M, X)$$

Notably, when  $M = R$ , this recovers the classical local cohomology modules:  $H_{\mathfrak{a}}^i(R, X) \cong H_{\mathfrak{a}}^i(X)$ . This naturally extends Hartshorne's original question:

\*Speaker.

QUESTION. Under what circumstances is  $\text{Ext}_R^i(R/\mathfrak{a}, H_a^j(M, L))$  finite for all  $i$  and  $j$ ?

In this work, we focus on this fundamental question, with particular attention to the finiteness properties of the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_a^t(M, X))$ . Our results aim to build upon and enhance previous works, including results such as [3, Theorem B], by developing these concepts within to the framework of generalized local cohomology modules.

## 2. Main results

We present sufficient conditions that guarantee the finiteness of  $\text{Ext}_R^2(R/\mathfrak{a}, H_a^t(M, X))$  in the following theorem.

THEOREM 2.1. Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that

- (i)  $\text{Ext}_R^{t+2-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  is finite for all  $i \leq t+2$ ,
- (ii)  $\text{Ext}_R^{t+3-i}(R/\mathfrak{a}, H_a^i(M, X))$  is finite for all  $i < t$ , and
- (iii)  $\text{Hom}_R(R/\mathfrak{a}, H_a^{t+1}(M, X))$  is finite.

Under these conditions, it follows that the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_a^t(M, X))$  is finite.

PROOF. Let

$$E^\bullet = 0 \longrightarrow X \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^i \longrightarrow \cdots$$

be an injective resolution of  $X$  and apply  $\text{Hom}_R(M, \Gamma_a(-))$  to its deletion  $E^{\bullet X}$  to get

$$\text{Hom}_R(M, \Gamma_a(E^{\bullet X})) = 0 \longrightarrow \text{Hom}_R(M, \Gamma_a(E^0)) \longrightarrow \cdots \longrightarrow \text{Hom}_R(M, \Gamma_a(E^i)) \longrightarrow \cdots$$

Let

$$0 \longrightarrow \text{Hom}_R(M, \Gamma_a(E^{\bullet X})) \longrightarrow T^{\bullet, 0} \longrightarrow T^{\bullet, 1} \longrightarrow \cdots \longrightarrow T^{\bullet, i} \longrightarrow \cdots$$

be a Cartan–Eilenberg injective resolution of  $\text{Hom}_R(M, \Gamma_a(E^{\bullet X}))$  and consider the third quadrant bicomplex  $\mathcal{T} = \{\text{Hom}_R(R/\mathfrak{a}, T^{p, q})\}$ . Denote the total complex of  $\mathcal{T}$  by  $\text{Tot}(\mathcal{T})$ .

The first filtration has  ${}^I E_2$  term the iterated cohomology  $H^p H^{p, q}(\mathcal{T})$ . We have

$$\begin{aligned} H^{p, q}(\mathcal{T}) &\cong H^q(\text{Hom}_R(R/\mathfrak{a}, T^{p, \bullet})) \\ &\cong \text{Ext}_R^q(R/\mathfrak{a}, \text{Hom}_R(M, \Gamma_a(E^p))) \\ &\cong \text{Hom}_R(\text{Tor}_q^R(R/\mathfrak{a}, M), \Gamma_a(E^p)). \end{aligned}$$

Therefore

$$\begin{aligned} {}^I E_2^{p, q} &\cong H^p H^{p, q}(\mathcal{T}) \\ &\cong H^p(\text{Hom}_R(\text{Tor}_q^R(R/\mathfrak{a}, M), \Gamma_a(E^{\bullet X}))) \\ &\cong H_a^p(\text{Tor}_q^R(R/\mathfrak{a}, M), X) \end{aligned}$$

which yields the third quadrant spectral sequence

$${}^I E_2^{p, q} := \text{Ext}_R^p(\text{Tor}_q^R(R/\mathfrak{a}, M), X) \rightrightarrows_p H^{p+q}(\text{Tot}(\mathcal{T})).$$

For all  $i$ ,  $0 \leq i \leq t+2$ , we have  ${}^I E_\infty^{t+2-i, i} = {}^I E_{t+4}^{t+2-i, i}$  because  ${}^I E_j^{t+2-i-j, i-1+j} = 0 = {}^I E_j^{t+2-i+j, i+1-j}$  for all  $j \geq t+4$ ; so that  ${}^I E_\infty^{t+2-i, i}$  is finite from the fact that  ${}^I E_{t+4}^{t+2-i, i}$  is a subquotient of  ${}^I E_2^{t+2-i, i}$  which is finite by assumption (i). There exists a finite filtration

$$0 = \phi^{t+3} H^{t+2} \subseteq \phi^{t+2} H^{t+2} \subseteq \cdots \subseteq \phi^1 H^{t+2} \subseteq \phi^0 H^{t+2} = H^{t+2}(\text{Tot}(\mathcal{T}))$$

such that  ${}^I E_\infty^{t+2-i, i} \cong \phi^{t+2-i} H^{t+2} / \phi^{t+3-i} H^{t+2}$  for all  $i$ ,  $0 \leq i \leq t+2$ . Now the exact sequences

$$0 \longrightarrow \phi^{t+3-i} H^{t+2} \longrightarrow \phi^{t+2-i} H^{t+2} \longrightarrow {}^I E_\infty^{t+2-i, i} \longrightarrow 0,$$

for all  $i$ ,  $0 \leq i \leq t+2$ , show that  $H^{t+2}(\text{Tot}(\mathcal{T}))$  is finite.

On the other hand, the second filtration has  ${}^{II}E_2$  term the iterated cohomology  $H''^p H^{q,p}(\mathcal{T})$ . Note that every short exact sequence of injective modules splits and so it remains split after applying the functor  $\text{Hom}_R(R/\mathfrak{a}, -)$ . By using this fact and the fact that  $T^{\bullet,\bullet}$  is a Cartan–Eilenberg injective resolution of  $\text{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X}))$ , we get

$$\begin{aligned} H^{q,p}(\mathcal{T}) &\cong H^q(\text{Hom}_R(R/\mathfrak{a}, T^{\bullet,p})) \\ &\cong \text{Hom}_R(R/\mathfrak{a}, H^q(T^{\bullet,p})) \\ &\cong \text{Hom}_R(R/\mathfrak{a}, H^{q,p}). \end{aligned}$$

Therefore

$$\begin{aligned} {}^{II}E_2^{p,q} &\cong H''^p H^{q,p}(\mathcal{T}) \\ &\cong H^p(\text{Hom}_R(R/\mathfrak{a}, H^{q,\bullet})) \\ &\cong \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M, X)) \end{aligned}$$

which gives the third quadrant spectral sequence

$${}^{II}E_2^{p,q} := \text{Ext}_R^p(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M, X)) \Longrightarrow_{\substack{\longrightarrow \\ p}} H^{p+q}(\text{Tot}(\mathcal{T})).$$

There exists a finite filtration

$$0 = \psi^{t+3} H^{t+2} \subseteq \psi^{t+2} H^{t+2} \subseteq \dots \subseteq \psi^1 H^{t+2} \subseteq \psi^0 H^{t+2} = H^{t+2}(\text{Tot}(\mathcal{T}))$$

such that  ${}^{II}E_{\infty}^{t+2-i,i} \cong \psi^{t+2-i} H^{t+2} / \psi^{t+3-i} H^{t+2}$  for all  $i$ ,  $0 \leq i \leq t+2$ . Since  $H^{t+2}(\text{Tot}(\mathcal{T}))$  is finite,  $\psi^2 H^{t+2}$  is finite. Hence  ${}^{II}E_{\infty}^{2,t} \cong \psi^2 H^{t+2} / \psi^3 H^{t+2}$  is finite. Therefore  ${}^{II}E_{t+4}^{2,t}$  is finite because  ${}^{II}E_{t+4}^{2,t} = {}^{II}E_{\infty}^{2,t}$  from the fact that  ${}^{II}E_j^{2-j,t-1+j} = 0 = {}^{II}E_j^{2+j,t+1-j}$  for all  $j \geq t+4$ . For all  $r \geq 2$ , let  ${}^{II}Z_r^{2,t} = \text{Ker}({}^{II}E_r^{2,t} \rightarrow {}^{II}E_r^{2+r,t+1-r})$  and  ${}^{II}B_r^{2,t} = \text{Im}({}^{II}E_r^{2-r,t-1+r} \rightarrow {}^{II}E_r^{2,t})$ . We have the short exact sequences

$$0 \rightarrow {}^{II}Z_r^{2,t} \rightarrow {}^{II}E_r^{2,t} \rightarrow {}^{II}E_r^{2,t} / {}^{II}Z_r^{2,t} \rightarrow 0$$

and

$$0 \rightarrow {}^{II}B_r^{2,t} \rightarrow {}^{II}Z_r^{2,t} \rightarrow {}^{II}E_{r+1}^{2,t} \rightarrow 0.$$

Since  ${}^{II}E_2^{2+r,t+1-r}$  and  ${}^{II}E_2^{2-r,t-1+r}$  are finite by assumptions (ii) and (iii),  ${}^{II}E_r^{2+r,t+1-r}$  and  ${}^{II}E_r^{2-r,t-1+r}$  are finite, and so  ${}^{II}E_r^{2,t} / {}^{II}Z_r^{2,t}$  and  ${}^{II}B_r^{2,t}$  are finite. Hence  ${}^{II}E_r^{2,t}$  is finite whenever  ${}^{II}E_{r+1}^{2,t}$  is finite. Therefore  ${}^{II}E_2^{2,t} = \text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^2(M, X))$  is finite.  $\square$

Building upon the previously mentioned theorem, we now present its subsequent corollaries. Recall that an  $R$ -module  $X$  is said to be  $\mathfrak{a}$ -cofinite if  $\text{Supp}_R(X) \subseteq \{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$  and the modules  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  are finite for all integers  $i$  [4].

**COROLLARY 2.2.** *Let  $M$  and  $X$  be finite  $R$ -modules and let  $t$  be a non-negative integer such that the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M, X))$  is finite and the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ . Under these assumptions, it can be concluded that the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite.*

**REMARK 2.3.** Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer. By employing methods similar to those used in the proof of Theorem 2.1, we observe the following results:

- (i) If  $\text{Ext}_R^{t-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  is finite for all  $i \leq t$  and  $\text{Ext}_R^{t+1-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  is finite for all  $i < t$ , then  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite. Furthermore, in the particular case where  $X$  is finite and  $H_{\mathfrak{a}}^i(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i < t$ , this finiteness property of  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  persists;

- (ii) If the module  $\text{Ext}_R^{t+1-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  is finite for all  $i \leq t+1$  and the modules  $\text{Ext}_R^{t+2-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  are finite for all  $i < t$ , then  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite. Moreover, in the special case where  $X$  is finite and  $H_{\mathfrak{a}}^i(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i < t$ , the finiteness of  $\text{Ext}_R^1(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  holds.

By considering Remark 2.3, Corollary 2.2, and [1, Theorem 3.5], and applying induction, we derive the following corollaries.

**COROLLARY 2.4.** *Let  $R$  be a local ring with  $\dim(R/\mathfrak{a}) \leq 2$ ,  $M$  and  $X$  finite  $R$ -modules, and  $t$  a non-negative integer. Then  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  is a finite  $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}}^i(M, X)$  is an  $\mathfrak{a}$ -cofinite  $R$ -module for all  $i < t$ .*

**COROLLARY 2.5.** *Let  $R$  be a local ring with  $\dim(R/\mathfrak{a}) \leq 2$ ,  $M$  and  $X$  finite  $R$ -modules, and  $t$  a non-negative integer. Then  $H_{\mathfrak{a}}^i(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i$  whenever  $H_{\mathfrak{a}}^{2i}(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i$  or  $H_{\mathfrak{a}}^{2i+1}(M, X)$  is  $\mathfrak{a}$ -cofinite for all  $i$ .*

By substituting  $M = R$  into Theorem 2.1 and Corollaries 2.2, 2.4, and 2.5, we obtain the corresponding results pertaining to standard local cohomology modules.

**COROLLARY 2.6.** (see [3, Theorem B]) *Let  $X$  be an arbitrary  $R$ -module and  $t$  a non-negative integer such that  $\text{Ext}_R^{t+2}(R/\mathfrak{a}, X)$  is finite,  $\text{Ext}_R^{t+3-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  is finite for all  $i < t$ , and  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(X))$  is finite. Under these conditions, it follows that the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite.*

**COROLLARY 2.7.** *Let  $X$  be a finite  $R$ -module and let  $t$  be a non-negative integer such that the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(X))$  is finite and the local cohomology modules  $H_{\mathfrak{a}}^i(X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ . Under these assumptions, it can be concluded that the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(X))$  is finite.*

**COROLLARY 2.8.** *Let  $R$  be a local ring with  $\dim(R/\mathfrak{a}) \leq 2$ ,  $X$  a finite  $R$ -module, and  $t$  a non-negative integer. Then  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(X))$  is a finite  $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}}^i(X)$  is an  $\mathfrak{a}$ -cofinite  $R$ -module for all  $i < t$ .*

**COROLLARY 2.9.** *Let  $R$  be a local ring with  $\dim(R/\mathfrak{a}) \leq 2$ ,  $X$  a finite  $R$ -module, and  $t$  a non-negative integer. Then  $H_{\mathfrak{a}}^i(X)$  is  $\mathfrak{a}$ -cofinite for all  $i$  whenever  $H_{\mathfrak{a}}^{2i}(X)$  is  $\mathfrak{a}$ -cofinite for all  $i$  or  $H_{\mathfrak{a}}^{2i+1}(X)$  is  $\mathfrak{a}$ -cofinite for all  $i$ .*

### 3. Conclusion

Our analysis demonstrates that the module  $\text{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is finite, assuming that both  $M$  and  $X$  are finite  $R$ -modules, the module  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M, X))$  is finite, and the generalized local cohomology modules  $H_{\mathfrak{a}}^i(M, X)$  are  $\mathfrak{a}$ -cofinite for all  $i < t$ .

### References

1. Bahmanpour, K., Naghipour, R., and Sedghi, M. (2015) *Cofiniteness with respect to ideals of small dimensions*, *Algebr. Represent. Theory*, **18**(2), 369–379.
2. Brodmann, M. P. and Sharp, R. Y. (1998) *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge.
3. Dibaei, M. T. and Yassemi, S. (2006) *Finiteness of extension functors of local cohomology modules*, *Comm. Algebra*, **34**(8), 3097–3101.
4. Hartshorne, R. (1969/70) *Affine duality and cofiniteness*, *Invent. Math.*, **9**, 145–164.
5. Herzog, J. (1970) *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Universität Regensburg, Habilitationsschrift.



## Finiteness dimensions of generalized local cohomology modules

Alireza Vahidi\*

Department of Mathematics, Payame Noor University, Tehran, Iran  
Email: vahidi.ar@pnu.ac.ir

**ABSTRACT.** Let  $R$  be a commutative Noetherian ring with non-zero identity,  $\mathfrak{a}$  an ideal of  $R$ ,  $M$  a finite  $R$ -module, and  $n$  a non-negative integer. In this paper, for an arbitrary  $R$ -module  $X$  which is not necessarily finite, we demonstrate that  $f_{\mathfrak{a}}^n(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}$  if  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is an  $\text{FD}_{<n} \text{ } R$ -module for all  $i$ . Here,  $f_{\mathfrak{a}}^n(M, X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$  is the  $n$ th finiteness dimension of  $M$  and  $X$  with respect to  $\mathfrak{a}$  and  $f_{\mathfrak{a}}(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not a finite } R\text{-module}\}$  is the finiteness dimension of  $M$  and  $X$  with respect to  $\mathfrak{a}$ .

**Keywords:** finiteness dimensions, generalized local cohomology modules, local cohomology modules

**AMS Mathematics Subject Classification [2020]:** 13D05, 13D45

### 1. Introduction

Throughout,  $R$  denotes a commutative Noetherian ring with non-zero identity,  $\mathfrak{a}$  is an ideal of  $R$ ,  $M$  and  $N$  are finite (i.e., finitely generated)  $R$ -modules, and  $n$  and  $t$  are non-negative integers. For foundational results, notation, and terminology not explicitly defined here, readers are encouraged to consult [3].

A central topic in local cohomology concerns the finiteness properties of local cohomology modules. An important result in this area is Faltings' Local-Global Principle:

**THEOREM 1.1.** (see [3, Theorem 9.6.1])  $H_{\mathfrak{a}}^i(N)$  is a finite  $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(N_{\mathfrak{p}})$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  and all  $i \leq t$ .

Faltings' Local-Global Principle can also be expressed in terms of the finiteness dimension  $f_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(N) \text{ is not a finite } R\text{-module}\}$  of  $N$  with respect to  $\mathfrak{a}$  with the understanding that the infimum over the empty set is interpreted as infinity. In this context, the principle states that  $f_{\mathfrak{a}}(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}$ . Bahmanpour et al., in [2], introduced the notion of the  $n$ th finiteness dimension of  $N$  with respect to  $\mathfrak{a}$  by  $f_{\mathfrak{a}}^n(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$ . Faltings' Local-Global Principle extends to this setting, implying that  $f_{\mathfrak{a}}(N) = f_{\mathfrak{a}}^0(N)$ , that is,

$$(1) \quad f_{\mathfrak{a}}^0(N) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(N) \text{ is not a finite } R\text{-module}\}.$$

\*Speaker.

Mehrvarz et al., in [6, Theorem 2.10], generalized Faltings' Local-Global Principle (1), establishing that

$$(2) \quad f_{\mathfrak{a}}^n(N) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(N) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\},$$

where an arbitrary  $R$ -module  $X$  is said to be an  $\text{FD}_{<n}$   $R$ -module if there exists a finite submodule  $X' \subseteq X$  such that  $\dim_R(X/X') < n$ .

The concept of the  $n$ th generalized local cohomology module for arbitrary  $R$ -modules  $X$  and  $Y$  with respect to  $\mathfrak{a}$  was introduced by Herzog in [4]:

$$H_{\mathfrak{a}}^n(X, Y) \cong \varinjlim_{i \in \mathbb{N}} \text{Ext}_R^n(X/\mathfrak{a}^i X, Y).$$

It is clear that  $H_{\mathfrak{a}}^n(R, Y)$  coincides with the standard  $n$ th local cohomology module  $H_{\mathfrak{a}}^n(Y)$  of  $Y$  with respect to  $\mathfrak{a}$ . In [5, Definition 2.3 and Theorem 2.4], Hoang introduced the concept of the  $n$ th finiteness dimension  $f_{\mathfrak{a}}^n(M, N)$  of  $M$  and  $N$  with respect to  $\mathfrak{a}$  by  $f_{\mathfrak{a}}^n(M, N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$ , where  $f_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, N) \text{ is not a finite } R\text{-module}\}$ , and extended (2) by showing that

$$(3) \quad f_{\mathfrak{a}}^n(M, N) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, N) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}.$$

Let  $X$  be an arbitrary  $R$ -module that may not necessarily be finite. Recently, in [1, Theorem 2.3], the authors improved Faltings' Local-Global Principle (1) and established that if  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is a finite  $R$ -module for all  $i$ , then  $f_{\mathfrak{a}}^0(X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(X) \text{ is not a finite } R\text{-module}\}$ . Building upon this, we refine and extend the existing results and the relationship shown in (3). Specifically, we demonstrate that the equality  $f_{\mathfrak{a}}^n(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}$  holds under the condition that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  are  $\text{FD}_{<n}$   $R$ -modules for all  $i$ .

## 2. Main results

The following lemmas are essential components in the demonstration of the main result of this paper.

LEMMA 2.1. *Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$ . Then  $\text{Ext}_R^i(\text{Tor}_j^R(R/\mathfrak{a}, M), X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$  and all  $j$ .*

LEMMA 2.2. *Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that  $\text{Ext}_R^{t-i}(\text{Tor}_i^R(R/\mathfrak{a}, M), X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$  and  $\text{Ext}_R^{t+1-i}(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M, X))$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i < t$ . Then  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is an  $\text{FD}_{<n}$   $R$ -module.*

LEMMA 2.3. *Let  $X$  be an  $\mathfrak{a}$ -torsion  $R$ -module such that  $X_{\mathfrak{p}}$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim_R(R/\mathfrak{p}) \geq n$  and  $\text{Hom}_R(R/\mathfrak{a}, X)$  is an  $\text{FD}_{<n}$   $R$ -module. Then  $X$  is an  $\text{FD}_{<n}$   $R$ -module.*

PROOF. Suppose, on the contrary, that  $X$  is not an  $\text{FD}_{<n}$   $R$ -module and seek a contradiction. Let  $A_1 = \{\mathfrak{p} \in \text{Ass}_R(X) : \dim_R(R/\mathfrak{p}) \geq n\}$  and  $\mathfrak{a}_1 = \bigcap_{\mathfrak{p} \in A_1} \mathfrak{p}$ . Since  $X$  is not an  $\text{FD}_{<n}$   $R$ -module,  $\dim_R(X) \geq n$ . Thus  $A_1$  is a non-empty and finite set because  $\text{Ass}_R(X) = \text{Ass}_R(\text{Hom}_R(R/\mathfrak{a}, X))$  and  $\text{Hom}_R(R/\mathfrak{a}, X)$  is an  $\text{FD}_{<n}$   $R$ -module. For all  $\mathfrak{p} \in A_1$ ,  $X_{\mathfrak{p}}$  is a finite  $R_{\mathfrak{p}}$ -module and so there exists a finite submodule  $N(\mathfrak{p})$  of  $X$  such that  $(N(\mathfrak{p}))_{\mathfrak{p}} = X_{\mathfrak{p}}$ . Let  $N_1 = \sum_{\mathfrak{p} \in A_1} N(\mathfrak{p})$ . Then  $N_1$  is a finite submodule of  $X$  such that  $A_1 \cap A_2 = \emptyset$  and  $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ , where  $A_2 = \{\mathfrak{p} \in \text{Ass}_R(X/N_1) : \dim_R(R/\mathfrak{p}) \geq n\}$  and

$\mathfrak{a}_2 = \bigcap_{\mathfrak{p} \in A_2} \mathfrak{p}$ . Since  $X$  is not an  $\text{FD}_{<n}$   $R$ -module,  $\dim_R(X/N_1) \geq n$ . Note that  $X/N_1$  is  $\mathfrak{a}$ -torsion and  $\text{Hom}_R(R/\mathfrak{a}, X/N_1)$  is  $\text{FD}_{<n}$  from the exact sequence

$$\text{Hom}_R(R/\mathfrak{a}, X) \longrightarrow \text{Hom}_R(R/\mathfrak{a}, X/N_1) \longrightarrow \text{Ext}_R^1(R/\mathfrak{a}, N_1).$$

Thus  $A_2$  is a non-empty and finite set, and so  $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2$ .

Therefore, using the above method on the  $R$ -module  $X/N_1$ , there is a finite submodule  $N_2$  ( $\supseteq N_1$ ) of  $X$  such that  $A_2 \cap A_3 = \emptyset$  and  $\mathfrak{a}_2 \subsetneq \mathfrak{a}_3$ , where  $A_3 = \{\mathfrak{p} \in \text{Ass}_R(X/N_2) : \dim_R(R/\mathfrak{p}) \geq n\}$  and  $\mathfrak{a}_3 = \bigcap_{\mathfrak{p} \in A_3} \mathfrak{p}$ .

Following the same approach, we observe that there exists an ascending chain of ideals within the Noetherian ring  $R$ :

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots \subsetneq \mathfrak{a}_i \subsetneq \cdots,$$

which does not stabilize. This contradiction implies that  $X$  is an  $\text{FD}_{<n}$   $R$ -module.  $\square$

We are now ready to present and substantiate the main result of this research, which extends and enhances the findings in [3, Theorem 9.6.1 and 9.6.2], [6, Theorem 2.10], [5, Theorem 2.4], and [1, Theorems 1.1(i $\Leftrightarrow$ ii) and 2.3].

**THEOREM 2.4.** *Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$ . Then  $H_{\mathfrak{a}}^i(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, X_{\mathfrak{p}})$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim_R(R/\mathfrak{p}) \geq n$  and for all  $i \leq t$ .*

**PROOF.** ( $\Rightarrow$ ). Let  $\mathfrak{p}$  be a prime ideal of  $R$  with  $\dim_R(R/\mathfrak{p}) \geq n$  and let  $i \leq t$ . Since  $H_{\mathfrak{a}}^i(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module, there exists a finite submodule  $N_i$  of  $H_{\mathfrak{a}}^i(M, X)$  such that  $\dim_R(H_{\mathfrak{a}}^i(M, X)/N_i) < n$ . Thus  $(H_{\mathfrak{a}}^i(M, X)/N_i)_{\mathfrak{p}} = 0$  and so  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, X_{\mathfrak{p}}) \cong (H_{\mathfrak{a}}^i(M, X))_{\mathfrak{p}} = (N_i)_{\mathfrak{p}}$  is a finite  $R_{\mathfrak{p}}$ -module.

( $\Leftarrow$ ). We prove by using induction on  $t$ . Let  $t = 0$ . Since  $\text{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M, X))$  is an  $\text{FD}_{<n}$   $R$ -module from Lemma 2.2,  $\Gamma_{\mathfrak{a}}(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module by Lemma 2.3. Suppose that  $t > 0$  and that  $t - 1$  is settled. It is enough to show that  $H_{\mathfrak{a}}^t(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module because  $H_{\mathfrak{a}}^i(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t - 1$  from the induction hypothesis on  $t - 1$ . Thus, by Lemmas 2.1 and 2.2,  $\text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, X))$  is an  $\text{FD}_{<n}$   $R$ -module. Therefore  $H_{\mathfrak{a}}^t(M, X)$  is an  $\text{FD}_{<n}$   $R$ -module from Lemma 2.3.  $\square$

**DEFINITION 2.5.** (see [5, Definition 2.3]) Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module (not necessarily finite), and  $n$  a non-negative integer. We set  $f_{\mathfrak{a}}(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not a finite } R\text{-module}\}$  and  $f_{\mathfrak{a}}^n(M, X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$  which are called finiteness dimension and  $n$ th finiteness dimension of  $M$  and  $X$  with respect to  $\mathfrak{a}$ , respectively. When  $M = R$ , we write  $f_{\mathfrak{a}}(X) = f_{\mathfrak{a}}(R, X)$  and  $f_{\mathfrak{a}}^n(X) = f_{\mathfrak{a}}^n(R, X)$  which are called finiteness dimension and  $n$ th finiteness dimension of  $X$  with respect to  $\mathfrak{a}$ , respectively. Thus  $f_{\mathfrak{a}}(X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(X) \text{ is not a finite } R\text{-module}\}$  and  $f_{\mathfrak{a}}^n(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$ .

**COROLLARY 2.6.** *Let  $M$  be a finite  $R$ -module and let  $X$  be an arbitrary  $R$ -module such that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i$  (in fact, for all  $i \leq f_{\mathfrak{a}}^n(M, X)$ ). Then*

$$f_{\mathfrak{a}}^n(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}.$$

By applying Theorem 2.4 and Corollary 2.6 with  $n = 0$ , we obtain the following results.

**COROLLARY 2.7.** *Let  $M$  be a finite  $R$ -module,  $X$  an arbitrary  $R$ -module, and  $t$  a non-negative integer such that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is a finite  $R$ -module for all  $i \leq t$ . Then  $H_{\mathfrak{a}}^i(M, X)$  is a finite  $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, X_{\mathfrak{p}})$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  and for all  $i \leq t$ .*

**COROLLARY 2.8.** *Let  $M$  be a finite  $R$ -module and let  $X$  be an arbitrary  $R$ -module such that  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is a finite  $R$ -module for all  $i$  (in fact, for all  $i \leq f_{\mathfrak{a}}^0(M, X)$ ). Then*

$$f_{\mathfrak{a}}^0(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not a finite } R\text{-module}\},$$

*that is,  $f_{\mathfrak{a}}(M, X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}$ .*

We present the following corollaries pertaining to the ordinary local cohomology modules.

**COROLLARY 2.9.** *Let  $X$  be an arbitrary  $R$ -module and let  $t$  be a non-negative integer such that  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$ . Then  $H_{\mathfrak{a}}^i(X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(X_{\mathfrak{p}})$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim_R(R/\mathfrak{p}) \geq n$  and for all  $i \leq t$ .*

**COROLLARY 2.10.** *Let  $X$  be an arbitrary  $R$ -module such that  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i$  (in fact, for all  $i \leq f_{\mathfrak{a}}^n(X)$ ). Then*

$$f_{\mathfrak{a}}^n(X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(X) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}.$$

**COROLLARY 2.11.** *Let  $X$  be an arbitrary  $R$ -module and let  $t$  be a non-negative integer such that  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is a finite  $R$ -module for all  $i \leq t$ . Then  $H_{\mathfrak{a}}^i(X)$  is a finite  $R$ -module for all  $i \leq t$  if and only if  $H_{\mathfrak{a}R_{\mathfrak{p}}}^i(X_{\mathfrak{p}})$  is a finite  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$  and for all  $i \leq t$ .*

**COROLLARY 2.12.** *Let  $X$  be an arbitrary  $R$ -module such that  $\text{Ext}_R^i(R/\mathfrak{a}, X)$  is a finite  $R$ -module for all  $i$  (in fact, for all  $i \leq f_{\mathfrak{a}}^0(X)$ ). Then*

$$f_{\mathfrak{a}}^0(X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(X) \text{ is not a finite } R\text{-module}\},$$

*that is,  $f_{\mathfrak{a}}(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}$ .*

### 3. Conclusion

Our analysis demonstrates that  $f_{\mathfrak{a}}^n(M, X) = \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M, X) \text{ is not an } \text{FD}_{<n} \text{ } R\text{-module}\}$  if  $\text{Ext}_R^i(M/\mathfrak{a}M, X)$  is an  $\text{FD}_{<n}$   $R$ -module for all  $i$ .

### References

1. Abdi, L., Naghipour, R., and Sedghi, M. (2019) *Faltings' local-global principle for finiteness dimension of cofinite modules*, Arch. Math. (Basel), **112**(1), 33–39.
2. Bahmanpour, K., Naghipour, R., and Sedghi, M. (2013) *Minimaxness and cofiniteness properties of local cohomology modules*, Comm. Algebra, **41**(8), 2799–2814.
3. Brodmann, M. P. and Sharp, R. Y. (1998) *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge.
4. Herzog, J. (1970) *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Universitat Regensburg, Habilitationsschrift.
5. Hoang, N. V. (2017) *On Faltings' local-global principle of generalized local cohomology modules*, Kodai Math. J., **40**(1), 58–62.
6. Mehrvarz, A. A., Naghipour, R., and Sedghi, M. (2015) *Faltings' local-global principle for the finiteness of local cohomology modules over Noetherian rings*, Comm. Algebra, **43**(11), 4860–4872.



## Isomorphism Theorems for Grey $S$ -Acts

Masoomeh Hezarjaribi

Department of Mathematics, Payame Noor University, Tehran, Iran.

Email: Masoomeh.hezarjaribi@pnu.ac.ir

**ABSTRACT.** In this paper, we introduce the notion of grey relations and grey congruences on  $S$ -acts. We investigate the behavior of grey congruences under  $S$ -homomorphisms, and we establish analogues of the classical isomorphism theorems for grey  $S$ -acts.

**Keywords:** Grey congruence, Grey relation, Monoid,  $S$ -act.

**AMS Mathematics Subject Classification [2020]:** 20M30, 08A30, 03E72

### 1. Introduction

In this study, we intend to define a grey relation and then define the grey congruence relations on  $S$ -act. For this, we provide some preliminaries.

We recall from [6] and [5], a *grey number* is characterized by having a well-defined range, with distinct upper and lower limits, but its exact value within that range remains indeterminate, reflecting uncertainty or imprecision. A grey number can be represented as  $g^\pm \in [g^-, g^+] = \{g^- \leq t \leq g^+\}$ , where  $g^\pm$  is a grey number,  $t$  is information,  $g^-$  and  $g^+$  are the lower and upper bounds of the uncertain value. In [6], it is shown a new representation of grey numbers to consider continuous and discrete grey numbers. Consider the real set  $\mathbb{R}$ , where a grey number  $g^\pm$  can be represented as a union of several intervals  $g^\pm = \bigcup_{i=1}^n [a_i^-, a_i^+]$ , where  $i$  ranges from 1 to  $n$ , with  $n$  being a finite integer. These intervals can be either closed or open depending on the context. For each interval  $[a_i^-, a_i^+]$ , the probability  $p_i$  represents the likelihood that a grey number  $g$  lies within this interval. A grey number  $g^\pm$  satisfies the following conditions:

- (i)  $p_i > 0$  if the interval  $[a_i^-, a_i^+]$  is contained within  $g^\pm$ ;
- (ii)  $p_i = 0$  if the interval  $[a_i^-, a_i^+]$  is not part of  $g^\pm$ ;
- (iii) the sum of all probabilities  $\sum_{i=1}^n p_i = 1$ , ensuring normalization. then  $g$  is called a grey number represented by  $g^\pm$ .  $g^- = \inf_{a_i^- \in g^\pm} a_i^-$  and  $g^+ = \sup_{a_i^+ \in g^\pm} a_i^+$  are called the lower and upper of  $g^\pm$ . For a set  $A \subseteq U$ , if the characteristic function value of  $x$  with respect to  $A$  can be expressed with a single white number  $v \in [0, 1]$ ,  $\chi_A : U \rightarrow [0, 1]$  then  $A$  is a *white set*. For a set  $A \subseteq U$ , if the characteristic function value of  $x$  with respect to  $A$  can be expressed with a grey number  $g_A^\pm(x) \in \bigcup_{i=1}^n [a_i^-, a_i^+] \in D[0, 1]^\pm$ ,  $\chi_A : U \rightarrow D[0, 1]^\pm$  then  $A$  is a *grey set*, where  $D[0, 1]^\pm$  refers to the set of all grey numbers within the interval

$[0, 1]$ . Similar to fuzzy set, a grey set  $A$  is shown  $A = \frac{g^\pm(x_1)}{x_1} + \frac{g^\pm(x_2)}{x_2} + \dots + \frac{g^\pm(x_n)}{x_n}$ , for  $x_i \in U$ . The grey set denoted as  $A = (U, \chi_A)$ . It is clear that the special case of a grey set is a white set and any fuzzy set is a special case of a white set. We recall grey lattice operation from [4], which for grey numbers  $x^\pm = [x^-, x^+]$  (lower  $x^-$  and upper  $x^+$ ) and  $y^\pm = [y^-, y^+]$  (lower  $y^-$  and upper  $y^+$ ), the operations of Join and Meet for two grey numbers are defined as follows: Join:  $x^\pm \vee y^\pm = [\min(x^-, y^-), \max(x^+, y^+)]$   
Meet:  $x^\pm \wedge y^\pm = [\max(x^-, y^-), \min(x^+, y^+)]$ .

Now according to the definition Join and Meet, the partial order  $\preceq$  on grey set  $(X, \chi_A)$  is shown as  $x^\pm \preceq y^\pm \iff x^+ \leq y^+ \text{ and } y^- \leq x^-$ . We recall from [1], [4] and [2], the category  $GSet$  of grey sets where morphism between two grey sets  $A = (U, \chi_A)$  and  $B = (U', \chi_B)$  is a function  $f : U \rightarrow U'$  such that upper  $\chi_A(x) \leq \text{upper } \chi_B f(x)$  and lower  $\chi_B f(x) \leq \text{lower } \chi_A(x)$  for any  $x \in U$ . In this paper, any grey number  $t$  with upper  $g^+$  and lower  $g^-$  is shown as  $t_{g^-}^{g^+}$ . Now, we recall a (right)  $S$ -act on monoid  $S$  is a non-empty set  $A$  together with a map  $A \times S \rightarrow A, (a, s) \mapsto as$ , such that for all  $a \in A, s, t \in S, (as)t = a(st)$  and  $a.1 = a$ . A non-empty subset  $B \subseteq A$  is called a *subact* of  $A$  if  $bs \in B$  for all  $b \in B$  and  $s \in S$ . Let  $A$  and  $B$  be two  $S$ -acts. A mapping  $f : A \rightarrow B$  is called an  $S$ -homomorphism if  $f(as) = f(a)s$  for all  $a \in A, s \in S$ . The category of all  $S$ -acts and  $S$ -homomorphisms between them denoted as **Act- $S$** . For more see [3]

## 2. Main Result

**DEFINITION 2.1.** Let  $A_1, A_2, \dots, A_n$  be non-empty  $S$ -acts. A grey  $n$ -ary relation  $\mu$  on  $A_1, A_2, A_3, \dots, A_n$  is defined as a grey set of the product set  $A_1, A_2, \dots, A_n$ , i.e., there exists a characteristic function  $\mu : A_1 \times A_2 \times \dots \times A_n \rightarrow D[0, 1]^\pm$  such that  $\mu(y)$  is a grey number for any  $y \in A_1 \times A_2 \times \dots \times A_n$ .

**DEFINITION 2.2.** Consider binary relation  $\mu : A \times A \rightarrow D[0, 1]^\pm$ . It is called  
(i) Reflexive if  $\mu^+(x, x) = 1$  and  $\mu^-(x, x) = 0$  for any  $x \in A$ ;  
(ii) Symmetric if  $\mu^+(x, y) = \mu^+(y, x)$  and  $\mu^-(x, y) = \mu^-(y, x)$  for any  $x, y \in A$ ;  
(iii) If  $(\mu o \mu)^+(x, y) \leq \mu^+(x, y)$  and  $\mu^-(x, y) \leq (\mu o \mu)^-(x, y)$ , for any  $x, y \in A$ , then  $\mu$  is called transitive if  $\mu o \mu(x, y) = \bigvee_{z \in X} (\mu(x, z) \wedge \mu(z, y))$

A grey relation  $\mu$  on an  $S$ -act  $A$  is considered a grey equivalence relation when it satisfies the properties of reflexivity, symmetry, and transitivity.

For two relations  $\theta, \phi$  on  $S$ -act  $A$  define  $\theta \preceq \phi$  if and only if  $\theta(x, y) \preceq \phi(x, y)$ . We define two relations on  $S$ -act  $A, \Delta : A \times A \rightarrow D[0, 1]^\pm$  and  $\nabla : A \times A \rightarrow D[0, 1]^\pm$  such that  $\Delta(x, y) = t_0^1$  if  $x = y, \Delta(x, y) = 0$  if  $x \neq y$  and  $\nabla(x, y) = t_0^1$  for any  $x, y \in A$ .

**PROPOSITION 2.3.** If  $\theta$  is a grey equivalence relation on  $S$ -act  $A$ , then  $\theta o \theta = \theta$ .

**PROOF.** For any  $x, y \in A, (\theta o \theta)^+(x, y) = \bigvee_{z \in A} (\theta(x, z) \wedge \theta(z, y)) \geq (\theta(x, x) \wedge \theta(x, y))^+ = \theta^+(x, y)$ . So  $(\theta o \theta)^+ = \theta^+$ . Similarly, we can show for lower bound.  $\square$

**PROPOSITION 2.4.** For a non-empty collection  $\{\theta_i\}_{i \in I}$  of grey equivalence relations on an  $S$ -act  $A$ , their intersection  $\bigwedge_{i \in I} \theta_i$  is a grey equivalence relation on  $A$ .

**THEOREM 2.5.** *Let  $A$  be an  $S$ -act. Then the set  $GC(A)$  of grey congruences on  $A$ , ordered by pointwise comparison, forms a complete lattice.*

**PROOF.** Given  $\{\theta_i\}_{i \in I} \subseteq GC(A)$ : The meet  $\bigwedge_{i \in I} \theta_i$  defined pointwise is in  $GC(A)$  (by Propoposition 2.4). The join  $\bigvee_{i \in I} \theta_i$  is obtained by taking the grey congruence closure of  $\bigcup \theta_i$ . Both operations preserve the grey congruence conditions.  $\square$

**DEFINITION 2.6.** Consider  $S$ -homomorphism  $f : A \rightarrow B$ , grey relation  $\alpha$  on  $S$ -act  $A$ , and  $\beta$  on  $S$ -act  $B$ . The inverse grey relation  $f^{-1}(\beta)$  on  $S$ -act  $A$  is defined as:  $f^{-1}(\beta)(a_1, a_2) = \beta(f(a_1), f(a_2))$ , for any  $a_1, a_2 \in A$ .

**LEMMA 2.7.** *Let  $f : A \rightarrow B$  be an  $S$ -homomorphism. Then*

- (i) *If  $\beta$  is a grey congruence on  $S$ -act  $B$ , then  $f^{-1}(\beta)$  is a grey congruence on  $S$ -act  $A$ .*
- (ii) *If  $\alpha$  is a grey congruence on  $S$ -act  $A$  and  $f$  is epimorphism, then  $f(\alpha)$  is a grey congruence on  $S$ -act  $B$ .*

For any grey congruence  $\theta$  on an  $S$ -act  $A$  and  $x \in A$ , define  $[x]_\theta = \{y \in A \mid \theta^+(x, y) = 1, \theta^-(x, y) = 0\}$ . Obviously,  $\frac{A}{\theta}$  is an  $S$ -act and  $\pi_\theta : A \rightarrow \frac{A}{\theta}$  is an epimorphism. Now, consider the characteristic function  $\theta_{[x]} : A \rightarrow D[0, 1]^\pm$  such that  $\theta_{[x]}(y) = \theta(x, y)$ .

**PROPOSITION 2.8.** *Let  $\alpha$  be a grey congruence on an  $S$ -act  $A$ . Then  $\alpha_{[a]} = \alpha_{[b]}$  if and only if  $\alpha^+(a, b) = 1$ , and  $\alpha^-(a, b) = 0$  for any  $a, b \in A$ .*

**PROOF.** Suppose that for any  $c \in A$ ,  $\alpha_{[a]}(c) = \alpha_{[b]}(c)$ . Then  $\alpha(a, c) = \alpha(b, c)$ . In particular,  $\alpha^+(a, b) = \alpha^+(b, b) = 1$  and  $\alpha^-(a, b) = \alpha^-(b, b) = 0$ . Conversely, for any  $c \in A$ , since  $\alpha(a, c) = (\alpha\alpha)(a, c) \succeq \alpha(a, b) \wedge \alpha(b, c)$  and by assumption, it follow  $\alpha(a, c) \succeq \alpha(b, c)$ . Similarly, we can show that  $\alpha(b, c) \succeq \alpha(a, c)$ . So  $\alpha(a, c) = \alpha(b, c)$  for any  $c \in A$ . Hence  $\alpha_{[a]} = \alpha_{[b]}$ .  $\square$

**THEOREM 2.9.** *If  $f : A \rightarrow B$  is an  $S$ -homomorphism and  $\beta$  is a grey congruence on grey  $S$ -act  $B$ . Then,  $\frac{A}{f^{-1}(\beta)} \cong \frac{B}{\beta}$ .*

**PROOF.** Clearly,  $f^{-1}(\beta)$  is a grey congruence on  $S$ -act  $A$  by Lemma 2.7. We define  $g : \frac{A}{f^{-1}(\beta)} \rightarrow \frac{B}{\beta}$  as  $g([a]_{f^{-1}(\beta)}) = [f(a)]_\beta$ . we have  $[a]_{f^{-1}(\beta)} = [a']_{f^{-1}(\beta)} \Leftrightarrow (f^{-1}(\beta))^+(a, a') = 1$  for any  $a, a' \in A$ , and  $(f^{-1}(\beta))^-(a, a') = 0 \Leftrightarrow \beta^+(f(a), f(a')) = 1$  and  $\beta^-(f(a), f(a')) = 0 \Leftrightarrow [f(a)]_\beta = [f(a')]_\beta$ . So  $g$  is well-defined and monomorphism. Obviously,  $g$  is an  $S$ -homomorphism and epimorphism. Thus,  $g$  is an isomorphism.  $\square$

Now by Theorem 2.9, we obtain the First Isomorphism Theorem.

**DEFINITION 2.10.** Let  $\alpha, \beta$  be two grey congruences on  $S$ -act  $A$  such that  $\alpha \preceq \beta$ , then define the grey relation  $\frac{\beta}{\alpha} : \frac{A}{\alpha} \rightarrow D[0, 1]^\pm$  such that  $\frac{\beta}{\alpha}([a]_\alpha, [a']_\alpha) = \beta(a, a')$  for any  $a, a' \in A$ .

In the following, we prove the Second Isomorphism Theorem.

THEOREM 2.11. Let  $\alpha \preceq \beta$  be two grey congruences on  $S$ -act  $A$ . Then  $\frac{A}{\beta} \cong \frac{\frac{A}{\alpha}}{\beta}$ .

PROOF. Define the mapping  $f : \frac{A}{\beta} \rightarrow \frac{\frac{A}{\alpha}}{\beta}$  such that  $f([a]_{\beta}) = [a]_{\beta}$ , for any  $a \in A$ .

Then, for any  $a, a' \in A$ ,  $[a]_{\beta} = [a']_{\beta} \Leftrightarrow \frac{\beta^+}{\alpha}([a]_{\alpha}, [a']_{\alpha}) = 1, \frac{\beta^-}{\alpha}([a]_{\alpha}, [a']_{\alpha}) = 0 \Leftrightarrow \beta^+(a, a') = 1, \beta^-(a, a') = 0 \Leftrightarrow [a]_{\beta} = [a']_{\beta}$  and hence  $f$  is well-defined and monomorphism. Obviously,  $f$  is an epimorphism. Thus  $f$  is an isomorphism.  $\square$

DEFINITION 2.12. Consider subact  $B$  of  $S$ -act  $A$  and grey congruence  $\theta$  on  $A$ . Define  $B^{\theta} = \{a \in A \mid B \cap [a]_{\theta} \neq \emptyset\}$ . Obviously,  $B^{\theta}$  is a subact of  $A$ .

DEFINITION 2.13. Let  $\theta$  be grey congruence on  $S$ -act  $A$ . Define  $\theta|_B = \theta \cap B^2$  which is the restriction of  $\theta$  to subact  $B$  of  $A$ . Clearly,  $\theta|_B$  is a grey congruence on  $S$ -act  $B$ .

In the following, we prove the Third Isomorphism Theorem.

THEOREM 2.14. Let  $B$  be a subact of the  $S$ -act  $A$  and  $\theta$  be a grey congruence on  $S$ -act  $A$ . Then  $\frac{B}{\theta|_B} \cong \frac{B^{\theta}}{\theta|_{B^{\theta}}}$ .

PROOF. Define a mapping  $f : \frac{B}{\theta|_B} \rightarrow \frac{B^{\theta}}{\theta|_{B^{\theta}}}$  such that for any  $b \in B$ ,  $f([b]_{\theta|_B}) = [b]_{\theta|_{B^{\theta}}}$ . It follows that  $f$  is an isomorphism.  $\square$

### 3. Conclusion

We introduced grey relations and grey congruences on  $S$ -acts, extending classical congruence theory to a framework that accounts for uncertainty. It was shown that the set of grey congruences forms a complete lattice and that the First, Second, and Third Isomorphism Theorems hold in this setting.

### References

1. Hezarjaribi. M. , Darvishi. D. and Habibi. Z.(2024) *Category of grey sets*, J. Prime Res. Math., **20**, 89–96.
2. Hezarjaribi. M. and Habibi. Z. (2022) *Some properties of grey S-acts over monoid*, New Math. Nat. Comput., **18**(2), 313-323.
3. Kilp. M. , Knauer. U. and Mikhalev. A. V. (2011) *Monoids, Acts and Categories*, Boston, De Gruyter, Berlin.
4. Yamaguchi. D. , Li. G. D. and Nagai. M. (2006) *On the combination of rough set theory and grey theory based on grey lattice operations*, in: S. Greco et al. (Eds.), Proceedings of the Fifth International Conference on Rough Sets and Current Trends in Computing, RSCTC 2006. Lecture Notes in Computer Science, vol 4259. Springer, Berlin, Heidelberg.
5. Yang. Y. (2007) *Extended grey numbers and their operations*, 2007 IEEE International Conference on Systems, Man and Cybernetics. Montreal.Que., 2181–2186.
6. Yang. Y and John. R. (2012) *Grey Sets and Greyness*, Information Sciences., **1**(185), 249–264.



## New Results on Injective Measurable $S$ -Acts

Masoomeh Hezarjaribi

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Masoomeh.hezarjaribi@pnu.ac.ir

**ABSTRACT.** In this paper, we present new results on measurable  $S$ -acts, focusing on injectivity and completeness. We establish the existence of complete injective envelopes, prove that products and retracts preserve injectivity under suitable conditions, and characterize minimal complete injective extensions.

**Keywords:** Complete injective, Measurable  $S$ -act, Injective,  $S$ -morphism

**AMS Mathematics Subject Classification [2020]:** 20M30, 06B15, 28A05

### 1. Introduction

The study of  $S$ -acts over monoids has a long history in algebra, providing a natural generalization of modules over rings. Injective and projective acts have been extensively investigated, and various constructions such as injective envelopes and products have been characterized in classical works (see, e.g., [1], [4], [5]). In this paper, we study some properties of measurable  $S$ -acts, where the underlying  $S$ -act is equipped with a  $\sigma$ -algebra compatible with the  $S$ -action, and we study injectivity and completeness in this context. Our main goal is to provide constructions of complete injective measurable  $S$ -acts, investigate their fundamental properties, and establish minimal injective envelopes, thus generalizing classical injectivity results to the measurable setting. Before presenting our main results, we recall some basic definitions and set up the necessary notation.

Let  $S$  be a monoid with identity element 1. A right  $S$ -act is a non-empty set  $A$  together with a map  $A \times S \rightarrow A$ ,  $(a, s) \mapsto as$ , satisfying  $1 \cdot a = a$  and  $a \cdot (st) = (a \cdot s) \cdot a$  for all  $s, t \in S$  and  $a \in A$  (see [1]). Left  $S$ -acts are defined similarly.

Let  $A$  and  $B$  be right  $S$ -acts. A map  $f : A \rightarrow B$  is an  $S$ -morphism (or  $S$ -homomorphism) if  $f(as) = f(a)s$ , for all  $s \in S, a \in A$ . The collection of all right  $S$ -acts together with  $S$ -morphisms forms a category, denoted by **Act-S**. A RIGHT  $S$ -act  $Q$  is injective if for every monomorphism  $f : B \rightarrow C$  in **S-Act** and every  $S$ -morphism  $g : B \rightarrow Q$ , there exists an  $S$ -morphism  $h : C \rightarrow Q$  such that  $hf = g$ . This category has products, coproducts, and enough injectives, i.e., every  $S$ -act embeds into an injective  $S$ -act.

A pair  $(A, \sigma_A)$  is called a measurable  $S$ -act if  $A$  is a right  $S$ -act and  $\sigma_A$  is a  $\sigma$ -algebra on  $A$  such that for each  $s \in S$ , the map  $\lambda_s : A \rightarrow A$  given by  $\lambda_s(a) = as$  is measurable. Let  $(A, \sigma_A)$  and  $(B, \sigma_B)$  be measurable  $S$ -acts. An  $S$ -homomorphism  $f : A \rightarrow B$  is called a *measurable  $S$ -morphism* (MS-morphism) if  $f^{-1}(B') \in \sigma_A$  for all  $B' \in \sigma_B$ .

An MS-morphism  $f : A \rightarrow B$  is called *complete* if  $f(A') \in \sigma_B$  for all  $A' \in \sigma_A$ . For more see [2] and [3].

## 2. Main Results

In this section, we investigate the structure of measurable  $S$ -acts with respect to injectivity and completeness. We introduce the notions of complete injective measurable  $S$ -acts and provide several fundamental results concerning their existence and properties. In particular, we show that for any measurable  $S$ -act, there exists a complete injective envelope, and we examine how products and retracts of injective acts behave in this setting.

DEFINITION 2.1. A measurable  $S$ -act  $(Q, \sigma_Q)$  is called *injective* if for every MS-monomorphism  $f : B \rightarrow C$  and MS-morphism  $g : B \rightarrow Q$ , there exists an MS-morphism  $h : C \rightarrow Q$  such that  $hf = g$ .

DEFINITION 2.2. A measurable  $S$ -act  $(Q, \sigma_Q)$  is *complete injective* if the property in 2.1 is hold whenever  $f$  is a complete MS-monomorphism.

DEFINITION 2.3. For any measurable  $S$ -act  $(A, \sigma_A)$ , define  $A^S = \{f : S \rightarrow A \mid f \text{ is an } S\text{-act homomorphism}\}$ ,  $\sigma_{A^S} = \{\nu^{-1}(X) \mid X \in \sigma_A\}$ , where  $\nu(f) = f(1)$ . Then  $(A^S, \sigma_{A^S})$  is a measurable  $S$ -act, with  $S$  acting by  $(s \cdot f)(t) = f(ts)$  for all  $s, t \in S$ .

PROPOSITION 2.4. Let  $(A, \sigma_A)$  be a measurable right  $S$ -act and define  $\iota : A \rightarrow A^S$  by

$$\iota(a)(s) = a \cdot s \quad (s \in S, a \in A).$$

Then  $\iota$  is an MS-monomorphism.

THEOREM 2.5. The measurable  $S$ -act  $(A^S, \sigma_{A^S})$  is complete injective.

PROOF. Let  $f : B \rightarrow C$  be a complete MS-monomorphism and  $g : B \rightarrow A^S$  be an MS-morphism. For each  $c \in C$ , define

$$h(c)(s) = \begin{cases} g(b)(s), & \text{if } c = f(b), \text{ for some } b \in B, \\ a_0, & \text{otherwise, where } a_0 \in A \text{ is fixed.} \end{cases}$$

Since  $f$  is complete, the image  $f(B)$  and its measurable complement in  $C$  are in  $\sigma_C$ , so the preimage of any  $X \in \sigma_A$  under  $h$  is measurable. The  $S$ -action on  $A^S$  is measurable by definition, so  $h$  is an MS-morphism. Since  $f$  is a complete monomorphism, each  $c \in C$  is either in the image of  $f$  or in a measurable complement. The map  $h : C \rightarrow A^S$  is measurable because preimages of measurable sets under  $h$  are unions of measurable sets in  $B$  and complements, both in  $\sigma_C$ . By construction,  $hf = g$ . Thus  $(A^S, \sigma_{A^S})$  is complete injective.  $\square$

THEOREM 2.6. Any measurable  $S$ -act can be embedded into a complete injective measurable  $S$ -act.

PROOF. Let  $(A, \sigma_A)$  be a measurable  $S$ -act. Consider  $A^S$  as above. Define the embedding  $\iota : A \rightarrow A^S$  by  $\iota(a)(s) = sa$ . Clearly,  $\iota$  is an  $S$ -homomorphism and measurable. By the previous theorem,  $(A^S, \sigma_{A^S})$  is complete injective, giving the desired embedding.  $\square$

**THEOREM 2.7.** *Let  $S$  be left reversible. Then any injective measurable  $S$ -act is a retract of a complete injective measurable  $S$ -act.*

**PROOF.** Let  $(Q, \sigma_Q)$  be injective. By the Theorem 2.6, there exists a complete injective measurable  $S$ -act  $(E(Q), \sigma_{E(Q)})$  containing  $Q$  as a subact via embedding  $i : Q \rightarrow E(Q)$ . Since  $Q$  is injective, there exists an MS-morphism  $r : E(Q) \rightarrow Q$  with  $ri = \text{id}_Q$ . Hence  $Q$  is a retract of  $E(Q)$ .  $\square$

**THEOREM 2.8.** *For any measurable  $S$ -act  $(A, \sigma_A)$ , there exists a minimal complete injective  $S$ -act  $(E(A), \sigma_{E(A)})$  containing  $A$  such that any MS-morphism from  $A$  to a complete injective  $S$ -act extends uniquely to  $E(A)$ .*

**PROOF.** Let  $\mathcal{F}$  be the family of all complete injective  $S$ -acts containing  $A$  as a subact via MS-monomorphisms. Consider  $E(A)$  as the intersection of all such  $S$ -acts in  $\prod_{F \in \mathcal{F}} F$  via the diagonal embedding of  $A$ . Then  $E(A)$  is complete injective, contains  $A$ , and is minimal by construction. Uniqueness of extensions follows from the universal property of intersections in the product.  $\square$

**THEOREM 2.9.** *Let  $\{(I_i, \sigma_{I_i})\}_{i \in I}$  be a family of injective measurable  $S$ -acts. Then their product  $\prod_{i \in I} (I_i, \sigma_{I_i})$  is complete injective.*

**PROOF.** Let  $f : B \rightarrow C$  be a complete MS-monomorphism and  $g : B \rightarrow \prod_{i \in I} I_i$  an MS-morphism. Write  $g = (g_i)_{i \in I}$  where  $g_i : B \rightarrow I_i$ . Since each  $I_i$  is injective, there exists  $h_i : C \rightarrow I_i$  such that  $h_i f = g_i$ . Define  $h : C \rightarrow \prod_{i \in I} I_i$  by  $h(c) = (h_i(c))_{i \in I}$ . Then  $h f = g$  and  $h$  is an MS-morphism because each  $h_i$  is. Therefore, the product is complete injective.  $\square$

**COROLLARY 2.10.** *If  $S$  is left reversible, then any coproduct of injective measurable  $S$ -acts is injective.*

**PROOF.** Let  $\{I_j\}_{j \in J}$  be injective measurable  $S$ -acts. The coproduct  $\coprod_{j \in J} I_j$  embeds into the product  $\prod_{j \in J} I_j$ , which is injective by the previous theorem. Since  $S$  is left reversible, retracts of injective acts are injective, so the coproduct is injective.  $\square$

**PROPOSITION 2.11.** *Let  $\{(A_i, \sigma_{A_i})\}_{i \in I}$  be a family of measurable right  $S$ -acts, and let  $(\prod_{i \in I} A_i, \sigma_{\prod})$  denote their product with the product  $\sigma$ -algebra. If each  $A_i$  is complete injective, then  $\prod_{i \in I} A_i$  is also a complete injective measurable right  $S$ -act.*

**PROOF.** Let  $f : B \rightarrow C$  be a complete MS-monomorphism and  $g : B \rightarrow \prod_{i \in I} A_i$  an MS-morphism. Write  $g = (g_i)_{i \in I}$  with  $g_i : B \rightarrow A_i$ .

Since each  $A_i$  is complete injective, there exists an MS-morphism  $h_i : C \rightarrow A_i$  such that  $h_i f = g_i$ .

Define  $h : C \rightarrow \prod_{i \in I} A_i$  by  $h(c) = (h_i(c))_{i \in I}$ .

$h$  is an  $S$ -morphism because each  $h_i$  is. Also,  $h$  is measurable because preimages of product-cylinder sets generate  $\sigma_{\prod}$ , and each  $h_i$  is measurable. By construction, we have  $h f = g$ . Hence  $\prod_{i \in I} A_i$  is complete injective.  $\square$

### 3. Conclusion

In this paper, we studied measurable  $S$ -acts and their morphisms, introducing the notions of injective and complete injective measurable  $S$ -acts. We showed that any measurable  $S$ -act embeds into a complete injective, and that products and retracts preserve injectivity under suitable conditions.

### References

1. Kilp, M., Knauer, U., and Mikhalev, A. V. (2000), *Monoids, Acts and Categories.*, Walter de Gruyter.
2. Hezarjaribi, M. and Habibi, Z. (2023) *Injectivity in the category of measurable  $S$ -acts*, Asian-European Journal of Mathematics, **16**(09), 2350168.
3. S. Resnick. (2019), *A Probability Path*, Germany: Birkhuser Boston.
4. Ahsan, J. (1987) *Monoids characterized by their quasi-injective  $S$ -systems*, Semigroup Forum, **36**,285–292.
5. P. Berthiaume. P. (1967) *The injective envelope of  $S$ -Sets*, Canad. Math. Bull., **10**(2), 261–273 .





## Some Properties of Flatness and Stability in $S$ -Posets over Pomonoids

Maomeh Hezarjaribi<sup>1,\*</sup> and Zeynab Abbas Zadeh Chari<sup>2†</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Masoomah.hezarjaribi@pnu.ac.ir

<sup>2</sup> M.Sc. Graduate in Pure Mathematics

Email: nahal.asady.1397@gmail.com

**ABSTRACT.** In this paper, we study several properties of flat and quasi-flat  $S$ -posets over pomonoids. We establish connections between reversibility of the pomonoid and structural features of  $S$ -posets, and examine how flatness and quasi-flatness are preserved under quotients.

**Keywords:** Flat, Pomonoid,  $S$ -poset, Stability

**AMS Mathematics Subject Classification [2020]:** 06F05, 16Y60, 18A30

### 1. Introduction

The study of  $S$ -posets over partially ordered monoids combines techniques from ordered algebraic structures and category theory. In recent years, various notions such as flatness, quasi-flatness, and stability of morphisms have been investigated to understand how algebraic properties of the underlying monoid influence categorical behaviors of posets equipped with  $S$ -actions. For more see [1] and [2]. In this paper, we focus on some fundamental properties related to flatness and stability, providing new insights into their interplay with reversibility and quotient constructions in  $S$ -posets. To set the stage for our main results, we first recall some fundamental definitions and preliminary concepts concerning  $S$ -posets and partially ordered monoids. A monoid  $S$  is called a partially ordered monoid if  $S$  is equipped with a partial order  $\leq$  such that this order is compatible with the monoid operation. That is, for all  $s, t, u \in S$ , if  $t \leq u$  then  $st \leq su$  and  $ts \leq us$ . If  $S$  is a monoid and  $A$  is a nonempty partially ordered set, then  $A$  is called a right  $S$ -poset provided that for all  $s, t \in S$  and  $a \in A$  the following conditions hold:  $a1 = a$ ,  $a(st) = (as)t$ , and for all  $a, b \in A$  and  $s \in S$ , if  $a \leq b$  then  $as \leq bs$ , and for all  $a \in A$  and  $s, t \in S$ , if  $s \leq t$  then  $as \leq at$ . A right  $S$ -poset is denoted by  $A_S$ . A left  $S$ -poset can be defined analogously and is denoted by  ${}_S A$ . If  $A$  and  $B$  are  $S$ -posets, a mapping  $f : A \rightarrow B$  is called an  $S$ -poset homomorphism if  $f$  preserves the order, and  $f$  is compatible with the  $S$ -action, i.e., for all  $a \in A$  and  $s \in S$  we have  $f(as) = f(a)s$ . If  $A$  and  $B$  are right (resp.

\*Speaker.

†Speaker.

left)  $S$ -posets, then  $f$  is called a right (resp. left)  $S$ -poset homomorphism. The category whose objects are  $S$ -posets and whose morphisms are  $S$ -poset homomorphisms is denoted by  $\text{Pos-}S$  (or  $S$ -pos). An  $S$ -poset monomorphism (resp. epimorphism) is precisely an  $S$ -poset homomorphism which is injective (resp. surjective). The kernel of an  $S$ -poset homomorphism  $f : A \rightarrow B$  is defined by  $\ker f = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ . Now, let  $H = \{(as, b), (a, sb) \mid a \in A, b \in B, s \in S\}$  where  $A$  and  $B$  are  $S$ -posets and  $S$  is a partially ordered monoid. Let  $\rho(H)$  denote the congruence on  $A \times B$  generated by  $H$ . Clearly,  $\rho(H)$  is the smallest  $S$ -poset congruence on  $A \times B$  which identifies all pairs  $((as, b), (a, sb))$  for  $a \in A, b \in B$  and  $s \in S$ , where  $S$  is regarded as a poset with trivial  $S$ -action. Therefore, the *tensor product* of  $A$  and  $B$  over  $S$ , denoted  $A \otimes_S B$ , is defined as the quotient  $(A \times B)/\rho(H)$ . The equivalence class of  $(a, b)$  is denoted  $a \otimes b$ , and for each  $a \in A, b \in B$ , and  $s \in S$ , we have  $as \otimes b = a \otimes sb$ .

Now, let  $S$  be a partially ordered monoid and  $B$  a left  $S$ -poset. Then  $B$  is said to be *flat* if and only if for every order-embedding  $f : A \rightarrow C$  of right  $S$ -posets, the induced map  $f \otimes id_B : A \otimes B \rightarrow C \otimes B$ ,  $(f \otimes id_B)(a \otimes b) = f(a) \otimes b$ , is injective. A right  $S$ -poset  $X$  is called *quasi-flat* if for every left  $S$ -poset  $B$ , every flat left  $S$ -poset  $C$ , and every  $S$ -poset monomorphism  $f : B \rightarrow C$ , the induced map  $1 \otimes f : X \otimes B \rightarrow X \otimes C$  is injective. Suppose  $f : X \rightarrow Y$  is a right  $S$ -poset homomorphism and  $g : A \rightarrow B$  is a left  $S$ -poset homomorphism. The pair  $(f, g)$  is said to be *stable* (or  $f$  stabilizes  $g$ ) if  $\text{im}(f \otimes 1) \cap \text{im}(1 \otimes g) = \text{im}(f \otimes g)$ . Equivalently, if  $y \otimes g(a) = f(u) \otimes b \in Y \otimes B$ , then there exist  $x' \in X$  and  $a' \in A$  such that  $y \otimes g(a) = f(x') \otimes g(a')$ . A monomorphism  $f$  of right  $S$ -posets is called *stable* if  $(f, g)$  is stable for all left  $S$ -poset monomorphisms  $g$ . A right  $S$ -poset monomorphism  $f : X \rightarrow Y$  is said to be *quasi-stable* if for all flat right  $S$ -posets  $B$  and all left  $S$ -poset monomorphisms  $\lambda : A \rightarrow B$ , the pair  $(f, \lambda)$  is stable. Let  $X$  be a left  $S$ -poset. A congruence relation on  $X$  is defined as follows:  $x \sim y$  if and only if either  $x = y$ , or there exist  $x_1, \dots, x_n \in X$  and  $s_1, t_1, \dots, s_n, t_n \in S$  such that

$$\begin{aligned} x &\leq s_1 x_1, \\ t_1 x_1 &\leq s_2 x_2, \\ &\vdots \\ t_n x_n &\leq y. \end{aligned}$$

We say that  $x$  is *connected* to  $y$  if  $x \sim y$ . A right  $S$ -poset  $A$  is called *reversible* if for all  $a, b \in A$ ,  $aS \cap bS \neq \emptyset$  implies that there exist  $s, t \in S$  such that  $as = bt$ . For more see [3], [4] and [5]

## 2. Main Results

In this section, we explore the connections between reversibility of partially ordered monoids and structural properties of  $S$ -posets. We show how left and right reversibility can be characterized via connectedness and reversibility of  $S$ -posets, and we study the effect of quotients on (quasi-)flatness. Furthermore, we examine the role of stable and quasi-stable morphisms in preserving flatness, leading to criteria that link algebraic conditions on  $S$  with categorical properties of  $S$ -posets.

LEMMA 2.1. *If  $S$  is a partially ordered monoid, then the following statements are equivalent:*

- (1)  $S$  is left reversible.

- (2) Every connected right  $S$ -poset is reversible.
- (3) Every sub- $S$ -poset of a connected right  $S$ -poset is connected.

LEMMA 2.2. Let  $f : X \rightarrow Y$  be a left  $S$ -poset monomorphism with  $X \neq \emptyset$ . If  $Y/X$  is quasi-flat, then  $S$  is left reversible.

LEMMA 2.3. Let  $f : X \rightarrow Y$  be an  $S$ -poset monomorphism and let  $A$  be a left partially ordered  $S$ -set. Then  $\bar{y} \otimes a = \bar{y}' \otimes a'$  in  $(Y/X) \otimes A$  if and only if either  $y \otimes a = y' \otimes a'$  in  $Y \otimes_S A$ , or there exist elements  $a_1, a'_1 \in A$  and  $x_1, x'_1 \in X$  such that  $y \otimes a = f(x_1) \otimes a_1$ ,  $y' \otimes a' = f(x'_1) \otimes a'_1$  in  $Y \otimes_S A$ .

COROLLARY 2.4. Let  $f : X \rightarrow Y$  be an  $S$ -poset monomorphism and let  $A$  be a left partially ordered  $S$ -set. Then  $\overline{f(x)} \otimes a = \overline{f(x')} \otimes a'$  in  $Y/X \otimes A$  if and only if  $a \sim a'$  in  $A$ .

LEMMA 2.5. Let  $f : X \rightarrow Y$  be an  $S$ -poset monomorphism and let  $B$  be a left partially ordered  $S$ -set. If  $y \otimes b = f(x) \otimes b'$  in  $Y \otimes_S B$ , then  $\bar{y} \otimes b = \overline{f(x)} \otimes b'$  in  $(Y/X) \otimes B$ .

COROLLARY 2.6. Let  $f : X \rightarrow Y$  be an  $S$ -poset monomorphism and suppose that  $Y$  is (quasi-)flat. Then  $Y/X$  is (quasi-)flat if and only if  $S$  is right reversible and  $f$  is (quasi-)stable.

PROPOSITION 2.7. Let  $f : X \rightarrow Y$  be an  $S$ -poset monomorphism and let  $A$  be a left partially ordered  $S$ -set. Then:

- (1)  $\ker(Y \otimes A \rightarrow (Y/X) \otimes A) = \{(f(x) \otimes a, f(x') \otimes a') \mid a \sim a'\} \cup 1_{X \otimes A}$ ;
- (2) If  $Y/X$  is quasi-flat, then  $S$  is right reversible;
- (3) If  $y \otimes a = f(x) \otimes a$  in  $Y \otimes A$ , then  $\bar{y} \otimes a = \overline{f(x)} \otimes a$  in  $Y/X \otimes A$ ;
- (4) If both  $X$  and  $Y/X$  are quasi-flat, then  $Y$  is quasi-flat;
- (5) If  $Y$  is (quasi-)flat, then  $Y/X$  is (quasi-)flat if and only if  $S$  is right reversible and  $f$  is (quasi-)stable.

PROOF. (1) Define the map  $\pi \otimes 1_A : Y \otimes A \rightarrow Y/X \otimes A$  by  $(y \otimes a) \mapsto \bar{y} \otimes a$ , where  $\bar{y} = \pi(y)$  and  $\pi : Y \rightarrow Y/X$  is the canonical projection, so that  $\bar{y}$  is the equivalence class of  $y$  in  $Y/X$ . The kernel of this map is  $\ker(\pi \otimes 1_A) = \{(y_1 \otimes a_1, y_2 \otimes a_2) \in (Y \otimes A) \times (Y \otimes A) \mid (\pi \otimes 1)(y_1 \otimes a_1) = (\pi \otimes 1)(y_2 \otimes a_2)\}$ , i.e.,  $\bar{y}_1 \otimes a_1 = \bar{y}_2 \otimes a_2$  in  $Y/X \otimes A$ .

Case 1: If  $y_1 = y_2$  and  $a_1 = a_2$ , then  $y_1 \otimes a_1 = y_2 \otimes a_2$ , i.e.,  $(y_1 \otimes a_1, y_2 \otimes a_2) \in 1_{Y \otimes A}$ .

If  $y_1, y_2 \in \text{im}(f)$ , suppose  $x_1, x_2 \in X$  exist such that  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ . Then  $\bar{y}_1 = \overline{f(x_1)} = \overline{f(x_2)} = \bar{y}_2$ , since all elements of  $\text{im}(f)$  lie in the same class. Hence, we have  $\overline{f(x_1)} \otimes a_1 = \overline{f(x_2)} \otimes a_2$  provided  $a_1 \sim a_2$ . Therefore,  $\ker(\pi \otimes 1_A) \subseteq \{(f(x_1) \otimes a_1, f(x_2) \otimes a_2) \mid a_1 \sim a_2\} \cup 1_{X \otimes A}$ , and the reverse inclusion is straightforward. This completes the proof of (1).

(2) Assume  $Y/X$  is semi-flat, i.e., for every left  $S$ -poset  $A$ , every flat left  $S$ -poset  $B$ , and every  $S$ -poset monomorphism  $k : A \rightarrow B$ , the induced map  $1_{Y/X} \otimes k : Y/X \otimes A \rightarrow Y/X \otimes B$  is injective. To show that  $S$  is right reversible, we need to show that any two principal left ideals of  $S$  intersect, i.e.,  $Sa \cap Sb \neq \emptyset$  for all  $a, b \in S$ . Since  $1_{Y/X} \otimes k$  is injective, if  $(1_{Y/X} \otimes k)(z_1) = (1_{Y/X} \otimes k)(z_2)$  then  $z_1 = z_2$ . Here  $z_1, z_2 \in Y/X \otimes A$  are of the form  $\bar{y} \otimes a$ ,

with  $\bar{y} \in Y/X$  and  $a \in A$ . Let  $A = S$  with left action  $s \cdot a = sa$ , and let  $B = S \times \{0, 1\}$  with action  $s \cdot (b, i) = (sb, i)$ . Define  $k : A \rightarrow B$  by  $k(a) = (a, 0)$ . Then  $k$  is injective. Since  $Y/X$  is semi-flat, the induced map  $1_{Y/X} \otimes k : Y/X \otimes S \rightarrow Y/X \otimes (S \times \{0, 1\})$  is injective. Suppose  $S$  is not right reversible, i.e., there exist  $u, v \in S$  such that  $Su \cap Sv = \emptyset$ . Consider  $\bar{y} \in Y/X$  arbitrary. For  $z_1 = \bar{y} \otimes ua$  and  $z_2 = \bar{y} \otimes vb$  in  $Y/X \otimes S$ , we have  $(1_{Y/X} \otimes k)(\bar{y} \otimes ua) = \bar{y} \otimes (ua, 0)$ ,  $(1_{Y/X} \otimes k)(\bar{y} \otimes vb) = \bar{y} \otimes (vb, 0)$ . If these are equal, then  $ua = vb$ , which contradicts  $Su \cap Sv = \emptyset$ . Hence  $S$  must be right reversible.

(3) Follows directly from Lemma (2.5).

(4) Assume  $f : X \rightarrow Y$  is an  $S$ -poset monomorphism with  $X$  and  $Y/X$  semi-flat. We show that  $Y$  is semi-flat. Let  $\lambda : A \rightarrow B$  be an  $S$ -poset monomorphism with  $B$  flat. Consider the induced map  $1_Y \otimes \lambda : Y \otimes A \rightarrow Y \otimes B$ ,  $y \otimes a \mapsto y \otimes \lambda(a)$ . Suppose  $(1_Y \otimes \lambda)(y \otimes a) = (1_Y \otimes \lambda)(y' \otimes a')$ , i.e.,  $y \otimes \lambda(a) = y' \otimes \lambda(a')$ . From Lemma (2.5), this implies  $\bar{y} \otimes \lambda(a) = \bar{y}' \otimes \lambda(a')$  in  $(Y/X) \otimes B$ . Since  $Y/X$  is semi-flat,  $1_{Y/X} \otimes \lambda$  is injective, so  $\bar{y} \otimes a = \bar{y}' \otimes a'$  in  $Y/X \otimes A$ . By Lemma (2.3), this occurs if and only if either  $y \otimes a = y' \otimes a'$  or there exist  $x_1, x_2 \in X$  and  $a_1, a_2 \in A$  such that

$$(1) \quad y \otimes a = f(x_1) \otimes a_1, \quad y' \otimes a' = f(x_2) \otimes a_2.$$

Since  $B$  is flat,  $f \otimes 1_B : X \otimes B \rightarrow Y \otimes B$  is injective. From (1) we obtain  $x_1 \otimes \lambda(a_1) = x_2 \otimes \lambda(a_2)$  in  $X \otimes B$ . Since  $X$  is semi-flat,  $1 \otimes \lambda$  is injective, giving

$$(2) \quad x_1 \otimes a_1 = x_2 \otimes a_2 \quad \text{in } X \otimes A.$$

Combining (1) and (2), we conclude  $y \otimes a = y' \otimes a'$ , so  $1_Y \otimes \lambda$  is injective. Therefore,  $Y$  is semi-flat.

(5) Follows from Corollary (2.6). □

### 3. Conclusion

We established the equivalence between reversibility of partially ordered monoids and connectedness properties of  $S$ -posets. Moreover, we showed how (quasi-)flatness is preserved under quotients and characterized its transfer via stable and quasi-stable morphisms. These results highlight the deep interplay between algebraic conditions on  $S$  and categorical properties of  $S$ -posets.

### References

1. Bulman-Fleming, S., Gutermuth, D., Gilmour, A., and Kilp, M. (2006) *Flatness Properties of S-Posets*, Communications in Algebra, **34**(4), 1291–1317.
2. Bulman-Fleming, S. and McDowell, K. (1983) *Absolutely flat semigroups*, Pacific J. Math., **107** (1983), 319–333.
3. Khosravi, R., Ershad, M., and Sedaghatjoo, M. (2010), *Strongly Flat and Condition (P) Covers of Acts Over Monoids*, Communications in Algebra, **38**(12), 4520–4530.
4. Rasouli, H. (2014), *Categorical properties of regular monomorphisms of S-posets*, European Journal of Pure and Applied Mathematics, **7**(2), 166–178.
5. Renshaw, J. (2001) *Stability and amalgamation in monoids*, Communications in Algebra, **29**(3), 1095–1110.



## On the $d$ -Dimension of Finitely Generated Modules

Mirsadegh Sayedsadeghi<sup>1</sup>, and SayedHamid Masoudi<sup>2\*</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: m\_sayedsadeghi@pnu.ac.ir

<sup>2</sup>Department of Mathematics, ImamAli univercity, Tehran, Iran.

Email: masoudipourlir@iamu.ac.ir

**ABSTRACT.** Let  $R$  be a commutative Noetherian ring with identity and  $d$  a nonzero integer. This paper investigates the  $d$ -dimension, denoted  $\dim(d, M)$ , for finitely generated  $R$ -modules  $M$ , within the framework of  $d$ -local cohomology. We establish key inequalities and equalities relating the  $d$ -dimensions of modules under support conditions and in short exact sequences. In particular, it is shown that if  $\text{Supp}(N) \subseteq \text{Supp}(M)$ , then  $\dim(d, N) \leq \dim(d, M)$ , and equality holds when the supports coincide. Furthermore, for an exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  of finitely generated modules, we prove that  $\dim(d, M) = \max\{\dim(d, L), \dim(d, N)\}$ . These results extend our understanding of the behavior of  $d$ -local cohomology and its associated dimension invariant.

**Keywords:** finitely generated,  $d$ -dim

**AMS Mathematics Subject Classification [2020]:** 13C05, 13C12, 13D45

### 1. Introduction

Throughout this paper,  $R$  denotes a commutative Noetherian ring with identity, and  $d$  is a nonzero integer. Our work operates within the framework of  $d$ -local cohomology, a theory initially developed by Banica and Soia [1]. For an  $R$ -module  $M$ , they defined the functor  $L_d(M)$  as the set of all elements  $m \in M$  such that  $\dim(Rm) \leq d$ . Equivalently,  $L_d(M) = \{m \in M \mid \mathfrak{a}m = 0 \text{ for some } \mathfrak{a} \in \Sigma\}$ , where  $\Sigma$  is the system of ideals  $\{\mathfrak{a} \in \mathcal{I}(R) \mid \dim(R/\mathfrak{a}) \leq d\}$  in the sense of Brodmann and Sharp [2, p.21]. An  $R$ -module  $M$  is  $d$ -torsion ( $d$ -torsion-free) when  $L_d(M) = M$  ( $L_d(M) = 0$ ). A key result is that  $L_d(M) \cong \varinjlim_{\mathfrak{a} \in \Sigma} \text{Hom}_R(R/\mathfrak{a}, M)$ , establishing a direct link to a direct limit of extension functors.

The  $i$ -th right derived functor of  $L_d(-)$ , denoted  $H_d^i(-)$ , was subsequently shown to be isomorphic to  $\varinjlim_{\mathfrak{a} \in \Sigma} \text{Ext}_R^i(R/\mathfrak{a}, M)$  [1]. This provides a concrete realization of these cohomology modules and connects them to a well-understood algebraic construction. Further foundational studies on the properties of  $H_d^i(M)$  have been conducted in [4] and [5], exploring their structure and behavior.

\*Speaker.

Let  $M$  be a finitely generated  $R$ -module. Then

$$\dim(d, M) = \sup\{\dim(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \Sigma\},$$

which is introduced in [1, p. 12] and it has been proven that

$$\dim(d, M) = \sup\{i \in \mathbb{N}_0 \mid H_d^i(M) \neq 0\}.$$

## 2. Main results

This section presents the principal findings of this paper, focusing on the behavior of the invariant  $\dim(d, -)$  in relation to finitely generated modules over a commutative Noetherian ring. We begin by establishing a fundamental inequality between the  $d$ -dimensions of two modules when one is supported within the support of the other. This result leads naturally to a corollary affirming the equality of  $d$ -dimensions for modules sharing the same support. Furthermore, we investigate the behavior of  $\dim(d, -)$  in short exact sequences, demonstrating that it satisfies a natural additivity property. These results extend our understanding of  $d$ -local cohomology and provide useful tools for further structural analysis.

**THEOREM 2.1.** *Let  $M, N$  two finitely generated  $R$ -modules such that*

$$\text{Supp}(N) \subseteq \text{Supp}(M).$$

*Then*

$$\dim(d, N) \leq \dim(d, M).$$

**PROOF.** It suffices to show that for every  $i > \dim(d, M)$ , we have  $H_d^i(N) = 0$ . If this is proven, then by definition,  $\dim(d, N) \leq \dim(d, M)$ . We now prove the claim by descending induction on  $i$ . For  $i = \dim M + 1$ , since

$$\dim(d, N) \leq \dim N \leq \dim M < i,$$

by [1], we conclude that  $H_d^i(N) = 0$ .

Now suppose  $\dim(d, M) < i \leq \dim M$ . We show that  $H_d^i(N) = 0$ . Since  $M$  and  $N$  are finitely generated  $R$ -modules and

$$\text{Supp}(N) \subseteq \text{Supp}(M),$$

we have

$$V(\text{Ann}_R N) \subseteq V(\text{Ann}_R M),$$

which implies

$$\sqrt{\text{Ann}_R M} \subseteq \sqrt{\text{Ann}_R N}.$$

Since  $R$  is a Noetherian ring, there exists  $n \in \mathbb{N}$  such that  $\mathfrak{J}^n \subseteq \text{Ann}_R N$ , where  $\mathfrak{J} = \text{Ann}_R M$ . Thus, we have a chain of submodules of  $N$ :

$$0 = \mathfrak{J}^n N \subset \mathfrak{J}^{n-1} N \subset \cdots \subset \mathfrak{J} N \subset N.$$

For each  $1 \leq i \leq n$ , the quotient module  $\mathfrak{J}^{i-1} N / \mathfrak{J}^i N$  is finitely generated, and by [3, Corollary 1.5], it has a finite chain such that each successive quotient is a homomorphic image of a finite direct sum of copies of  $M$ .

We assume  $\dim M < i \leq \dim M$  (this seems to be a typo; likely it should be  $\dim(d, M) < i \leq \dim M$ ). We show that  $H_d^i(T) = 0$  for  $T = N / \mathfrak{J}^i N$ . We proceed by induction on the length of the chain. For  $k = 1$ , we have an exact sequence

$$0 \rightarrow K \rightarrow M^t \rightarrow T \rightarrow 0$$

for some positive integer  $t$ . This yields the long exact sequence

$$\cdots \rightarrow H_d^i(K) \rightarrow H_d^i(M^t) \rightarrow H_d^i(T) \rightarrow H_d^{i+1}(K) \rightarrow \cdots$$

Since  $K$  is a submodule of  $M^t$ , it is finitely generated, and

$$\text{Supp}(K) \subseteq \text{Supp}(M^t) = \text{Supp}(M).$$

By the induction hypothesis,  $H_d^i(M^t) = \bigoplus H_d^i(M) = 0$  (since  $i > \dim(d, M)$ ) and similarly  $H_d^{i+1}(K) = 0$ . Therefore,  $H_d^i(T) = 0$ .

Continuing this process, we eventually obtain a long exact sequence

$$\cdots \rightarrow H_d(N_{k-1}) \rightarrow H_d(N_k) \rightarrow H_d(N_{k-1}) \rightarrow \cdots$$

from which it follows that  $H_d(N_k) = 0$ . Hence, the claim holds, i.e.,

$$\dim(d, N) \leq \dim(d, M).$$

□

**COROLLARY 2.2.** *Let  $M, N$  two finitely generated  $R$ -modules such that*

$$\text{Supp}N = \text{Supp}M.$$

*Then*

$$\dim(d, N) = \dim(d, M).$$

**PROOF.** This is clear by Theorem 2.1. □

**THEOREM 2.3.** *Let*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

*an exact sequence of finitely generated  $R$ -modules. Then*

$$\dim(d, M) = \max\{\dim(d, N), \dim(d, L)\}.$$

**PROOF.** We have  $\text{Supp}(L) \subseteq \text{Supp}(M)$ ,  $\text{Supp}(N) \subseteq \text{Supp}(M)$ . Then by Theorem 2.1, we have  $\dim(d, L) \leq \dim(d, M)$  and  $\dim(d, N) \leq \dim(d, M)$ , so

$$\max\{\dim(d, N), \dim(d, L)\} \leq \dim(d, M).$$

Conversely, suppose that for  $t = \max\{\dim(d, N), \dim(d, L)\}$ , we have  $H_d^i(N) = 0$  and  $H_d^i(L) = 0$  for all  $i > t$ . From the exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0,$$

we obtain the long exact sequence

$$\cdots \rightarrow H_d^i(L) \rightarrow H_d^i(M) \rightarrow H_d^i(N) \rightarrow \cdots$$

It follows that  $H_d^i(M) = 0$  for all  $i > t$ . Therefore,

$$\dim(d, M) \leq \max\{\dim(d, N), \dim(d, L)\}.$$

This completes the proof. □

**COROLLARY 2.4.** *Let  $M$  be a finitely generated  $R$ -module and let  $\mathfrak{a}$  be an ideal of  $R$  such that  $V(\mathfrak{a}) \subseteq \text{Supp}(M)$ . Then*

$$\dim(d, M/\mathfrak{a}M) \leq \dim(d, M).$$

**PROOF.** Consider the canonical surjection  $M \rightarrow M/\mathfrak{a}M$ . Since

$$\text{Supp}(M/\mathfrak{a}M) \subseteq \text{Supp}(M),$$

the result follows immediately from Theorem 2.1. □

## References

1. Banica, C. and Soia, M. (1976) *Singular sets of a module on local cohomology*, Boll. Un. Mat. Ital. B., **16**, 923–934.
2. Brodmann, N. P. and Sharp, R. Y. (1998) *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge.
3. Johnson, B. (2009) *A Theorem of Gruson*.
4. Zamani, N., Bijan-zadeh, M. H. and Sayedsadeghi, M. S. (2016) *Cohomology with supports of dimension  $\leq d$* , J. Algebra Appl., **15**(4), 1650042, 10 pp.
5. Zamani, N., Bijan-Zadeh, M. H. and Sayedsadeghi, M. S. (2013)  *$d$ -Transform functor and some finiteness and isomorphism results*, Vietnam J. Math., **41**(2), 179–186.







## $p$ -solvable Finite Groups, under special prime $p = 11$

Sara Pouyandeh\*

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: s.pouyandeh124@pnu.ac.ir

**ABSTRACT.** Let  $G$  be a finite group, and let  $m(G) = \sum_{g \in G} 1/o(g)$ , where  $o(g)$  denotes the order of  $g$ . In this paper, we determine all groups such that  $m(G) < 4$ . we provide a criterion for  $p$ -solvability using the function  $m(G)$ , where  $p \in \{11\}$  then  $G$  is a  $p$ -solvable group.

**Keywords:** Finite group, element order,  $p$ -solvability,

**AMS Mathematics Subject Classification [2020]:** 20D10, 20D15, 20D20.

### 1. Introduction

Let  $G$  be a finite group and  $\psi(G) = \sum_{g \in G} o(g)$ , where  $o(g)$  denotes the order of  $g \in G$ , which was introduced by Amiri et al. They showed that  $C_n$  is the unique group of order  $n$  with the largest value of  $\psi(G)$  for groups of that order. In [3], Herzog, Longobardi and Maj determined the exact upper bound for  $\psi(G)$  for non-cyclic groups  $G$ . There are some applications for  $\psi(G)$ , for example,  $\psi(G)$  is equal to the sum of the number of arcs and the number of vertices of a directed power graph.

A finite group  $G$  is a  $B_\psi$ -group if  $\psi(H) < |G|$  for all proper subgroups  $H$  of  $G$ . Baniasad Azad showed that if  $S$  is a finite simple group, such that  $S \neq Alt(n)$  for any  $n \geq 14$ , then  $S$  is a  $B_\psi$ -group. The function  $\psi$  has been considered in various works (see [4]).

The functions  $m(G) = \sum_{g \in G} 1/o(g)$ ,  $l(G) = \sqrt[\nu]{\prod_{g \in G} o(g)/|G|}$  and  $\psi'(G) = \psi(G)/\psi(C_n)$  were introduced in [4]. Many authors investigate the influence of these functions on the structure of a finite group  $G$ . For example, if  $g \in \{\psi', l, m\}$ , and  $g(G) > g(C_2 \times C_2)$ ,  $g(G) > g(S_3)$ ,  $g(G) > g(A_4)$  or  $g(G) > g(A_5)$ , then  $G$  is cyclic, nilpotent, supersolvable or solvable, respectively (see [3, 4]).

Tărnăuceanu in [6], introduced  $\psi''(G) = \psi(G)/|G|^2$  and also proved the following theorem:

**THEOREM 1.1.** [6, Theorem 1.1] *Let  $G$  be a finite group. Then the following holds:*

- (a) *If  $\psi''(G) > 7/16 = \psi''(C_2 \times C_2)$ , then  $G$  is cyclic.*
- (b) *If  $\psi''(G) > 27/64 = \psi''(Q_8)$ , then  $G$  is abelian.*

\*Speaker.

- (c) If  $\psi''(G) > 13/36 = \psi''(S_3)$ , then  $G$  is nilpotent.
- (d) If  $\psi''(G) > 31/144 = \psi''(A_4)$ , then  $G$  is supersolvable.
- (e) If  $\psi''(G) > 211/3600 = \psi''(A_5)$ , then  $G$  is solvable.

In [1], Baniasad Azad and Khosravi proved the following theorem:

**THEOREM 1.2.** [1, Main Theorem] *Let  $G$  be a finite group such that  $\psi''(G) > \psi''(D_{2p})$ , where  $p$  is a prime number. Then  $G \cong O_p(G) \times O_{p'}(G)$  and  $O_p(G)$  is cyclic.*

In this paper, we focus on the function  $\psi''(G)$ . We give a criterion for  $p$ -solvability by the function  $\psi''$ , where  $p \in \{11\}$ . We prove that if  $G$  is a finite group and  $\psi''(G) > \psi''(\text{PSL}(2, p))$ , where  $p \in \{11\}$ , then  $G$  is a  $p$ -solvable group.

## 2. Main results

We need the following lemmas.

**LEMMA 2.1.** [7, Lemma 1] *Let  $G$  be a non-solvable group. Then  $G$  has a normal series  $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$  such that  $K/H$  is a direct product of isomorphic non-abelian simple groups and  $|G/K| \mid |\text{Out}(K/H)|$ .*

**LEMMA 2.2.** [5] *Let  $A$  be a cyclic proper subgroup of a finite group  $G$ , and let  $K = \text{core}_G(A)$ . Then  $|A : K| < |G : A|$ , and in particular, if  $|A| > |G : A|$ , then  $K > 1$ .*

**LEMMA 2.3.** [1, Lemma 2.1] *If  $\psi''(G) > t$ , then  $G$  has an element  $x$  such that  $|G : \langle x \rangle| < 1/t$ .*

**LEMMA 2.4.** [6] *Let  $H$  be a normal subgroup of the finite group  $G$ . Then  $\psi''(G) \leq \psi''(G/H)$ .*

**THEOREM 2.5.** *Let  $G$  be a finite group such that  $m(G) < m(\text{PSL}(2, 11))$ . Then  $G$  is an 11-solvable group.*

**PROOF.** We prove that  $G$  is solvable by induction on  $|G|$ . If  $|G| < 660$  or  $11 \nmid |G|$ , then  $G$  is an 11-solvable group. If  $G$  has a non-trivial normal 11-solvable subgroup  $K$ , then by Lemma 2.4,

$$m(G/K) \leq m(G) < m(\text{PSL}(2, 11)),$$

and so by the inductive hypothesis,  $G/K$  is an 11-solvable group, and consequently,  $G$  is 11-solvable. Therefore suppose that  $G$  has no non-trivial normal 11-solvable subgroup.

Since  $m(G) < m(\text{PSL}(2, 11)) = 16193/110 \simeq 147.2$ , using Lemma 2.3 we get that there exists an element  $x \in G$  such that

$$(1) \quad |G : \langle x \rangle| \leq 147.$$

Using Lemma 2.2,  $|\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 146$ . Therefore

$$|G : \text{core}_G(\langle x \rangle)| = |G : \langle x \rangle| \cdot |\langle x \rangle : \text{core}_G(\langle x \rangle)| \leq 147 \cdot 146 = 21462.$$

Since  $\text{core}_G(\langle x \rangle) = 1$ , so  $|G| \leq 21462$  and  $G$  is not an 11-solvable group. By Lemma 2.1,  $G$  has a normal series  $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$  such that  $N/M$  is isomorphic to a direct product of a non-abelian simple group  $S$  and  $|G/N| \mid |\text{Out}(N/M)|$ . If  $M$  is not an 11-solvable, then  $|N| = |N/M| \cdot |M|$  divides  $G$ . Therefore  $60 \cdot 660 \leq |G|$ , which is a contradiction. Thus  $M$  is 11-solvable and so  $M = 1$ .

By [2], we have

$$S \in \{\text{PSL}(2, q) \mid q = 5, 7, 8, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31\}$$

$$\cup \{A_6, A_7, A_8, \text{PSL}(3, 3), \text{PSL}(3, 4), \text{PSU}(3, 3), M_{11}\}.$$

Now, we consider two following cases:

- (1) Let  $11 \nmid |S|$ . Then by [2]  $11 \nmid |\text{Out}(N/M)| = |\text{Out}(N)|$ . Therefore  $|G| = |G/N| \cdot |N|$  and also  $|G/N| \mid |\text{Out}(N)|$ . Therefore  $11 \nmid |G|$  and so  $G$  is 11-solvable.
- (2) Let  $11 \mid |S|$ . Then by [2], we have

$$S \in \{\text{PSL}(2, 11), \text{PSL}(2, 23), M_{11}\}$$

$G$	$PSL(2, 11)$	$PSL(2, 23)$	$M_{11}$
$m(G)$	$\frac{16193}{110}$	$\frac{656686}{759} \simeq 865.1$	$\frac{459649}{330} \simeq 1392.8$

TABLE 1. The value of  $m(G)$  for some simple groups.

We see that  $m(G) \geq m(\text{PSL}(2, 11))$ , which is a contradiction.

The proof is now complete. □

### 3. Conclusion

In this paper, we proved that if  $G$  is a finite group and  $m(G) < m(\text{PSL}(2, 11))$ , where  $p \in \{11\}$ , then  $G$  is a  $p$ -solvable group.

### Acknowledgement

The author would like to thank the referees for their valuable comments which improved the manuscript.

### References

1. Baniasad Azad, M., & Khosravi, B. (2020). A criterion for  $p$ -nilpotency and  $p$ -closedness by the sum of element orders. *Comm. Algebra* **48**(12) 5391–5395. <https://doi.org/10.1080/00927872.2020.1788571>
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups, Oxford Univ. Press, Oxford, 1985.
3. Herzog, M., Longobardi, P., & Maj, M. (2018). An exact upper bound for sums of element orders in non-cyclic finite groups. *J. Pure Appl. Algebra* **222**(7) 1628–1642. <https://doi.org/10.1016/j.jpaa.2017.07.015>
4. Herzog, M., Longobardi, P., & Maj, M. (2018). Two new criteria for solvability of finite groups. *J. Algebra* **511**, 215–226. <https://doi.org/10.1016/j.jalgebra.2018.06.015>
5. Lucchini, A. (1998). On the order of transitive permutation groups with cyclic point-stabilizer. *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.*, **9**(4), 241–243.
6. Tărnăuceanu, M. (2020). Detecting structural properties of finite groups by the sum of element orders. *Israel J. Math.* **238**(2), 629–637. DOI:10.1007/s11856-020-2033-9
7. Xu, H., Chen, G., & Yan, Y. (2014). A new characterization of simple  $K_3$ -groups by their orders and large degrees of their irreducible characters. *Comm. Algebra* **42**(12), 5374–5380. DOI:10.1080/00927872.2013.842242



## A note on commuting mappings

A. H. Mokhtari\*

Technical Faculty of Ferdows, University of Birjand, Birjand, Iran.

Email: a.mokhtari@birjand.ac.ir

---

**ABSTRACT.** This extended abstract is based on the journal paper [4] published in the International Journal of Nonlinear Analysis and Applications. We present the main results with focus on applications to commutant preservers. This paper studies proper linear mappings on generalized matrix algebras. We characterize their general form and establish necessary and sufficient conditions for properness. Applications to commuting mappings, Lie centralizers, and Lie triple centralizers are investigated. Our results extend and unify several known characterizations for triangular algebras and nest algebras.

**Keywords:** Generalized matrix algebra, commuting mapping, Lie centralizer, Lie triple centralizer, proper mapping

**AMS Mathematics Subject Classification [2020]:** 15B33, 16W25, 47L35

---

### 1. Introduction

The study of linear mappings with special properties on algebras has been an active research area in mathematics [1–3, 5]. Among these, commuting mappings, Lie centralizers, and their various generalizations have attracted considerable attention due to their fundamental importance in the structure theory of algebras and operator algebras.

Let  $\mathcal{R}$  be a unital commutative ring and  $\mathfrak{A}$  be a unital algebra over  $\mathcal{R}$ . A linear mapping  $\phi : \mathfrak{A} \rightarrow \mathfrak{A}$  is called a *commuting mapping* if  $[\phi(x), x] = 0$  for all  $x \in \mathfrak{A}$ , where  $[x, y] = xy - yx$  denotes the Lie product. A linear mapping  $\phi$  is called a *Lie centralizer* if  $\phi([x, y]) = [\phi(x), y]$  for all  $x, y \in \mathfrak{A}$ , and a *Lie triple centralizer* if  $\phi([[x, y], z]) = [[\phi(x), y], z]$  for all  $x, y, z \in \mathfrak{A}$ .

A linear mapping  $\phi : \mathfrak{A} \rightarrow \mathfrak{A}$  is called *proper* if there exist  $\lambda \in Z(\mathfrak{A})$  and a linear mapping  $\tau : \mathfrak{A} \rightarrow Z(\mathfrak{A})$  such that  $\phi(x) = \lambda x + \tau(x)$  for all  $x \in \mathfrak{A}$ .

Generalized matrix algebras provide a unified framework that includes many important classes of algebras such as full matrix algebras, triangular algebras, nest algebras, and various operator algebras. In this paper, we investigate proper linear mappings on generalized matrix algebras and establish conditions under which various types of linear mappings are proper.

---

\*Speaker.

## 2. Preliminaries

A *Morita context* consists of two algebras  $\mathcal{A}$  and  $\mathcal{B}$ , two bimodules  $\mathcal{M}$  (an  $(\mathcal{A}, \mathcal{B})$ -bimodule) and  $\mathcal{N}$  (a  $(\mathcal{B}, \mathcal{A})$ -bimodule), and two bimodule homomorphisms satisfying certain compatibility conditions. The associated *generalized matrix algebra* is:

$$\mathcal{G} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

with matrix operations defined in the natural way.

When  $\mathcal{N} = 0$ , the algebra  $\mathcal{G}(\mathcal{A}, \mathcal{M}, 0, \mathcal{B})$  is called a *triangular algebra*. Generalized matrix algebras include a wide range of algebras, making them of great importance in mathematics.

A generalized matrix algebra  $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$  is called *weakly faithful* if for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ :

$$\begin{aligned} a\mathcal{M} = 0, \mathcal{N}a = 0 &\Rightarrow a = 0 \\ \mathcal{M}b = 0, b\mathcal{N} = 0 &\Rightarrow b = 0 \end{aligned}$$

The center of a weakly faithful generalized matrix algebra is characterized by:

$$Z(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : am = mb, na = bn \text{ for all } m \in \mathcal{M}, n \in \mathcal{N} \right\}$$

There exists a unique algebra isomorphism  $\eta : \pi_{\mathcal{A}}(Z(\mathcal{G})) \rightarrow \pi_{\mathcal{B}}(Z(\mathcal{G}))$  such that  $am = m\eta(a)$  and  $na = \eta(a)n$  for all  $a \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$ ,  $m \in \mathcal{M}$ ,  $n \in \mathcal{N}$ .

## 3. Main Results

We begin by characterizing the structure of proper linear mappings on generalized matrix algebras.

**THEOREM 3.1.** *Let  $\mathcal{G} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$  be a generalized matrix algebra. A linear mapping  $\Phi : \mathcal{G} \rightarrow \mathcal{G}$  is proper if and only if it has the form:*

$$\Phi \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & cm \\ nc & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

where the component mappings satisfy specific compatibility conditions and  $c \in \pi_{\mathcal{A}}(Z(\mathcal{G}))$ .

For weakly faithful generalized matrix algebras, we obtain a more precise characterization:

**THEOREM 3.2.** *Let  $\mathcal{G}$  be a weakly faithful generalized matrix algebra. A linear mapping  $\Phi : \mathcal{G} \rightarrow \mathcal{G}$  is proper if and only if:*

$$\Phi \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} f_1(a) + g_1(m) + h_1(n) + k_1(b) & f_1(1_{\mathcal{A}})m - mf_4(1_{\mathcal{A}}) \\ k_4(1_{\mathcal{B}})n - nk_1(1_{\mathcal{B}}) & f_4(a) + g_4(m) + h_4(n) + k_4(b) \end{bmatrix}$$

with the component mappings satisfying compatibility conditions.

Our main application provides sufficient conditions for various types of linear mappings to be proper:

**THEOREM 3.3.** *Let  $\mathcal{G}$  be a weakly faithful generalized matrix algebra and assume:*

- (1)  $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{G}))$  or  $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$
- (2)  $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{G}))$  or  $[\mathcal{B}, \mathcal{B}] = \mathcal{B}$

Then the following mappings on  $\mathcal{G}$  are proper:

- Every commuting mapping
- Every Lie centralizer
- Every Lie triple centralizer

SKETCH OF PROOF. The proof proceeds by showing that under the given conditions, the component mappings in the general form satisfy the necessary conditions for properness. The weakly faithfulness condition ensures that the center has the desired structure, while the conditions on  $Z(\mathcal{A})$  and  $Z(\mathcal{B})$  guarantee that the appropriate ideals are large enough to contain the commutator subspaces.  $\square$

#### 4. Applications to Specific Algebras

The results obtained for generalized matrix algebras have important consequences for specific classes of algebras:

COROLLARY 4.1. *Every commuting mapping, Lie centralizer, and Lie triple centralizer on  $n \times n$  triangular matrix algebras is proper.*

COROLLARY 4.2. *Every commuting mapping, Lie centralizer, and Lie triple centralizer on nest algebras is proper.*

COROLLARY 4.3. *Every commuting mapping, Lie centralizer, and Lie triple centralizer on full matrix algebras  $M_n(\mathcal{A})$  ( $n \geq 2$ ) is proper.*

These results extend and unify several known theorems in the literature, including classical results by Cheung for triangular algebras and recent work by various authors for generalized matrix algebras.

The conditions in Theorem 3.3 are satisfied by many important classes of algebras, including:

- Full matrix algebras  $M_n(\mathcal{A})$  ( $n \geq 2$ )
- Upper triangular matrix algebras  $T_n(\mathcal{A})$
- Standard operator algebras on Banach spaces
- Factor von Neumann algebras
- Non-trivial nest algebras

#### Acknowledgement

The author would like to express sincere thanks to the referee for valuable comments and suggestions.

#### References

- [1] Benkovi, D. (2015) *Lie triple derivations of unital algebras with idempotents*, Linear and Multilinear Algebra, **63**, 141–165.
- [2] Brear, M. (2004) *Commuting maps: a survey*, Taiwanese J. Math., **8**, 361–397.
- [3] Cheung, W.S. (2001) *Commuting maps of triangular algebras*, J. London Math. Soc., **63**, 117–127.
- [4] Mokhtari, A.H. (2023) *More on proper commuting and Lie mappings on generalized matrix algebras*, International Journal of Nonlinear Analysis and Applications, **14** **8**, 311–325.
- [5] Xiao Z.K. and Wei F. (2010) *Commuting mappings of generalized matrix algebras*, Linear Algebra Appl., **433**, 2178–2197.

# Some results on the top general local cohomology modules

Seadat Ollah Faramarzi \*

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.  
Email: s.o.faramarzi@pnu.ac.ir, s.o.faramarzi@gmail.com

---

ABSTRACT. Let  $R$  be a commutative Noetherian local ring,  $M$  be a non-zero finitely generated  $R$ -module of dimension  $d$  and  $\Phi$  be a system of ideals of  $R$ . In this paper, we determine the annihilators and the set of attached prime ideals of top general local cohomology module  $H_{\Phi}^d(M)$ .

AMS Mathematics Subject Classification [2020]: 13D45, 14B15, 13E05

---

## 1. Introduction

Let  $R$  stand a commutative Noetherian with non-zero identity ring, throughout this paper. Let  $\Phi$  be a non-empty set of ideals of  $R$ . Suppose that  $I, J \in \Phi$  are arbitrary. If there is an ideal  $K \in \Phi$  such that  $K \subseteq IJ$ , then  $\Phi$  is said to be a system of ideals; see [2]. Moreover, for an  $R$ -module  $B$ ,  $\Gamma_{\Phi}(B)$  is a  $\Phi$ -torsion submodule of  $B$ , which is defined by

$$\Gamma_{\Phi}(B) = \{x \in B \mid \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}.$$

We denote the  $i$ th right derived functor of the functor  $\Gamma_{\Phi}$  by  $H_{\Phi}^i$ . Obviously, if  $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}\}$ , then this functor coincides with the ordinary local cohomology functor  $H_{\mathfrak{a}}^i$ .

The investigation of the annihilators of the local cohomology modules (for short,  $\text{Ann} H_{\Phi}^t$ ) is one of the important problems of local cohomology modules. This topic has been studied by various researchers, for example, [1], [5], [6], [7] and has led to some interesting results. One of the methods to investigate the  $\text{Ann} H_{\Phi}^t(M)$  is applying the cd-filtration  $\{M_i\}_{i=1}^c$ , which is one of the main results done by Lynch [6]. In section 3, for the highest non-vanishing local cohomology modules, we study the annihilators and attached primes w.r.t the system of ideals  $\Phi$  of  $R$  by using the cd-filtration w.r.t  $\Phi$ . First of all, we prove the following theorem that is one of the main results of this paper.

**Theorem 1.1.** Suppose that  $\Phi$  is a system of ideals of a local ring  $(R, \mathfrak{m})$  and that  $M$  is a  $d$ -dimensional finitely generated  $R$ -module. Then

$$\text{Att}_R(H_{\Phi}^d(M)) = \{\mathfrak{p} \in \text{Assh}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

Here  $\text{Assh}_R(M) = \{\mathfrak{p} \in \text{Ass}_R(M) : \dim_R(R/\mathfrak{p}) = d\}$ .

Next we compute  $\text{Ann} H_{\Phi}^d(M)$  for a complete local ring in terms of  $\mathfrak{m}$ -primary ideals. Finally we determine  $\text{Ann} H_{\Phi}^d(M)$  without completion condition on  $R$ . In this regard, we established the following theorem.

**Theorem 1.2.** Suppose that  $\Phi$  is a system of ideals of a local ring  $(R, \mathfrak{m})$  and that  $M$  is a nonzero finitely generated  $R$ -module of finite dimension  $d$  that satisfies  $H_{\Phi}^d(M) \neq 0$ . Then

---

\*Speaker.

(ii)  $\text{Ann}_R(H_\Phi^d(M)) = \text{Ann}_R(M/\Gamma_{\Phi_{d-1}(M)}(M)) = \text{Ann}_R(M/\bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j)=d} N_j)$ .

(ii)  $\text{Rad}(\text{Ann}_R(H_\Phi^d(M))) = \bigcap_{\mathfrak{p} \in \text{Ass}_R M, \text{cd}(\Phi, R/\mathfrak{p})=d} \mathfrak{p}$ . (See 3.2.)

(iii)  $\text{Supp}(H_\Phi^d(M)) \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}_R M, \text{cd}(\Phi, R/\mathfrak{p})=d} V(\mathfrak{p}) \cap \Omega$ . (See 3.2.)

Here  $\Omega = \bigcup_{\mathfrak{a} \in \Phi} V(\mathfrak{a})$  and  $\Phi_{d-1}(M) = \{\mathfrak{a}_j^k : j \leq d-1, k \in \mathbb{N}\}$ , where  $\mathfrak{a}_j := \prod_{\mathfrak{p} \in \text{Ass}_R(M), \text{cd}(\Phi, \frac{R}{\mathfrak{p}}) \leq d-1} \mathfrak{p}$ .

## 2. Preliminaries

The concept of cd-filtration of a finitely generated R-module  $M$  with respect to a system of ideals introduced in [4]. In this section we remind some results, which are used in the next section. For more details, see [4].

**Definition 2.1.** Let  $\Phi$  be a system of ideals of  $R$  and  $M$  be an R-module. The cohomological dimension of  $M$  with respect to  $\Phi$  is defined as

$$\text{cd}(\Phi, M) := \sup\{i \in \mathbb{N}_0 \mid H_\Phi^i(M) \neq 0\}.$$

We adopt the convention that the supremum of empty set of integers is  $-\infty$ .

**Definition 2.2.** ([4], Definition 2.3) Suppose that  $M$  is a finitely generated R-module and that  $\Phi$  is a system of ideals of  $R$ . The increasing filtration  $\mathcal{M} = \{M_j\}_{j=0}^c$  of submodules of  $M$  when  $c := \text{cd}(\Phi, M)$ , is called the cd-filtration of  $M$  w.r.t  $\Phi$ . Here  $M_j$  is the largest submodule of  $M$  satisfying  $\text{cd}(\Phi, M_j) \leq j$  for any integer  $0 \leq j \leq c$ .

**Lemma 2.3.** ([4], Lemma 2.4) Let  $\Phi$  be a system of ideals of  $R$ . Let  $M$  and  $N$  be two finitely generated R-modules such that  $\text{Supp}(N) \subseteq \text{Supp}(M)$ . Then

$$\text{cd}(\Phi, N) \leq \text{cd}(\Phi, M).$$

**Lemma 2.4.** ([4], corollary 2.6) Let  $M$  be a finitely generated R-module. Then

$$\text{cd}(\Phi, M) = \text{cd}\left(\Phi, \frac{R}{\text{Ann}(M)}\right) = \max\{\text{cd}\left(\Phi, \frac{R}{\mathfrak{p}}\right) : \mathfrak{p} \in \text{MinSupp}(M)\}.$$

**Notation 2.5.** ([4], Remark 2.8) Let  $\Phi$  be a system of ideals of  $R$  and  $M$  be a non-zero finitely generated R-module. For each non-negative integer  $i$ , the set  $\{\mathfrak{a}_j^k : 0 \leq j \leq i, k \in \mathbb{N}\}$ , where  $\mathfrak{a}_j := \prod_{\mathfrak{p} \in \text{Ass}_R(M), \text{cd}(\Phi, \frac{R}{\mathfrak{p}}) \leq j} \mathfrak{p}$ , is a system of ideals of  $R$ . We will denote this set by  $\Phi_i(M)$ .

**Proposition 2.6.** ([4], Proposition 2.9) Let  $\Phi$  be a system of ideals of  $R$ , let  $M$  be a nonzero finitely generated R-module with finite cohomological dimension  $c := \text{cd}(\Phi, M)$  and let  $\mathcal{M} = \{M_i\}_{i=0}^c$  be the cd-filtration of  $M$  with respect to  $\Phi$ . Then for all integers  $0 \leq i \leq c$ , we have

$$M_i = \Gamma_{\Phi_i(M)}(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j.$$

Here  $0 = \bigcap_{j=1}^n N_j$  denotes a reduced primary decomposition of the zero submodule in  $M$  with  $\text{Ass}(M/N_j) = \{\mathfrak{p}_j\}$ .

**Definition 2.7.** Let the notations and assumptions be as in Proposition 2.6. Then by Proposition 2.6,  $M_{c-1} = \Gamma_{\Phi_{c-1}(M)}(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j)=c} N_j$  is the largest submodule  $T$  of  $M$  with the property that  $\text{cd}(\Phi, T) < c$ . We will denote this submodule of  $M$  by  $T(\Phi, M)$ .

### 3. Main results

One of the main results done by Lynch [6], is the investigation of the annihilators of top general local cohomology modules and using it in the vanishing problem. In this section we investigate the annihilators of top general local cohomology modules for a system of ideals of  $R$ . To do this, first of all we obtain the  $\text{Att}(H_{\Phi}^d(M))$  for a  $d$ -dimensional finitely generated  $R$ -module  $M$ . Next we compute  $\text{Ann}(H_{\Phi}^d(M))$  for a complete local ring in terms of  $\mathfrak{m}$ -primary ideals. Finally we determine  $\text{Ann}(H_{\Phi}^d(M))$  without completion condition on  $R$ .

Now, we state and prove the first main theorem of this paper.

**Theorem 3.1.** Suppose that  $\Phi$  is a system of ideals of a local ring  $(R, \mathfrak{m})$  and that  $M$  is a  $d$ -dimensional finitely generated  $R$ -module. Then

$$\text{Att}_R(H_{\Phi}^d(M)) = \{\mathfrak{p} \in \text{Assh}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

PROOF. There is nothing to prove when  $\text{cd}(\Phi, M) < d$ , so assume, in addition, that  $\text{cd}(\Phi, M) = d$ . The claimed equality holds easily, when  $M = R$ .

So, assume that  $M$  is an arbitrary finitely generated  $R$ -module with  $\text{cd}(\Phi, M) = \dim M = d$ . We set  $\bar{R} := R/\text{Ann}_R(M)$ . Hence there are isomorphisms  $H_{\Phi}^d(M) \cong H_{\Phi\bar{R}}^d(M) \cong H_{\Phi\bar{R}}^d(\bar{R}) \otimes_{\bar{R}} M$  of  $\bar{R}$ -modules. Hence we have

$$\begin{aligned} \text{Att}_{\bar{R}}(H_{\Phi}^d(M)) &= \text{Att}_{\bar{R}}(H_{\Phi\bar{R}}^d(\bar{R})) \cap \text{Supp}_{\bar{R}}(M) \\ &= \text{Att}_{\bar{R}}(H_{\Phi\bar{R}}^d(\bar{R})) \\ &= \{P \in \text{Spec}(\bar{R}) : \dim \bar{R}/P = \text{cd}(\Phi\bar{R}, \bar{R}/P) = d\}. \end{aligned}$$

Now, the Independence Theorem yields  $H_{\Phi\bar{R}}^i(\bar{R}/\mathfrak{p}\bar{R}) \cong H_{\Phi}^i(\bar{R}/\mathfrak{p}\bar{R}) \cong H_{\Phi}^i(R/\mathfrak{p})$  for all  $i \in \mathbb{N}_0$ . Hence  $\text{cd}(\Phi\bar{R}, \bar{R}/P) = \text{cd}(\Phi, R/\mathfrak{p})$ . Therefore

$$\begin{aligned} \text{Att}_{\bar{R}}(H_{\Phi}^d(M)) &= \{\mathfrak{p}\bar{R} : \mathfrak{p} \in \text{Supp}_R(M), \dim R/\mathfrak{p} = \text{cd}(\Phi, R/\mathfrak{p}) = d\} \\ &= \{\mathfrak{p}\bar{R} : \mathfrak{p} \in \text{Assh}_R(M), \text{cd}(\Phi, R/\mathfrak{p}) = d\}. \end{aligned}$$

Also, by [3, Exercise 8.2.5], we have

$$\text{Att}(H_{\Phi}^d(M)) = \{\mathfrak{p} \in \text{Assh}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = d\}.$$

□

**Corollary 3.2.** With the assumptions of Theorem 1.2, it holds that

- (i)  $\text{Rad}(\text{Ann}_R(H_{\Phi}^d(M))) = \bigcap_{\mathfrak{p} \in \text{Assh}_R M, \text{cd}(\Phi, R/\mathfrak{p}) = d} \mathfrak{p}$ .
- (ii)  $\text{Supp}(H_{\Phi}^d(M)) \subseteq \bigcup_{\mathfrak{p} \in \text{Assh}_R M, \text{cd}(\Phi, R/\mathfrak{p}) = d} V(\mathfrak{p}) \cap \Omega$ .

The following corollary, extend the result of Hellus to general local cohomology modules.

**Corollary 3.3.** Suppose that  $(R, \mathfrak{m})$  is a complete local ring, that  $\Phi$  is a system of ideals of  $R$ , and that  $M$  is a nonzero finitely generated  $R$ -module with finite dimension  $d$  such that  $H_{\Phi}^d(M) \neq 0$ . Then

$$\text{Att}_R(H_{\Phi}^d(M)) = \{\mathfrak{p} \in \text{Assh}_R(M) \mid \text{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}, \text{ for all } \mathfrak{a} \in \Phi\}.$$

PROOF. Suppose that  $\mathfrak{p} \in \text{Att}_R(H_{\Phi}^d(M))$ . Then by virtue of Theorem 3.1,  $\mathfrak{p} \in \text{Assh}_R M$  and  $\text{cd}(\Phi, R/\mathfrak{p}) = d$ . Thus,  $H_{\Phi}^d(R/\mathfrak{p}) \neq 0$ . Let  $\Psi := \Phi(R/\mathfrak{p}) = \{\mathfrak{a}(R/\mathfrak{p}) : \mathfrak{a} \in \Phi\}$  be a system of ideals of  $R/\mathfrak{p}$ . Clearly,  $H_{\Psi}^d(R/\mathfrak{p}) \neq 0$ . As  $R/\mathfrak{p}$  is a complete local domain with

dimension  $d$ , Lichtenbaum–Hartshorn Theorem implies that  $\text{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ , for all  $\mathfrak{a} \in \Phi$ . Conversely, let  $\mathfrak{p} \in \text{Assh}_R M$  be such that for all  $\mathfrak{a} \in \Phi$ ,  $\text{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ . Since  $\text{Rad}(\mathfrak{a}(R/\mathfrak{p})) = \mathfrak{m}/\mathfrak{p}$ , it follows from [3, Theorems 4.2.1 and 6.1.4] that

$$H_{\Phi}^d(R/\mathfrak{p}) \cong \varinjlim_{\mathfrak{a} \in \Phi} H_{\mathfrak{a}}^d(R/\mathfrak{p}) \cong \varinjlim_{\mathfrak{a} \in \Phi} H_{\mathfrak{a}(R/\mathfrak{p})}^d(R/\mathfrak{p}) \cong H_{\mathfrak{m}/\mathfrak{p}}^d(R/\mathfrak{p}) \neq 0.$$

Hence  $\text{cd}(\Phi, R/\mathfrak{p}) = d$ . According to Theorem 3.1,  $\mathfrak{p} \in \text{Att}_R(H_{\Phi}^d(M))$ .  $\square$

In Theorem 3.6, we compute the annihilator of top general local cohomology module. But before that we need some lemmas.

**Lemma 3.4.** Suppose that  $(R, \mathfrak{m})$  is a local ring, that  $\Phi$  is a system of ideals of  $R$ , and that  $M$  is a nonzero finitely generated  $R$ -module with finite dimension  $d$  such that  $H_{\Phi}^d(M) \neq 0$ . Moreover, suppose that  $G := M/T_R(\Phi, M)$ . Then the following conditions hold:

- (i)  $\text{cd}(\Phi, G) = d$ ,
- (ii) If  $N$  is a non-zero submodule of  $G$ , then  $\text{cd}(\Phi, N) = d$ ,
- (iii)  $\text{Ass}_R G = \text{Att}_R H_{\Phi}^d(G) = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = d\}$ ,
- (iv)  $H_{\Phi}^d(G) \cong H_{\Phi}^d(M)$ .

**Lemma 3.5.** With the assumptions of Corollary 3.3, it holds that

$$\text{Ann}_R(H_{\Phi}^d(M)) = \text{Ann}_R(M/T_R(\Phi, M)).$$

PROOF. Suppose that  $G := M/T_R(\Phi, M)$ . Then, by Lemma 3.4, it suffices to show that  $\text{Ann}_R(H_{\Phi}^d(G)) = \text{Ann}_R(G)$ . By Lemma 3.4(iii) and Corollary 3.3, for all  $\mathfrak{a} \in \Phi$  and all  $\mathfrak{p} \in \text{Att}_R(H_{\Phi}^d(M))$ , we have

$$\begin{aligned} \mathfrak{m} &= \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{\mathfrak{p} \in \text{Att}_R(H_{\Phi}^d(M))} \text{Rad}(\mathfrak{a} + \mathfrak{p}) = \bigcap_{\mathfrak{a} \in \Phi} \text{Rad}(\mathfrak{a} + \bigcap_{\mathfrak{p} \in \text{Ass}_R(G)} \mathfrak{p}) \\ &= \bigcap_{\mathfrak{a} \in \Phi} \text{Rad}(\mathfrak{a} + \text{Rad}(\text{Ann}_R(G))). \end{aligned}$$

Thus  $\mathfrak{m} = \text{Rad}(\mathfrak{a} + \text{Rad}(\text{Ann}_R(G)))$  for all  $\mathfrak{a} \in \Phi$ . Therefore

$$H_{\mathfrak{m}}^d(G) \cong H_{\mathfrak{a} + \text{Rad}(\text{Ann}_R(G))}^d(G) \cong H_{\mathfrak{a}}^d(G)$$

for all  $\mathfrak{a} \in \Phi$ . So  $H_{\Phi}^d(G) \cong H_{\mathfrak{m}}^d(G)$ . Therefore, [1, Theorem 2.6] implies that

$$\text{Ann}_R(H_{\Phi}^d(G)) = \text{Ann}_R(H_{\mathfrak{m}}^d(G)) = \text{Ann}_R(G/T_R(G)).$$

The fact

$$\text{cd}(\Phi, T_R(G)) \leq \dim T_R(G) < \dim G,$$

and Lemma 3.4 (ii) imply that  $T_R(G) = 0$ . Hence  $\text{Ann}_R(H_{\Phi}^d(G)) = \text{Ann}_R(G)$ .  $\square$

**Theorem 3.6.** Let  $(R, \mathfrak{m})$  be a local ring,  $\Phi$  be a system of ideals of  $R$  and  $M$  be a non-zero finitely generated  $R$ -module of dimension  $d$  such that  $H_{\Phi}^d(M) \neq 0$ . Then there is the following equality:

$$\text{Ann}_R(H_{\Phi}^d(M)) = \text{Ann}_R(M/T_R(\Phi, M)).$$

PROOF. By Lemma 3.4, without loss of generality, let  $T_R(\Phi, M) = 0$ . It is clear that  $\text{Ann}_R(M) \subseteq \text{Ann}_R(H_{\Phi}^d(M))$ , thus it suffices to prove  $\text{Ann}_R(H_{\Phi}^d(M)) \subseteq \text{Ann}_R(M)$ .

In this regard, suppose that  $x \in \text{Ann}_R(H_{\Phi}^d(M))$ . Now, we prove that  $xM = 0$ . Since  $xH_{\Phi}^d(M) = 0$ , clearly  $xH_{\Phi\hat{R}}^d(\hat{M}) = 0$ . Therefore,  $x\hat{R} \subseteq \text{Ann}_{\hat{R}}(H_{\Phi\hat{R}}^d(\hat{M}))$ . Now by Lemma 3.5,  $x\hat{R} \subseteq \text{Ann}_{\hat{R}}(\hat{M}/T_{\hat{R}}(\Phi\hat{R}, \hat{M}))$ . Therefore,  $x\hat{M} \subseteq T_{\hat{R}}(\Phi\hat{R}, \hat{M})$ . Hence,  $\text{cd}(\Phi\hat{R}, x\hat{M}) < d$ . Consequently,  $\text{cd}(\Phi, xM) < d$ . Thus, by Definition 2.7,  $xM \subseteq T_R(\Phi, M)$ .  $\square$

The following corollaries are obtained from Theorem 3.6.

**Corollary 3.7.** With the assumptions of Theorem 3.6, it holds that

$$V(\text{Ann}_R(H_\Phi^d(M))) = \text{Supp}(M/T_R(\Phi, M)).$$

**Corollary 3.8.** Suppose that  $(R, \mathfrak{m})$  is a local ring, that  $\Phi$  is a system of ideals of  $R$ , and that  $M$  is a nonzero finitely generated  $R$ -module of finite dimension  $d$  with  $\text{Ass}_R M = \text{Att}_R H_\Phi^d(M)$ . Then  $\text{Ann}_R(H_\Phi^d(M)) = \text{Ann}_R M$ .

**Corollary 3.9.** With the assumptions of Theorem 3.6, for each  $x \in R$ ,  $xH_\Phi^d(M) = 0$  if and only if  $H_\Phi^d(xM) = 0$ . Particularly,  $\text{cd}(\Phi, rM) = d$  for all  $0 \neq r \in R$ , if and only if  $\text{Ann}_R(H_\Phi^d(M)) = 0$ .

**Corollary 3.10.** Suppose that  $(R, \mathfrak{m})$  is a local ring of finite dimension  $d > 0$  and that  $\Phi$  is a system of ideals of  $R$ . Then the following statements are equivalent:

- (i)  $\text{Ann}_R H_\Phi^d(R) = 0$ .
- (ii)  $\text{Ass}_R R = \text{Att}_R H_\Phi^d(R)$ .

**Corollary 3.11.** Suppose that  $(R, \mathfrak{m})$  is a local domain of finite dimension  $d$  and that  $\Phi$  is a system of ideals of  $R$  with  $H_\Phi^d(R) \neq 0$ . Then  $\text{Ann}_R(H_\Phi^d(R)) = 0$ .

PROOF OF THEOREM 1.2. Theorem 1.2 follows from Theorem 3.6, Proposition 2.6 (or Definition 2.7) and Corollary 3.2.  $\square$

## Acknowledgement

## References

1. K. Bahmanpour, J. Azami and G. Ghasemi, On the annihilators of local cohomology modules, *J. Algebra*, 363(2012), 8-13.
2. M. H. Bijan-Zadeh, Torsion theories and local cohomology over commutative and Noetherian rings, *J. London Math. Soc.* (2) 19 (1979), no. 3, 402-410.
3. M. P. Brodmann and R. Y. Sharp, *Local cohomology; An algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.
4. S.O. Faramarzi and H. valadbeigi, On the cd-filtration of modules with respect to a system of ideals, *Journal of Algebra and Related Topics*, Vol. 12, No 1, (2024), 127-136.
5. M. Hellus, Attached primes of Matlis duals of local cohomology modules, *Arch. Math.*, (2007), 89, 202-210.
6. L. R. Lynch, Annihilators of top local cohomology, *Comm. Algebra*, 40(2012), 542-551.
7. P. Schenzel, Cohomological annihilators, *Math. Proc. Cambridge Philos. Soc.* 91(1982), 345-350.

6-7 November 2025

# Cohomological dimension filtration of an R-module

Seadat Ollah Faramarzi \*

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.  
Email: s.o.faramarzi@pnu.ac.ir, s.o.faramarzi@gmail.com

---

ABSTRACT. In this paper, we introduce the concept of the cohomological dimension filtration with respect to a system of ideals. As a main result, we provide a necessary and sufficient condition for an ascending chain of submodules of an R-module  $M$  to be the cd-filtration of  $M$ , with respect to a system of ideals.

AMS Mathematics Subject Classification [2020]: 13D45, 14B15, 13E05

---

## 1. Introduction

Throughout this paper, let  $R$  denotes a commutative Noetherian (non-zero identity) ring and  $\Phi$  is a system of ideals of  $R$ . In [2] a non-empty set of ideals  $\Phi$  of  $R$  is said to be a system of ideals, if whenever  $\mathfrak{a}, \mathfrak{b} \in \Phi$ , then there is an ideal  $\mathfrak{c} \in \Phi$  such that  $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$ . For every R-module  $B$ , we have

$$\Gamma_{\Phi}(B) = \{x \in B \mid \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}.$$

Thus,  $\Gamma_{\Phi}(B)$  is a  $\Phi$ -torsion submodule of  $B$ . The  $i$ -th right derived functor of the functor  $\Gamma_{\Phi}$  is denoted by  $H_{\Phi}^i$ . It is clear that when  $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}\}$ , the functor  $H_{\Phi}^i$  coincides with the ordinary local cohomology functor  $H_{\mathfrak{a}}^i$ .

One of the main topics in commutative algebra is the study of module properties using the concept of *dimension filtration*, which is introduced by P. Schenzel in [5]. Atazadeh and et al [1], generalize Schenzel's results to *cohomological dimension filtration* (abbreviated as cd-filtration) with respect to an ideal. In this paper, we generalize the above results and introduce the concept of the cohomological dimension filtration with respect to the system of ideals  $\Phi$  of  $R$ . In this regard, we provide a necessary and sufficient condition for an ascending chain of submodules of  $M$  to be a cd-filtration of  $M$ , with respect to the system of ideals  $\Phi$  of  $R$ .

For any system of ideals  $\Phi$  of  $R$ , we denote  $\Omega := \cup_{\mathfrak{a} \in \Phi} V(\mathfrak{a})$ , where  $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}R : \mathfrak{p} \supseteq \mathfrak{a}\}$  and the set  $\{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = i\}$ , denote by  $\text{Ass}_R^i(M)$ .

## 2. Preliminaries

The concept of cd-filtration of a finitely generated R-module  $M$  with respect to a system of ideals introduced in [4]. In this section we remind some results, which are used in the next section. For more details, see [4].

**Definition 2.1.** Let  $\Phi$  be a system of ideals of  $R$  and  $M$  be an R-module. The cohomological dimension of  $M$  with respect to  $\Phi$  is defined as

$$\text{cd}(\Phi, M) := \sup\{i \in \mathbb{N}_0 \mid H_{\Phi}^i(M) \neq 0\}.$$

---

\*Speaker.

We adopt the convention that the supremum of empty set of integers is  $-\infty$ .

**Definition 2.2.** ([4], Definition 2.3) Suppose that  $M$  is a finitely generated  $R$ -module and that  $\Phi$  is a system of ideals of  $R$ . The increasing filtration  $\mathcal{M} = \{M_j\}_{j=0}^c$  of submodules of  $M$  when  $c := \text{cd}(\Phi, M)$ , is called the  $\text{cd}$ -filtration of  $M$  w.r.t  $\Phi$ . Here  $M_j$  is the largest submodule of  $M$  satisfying  $\text{cd}(\Phi, M_j) \leq j$  for any integer  $0 \leq j \leq c$ .

**Lemma 2.3.** ([4], Lemma 2.4) Let  $\Phi$  be a system of ideals of  $R$ . Let  $M$  and  $N$  be two finitely generated  $R$ -modules such that  $\text{Supp}(N) \subseteq \text{Supp}(M)$ . Then

$$\text{cd}(\Phi, N) \leq \text{cd}(\Phi, M).$$

**Lemma 2.4.** ([4], corollary 2.6) Let  $M$  be a finitely generated  $R$ -module. Then

$$\text{cd}(\Phi, M) = \text{cd}\left(\Phi, \frac{R}{\text{Ann}(M)}\right) = \max\left\{\text{cd}\left(\Phi, \frac{R}{\mathfrak{p}}\right) : \mathfrak{p} \in \text{MinSupp}(M)\right\}.$$

**Notation 2.5.** ([4], Remark 2.8) Let  $\Phi$  be a system of ideals of  $R$  and  $M$  be a non-zero finitely generated  $R$ -module. For each non-negative integer  $i$ , the set  $\{\mathfrak{a}_j^k : 0 \leq j \leq i, k \in \mathbb{N}\}$ , where  $\mathfrak{a}_j := \prod_{\mathfrak{p} \in \text{Ass}_R(M), \text{cd}(\Phi, \frac{R}{\mathfrak{p}}) \leq j} \mathfrak{p}$ , is a system of ideals of  $R$ . We will denote this set by  $\Phi_i(M)$ .

**Proposition 2.6.** ([4], Proposition 2.9) Let  $\Phi$  be a system of ideals of  $R$ , let  $M$  be a nonzero finitely generated  $R$ -module with finite cohomological dimension  $c := \text{cd}(\Phi, M)$  and let  $\mathcal{M} = \{M_i\}_{i=0}^c$  be the  $\text{cd}$ -filtration of  $M$  with respect to  $\Phi$ . Then for all integers  $0 \leq i \leq c$ , we have

$$M_i = \Gamma_{\Phi_i(M)}(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j.$$

Here  $0 = \bigcap_{j=1}^n N_j$  denotes a reduced primary decomposition of the zero submodule in  $M$  with  $\text{Ass}(M/N_j) = \{\mathfrak{p}_j\}$ .

**Definition 2.7.** Let the notations and assumptions be as in Proposition 3.6. Then by Proposition 3.6,  $M_{c-1} = \Gamma_{\Phi_{c-1}(M)}(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) = c} N_j$  is the largest submodule  $T$  of  $M$  with the property that  $\text{cd}(\Phi, T) < c$ . We will denote this submodule of  $M$  by  $T(\Phi, M)$ .

### 3. Main results

In this section, we introduce the concept of *cohomological dimension filtration* (briefly as  $\text{cd}$ -filtration) of a finite  $R$ -module  $M$ , with respect to the system of ideals  $\Phi$  of  $R$ . Also, we determine the factors of this filtration by a reduced primary decomposition of the zero submodule in  $M$ . Next, we determine the associated prime ideals of factors of the  $\text{cd}$ -filtration of  $M$ . Finally, a necessary and sufficient condition for a filtration to be the  $\text{cd}$ -filtration of a module is provided.

**Definition 3.1.** Let  $M$  be a finitely generated  $R$ -module and  $\Phi$  be a system of ideals of  $R$ . The increasing filtration  $\mathcal{M} = \{M_j\}_{j=0}^c$  of submodules of  $M$ , when  $c := \text{cd}(\Phi, M)$  is called the *cohomological dimension filtration* of  $M$  with respect to  $\Phi$ . Here  $M_j$  is the largest submodule of  $M$  such that  $\text{cd}(\Phi, M_j) \leq j$  for any integer  $0 \leq j \leq c$ .

**Lemma 3.2.** Let  $R$  be a Noetherian ring and  $\Phi$  be a system of ideals of  $R$ . Let  $M$  and  $N$  be two finitely generated  $R$ -modules such that  $\text{Supp}N \subseteq \text{Supp}M$ . Then  $\text{cd}(\Phi, N) \leq \text{cd}(\Phi, M)$ .

**Lemma 3.3.** Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence of finitely generated  $R$ -modules. Then

$$\text{cd}(\Phi, M) = \max\{\text{cd}(\Phi, L), \text{cd}(\Phi, N)\}.$$

**Corollary 3.4.** Let  $M$  be a finitely generated  $R$ -module. Then

$$\text{cd}(\Phi, M) = \text{cd}(\Phi, R/\text{Ann}_R(M)) = \max\{\text{cd}(\Phi, \frac{R}{\mathfrak{p}}) \mid \mathfrak{p} \in \text{minSupp}_R(M)\}.$$

**Remark 3.5.** let  $R$  be a Noetherian ring and  $M$  be a finitely generated  $R$ -module, the set  $\Phi_i = \{\mathfrak{a}_j \mid 0 \leq j \leq i\}$  be a system of ideals of  $R$  where  $\mathfrak{a}_j := \prod_{\text{cd}(\Phi, R/\mathfrak{p}) \leq j, \mathfrak{p} \in \text{Ass}_R M} \mathfrak{p}$ . Obviously, we have a descending chain of ideals

$$\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_c,$$

and so for two ideals  $\mathfrak{a}_k, \mathfrak{a}_j$  where  $0 \leq j, k \leq i$  there exists an ideal  $\mathfrak{a}_l$  such that  $\mathfrak{a}_l \subseteq \mathfrak{a}_j \mathfrak{a}_k$  when  $l := \max\{j, k\}$ .

We recall that  $\text{Ass}_R^i(M) = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = i\}$ , for any  $0 \leq i \leq c$  and  $\Omega_i := \bigcup_{I \in \Phi_i} V(I)$ .

**Proposition 3.6.** Let  $R$  be a Noetherian ring,  $\Phi$  be a system of ideals of  $R$  and  $M$  be a non-zero finitely generated  $R$ -module with finite cohomological dimension  $c := \text{cd}(\Phi, M)$  and let  $\mathcal{M} = \{M_i\}_{i=0}^c$  be the cd-filtration of  $M$ , with respect to  $\Phi$ . Then for all integers  $0 \leq i \leq c$ , we have

$$M_i = H_{\Phi_i}^0(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j.$$

Here  $0 = \bigcap_{j=1}^n N_j$  denotes a reduced primary decomposition of the zero submodule in  $M$  and  $N_j$  is a  $\mathfrak{p}_j$ -primary submodule of  $M$ .

**PROOF.** First, we show the equality  $H_{\Phi_i}^0(M) = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j$ . Suppose that  $x \in \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j$  since  $N_j$  is a  $\mathfrak{p}_j$ -primary submodule of  $M$ , there is an integer  $s_j$  such that  $\mathfrak{p}_j^{s_j} M \subseteq N_j$ . Now let  $s := \max\{s_j \mid 1 \leq j \leq n\}$ , thus for all  $1 \leq j \leq n$ , we have  $\mathfrak{p}_j^s M \subseteq N_j$ . Since  $x \in M$ , then  $\mathfrak{p}_j^s x \subseteq N_j$ , for all  $1 \leq j \leq n$ . Also for any  $I \in \Phi_i$ , we have  $I^s x \subseteq \mathfrak{a}_0^s x \subseteq N_j$ , for all  $0 \leq j \leq n$ , thus,  $I^s x \subseteq \bigcap_{j=1}^n N_j = 0$ . Therefore by Remark 3.5,  $x \in H_{\Phi_i}^0(M)$ .

In order to prove the reverse, assume the contrary holds. Then there exists  $x \in H_{\Phi_i}^0(M)$  such that  $x \notin \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j$ . Hence there is an integer  $t$  such that  $x \notin N_t$  and  $\text{cd}(\Phi, R/\mathfrak{p}_t) > i$ . Now since  $x \in H_{\Phi_i}^0(M)$ , there exists an ideal  $\mathfrak{b} \in \Phi_i$  such that  $x\mathfrak{b} = 0$ . Because of  $x \notin N_t$  and  $N_t$  is a  $\mathfrak{p}_t$ -primary submodule,  $\mathfrak{b} \subseteq \mathfrak{p}_t$ . Thus there is an integer  $j$  such that  $\mathfrak{p}_j \subseteq \mathfrak{p}_t$  and  $\text{cd}(\Phi, R/\mathfrak{p}_j) \leq i$ . Therefore, by virtue of Lemma 3.2, we get  $\text{cd}(\Phi, R/\mathfrak{p}_t) \leq \text{cd}(\Phi, R/\mathfrak{p}_j) \leq i$ , which is a contradiction. The first equality holds clearly.  $\square$

In the next proposition, we investigate some properties of associated primes of cohomological dimension filtration of  $M$ , with respect to a system of ideals of  $R$ .

**Proposition 3.7.** Let  $R$  be a Noetherian ring, and  $\Phi$  be a system of ideals of  $R$  and  $M$  be a non-zero finitely generated  $R$ -module with finite cohomological dimension  $c := \text{cd}(\Phi, M)$ . Let  $\{M_i\}_{i=0}^c$  be a cd-filtration of  $M$  with respect to  $\Phi$ . Then for all integers  $0 \leq i \leq c$ ,

- (i)  $\text{Ass}_R M_i = \Omega_i = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) \leq i\}$ ,
- (ii)  $\text{Ass}_R(M/M_i) = \text{Ass}_R M \setminus \Omega_i = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) > i\}$ ,
- (iii)  $\text{Ass}_R M_i/M_{i-1} = \Omega_i \setminus \Omega_{i-1} = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = i\}$ .

PROOF. By Proposition 3.6,  $M_i = H_{\Phi_i}^0(M)$ . Thus, we have

$$\text{Ass}_R M_i = \text{Ass}_R M \cap \Omega_i.$$

Now (i), obtain easily from Lemma 3.2.

Using [3], Exercise 2.1.14] (ii) holds. To show (iii), as  $M_i/M_{i-1} \subseteq M/M_{i-1}$ , so  $\text{Ass}_R M_i/M_{i-1} \subseteq \text{Ass}_R M/M_{i-1}$ , and it follows from part (ii) that  $\text{cd}(\Phi, R/\mathfrak{p}) \geq i$ , for all  $\mathfrak{p} \in \text{Ass}_R M_i/M_{i-1}$ . Moreover, with the short exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

and Lemma 3.2, we have

$$\text{cd}(\Phi, M_i/M_{i-1}) \leq \text{cd}(\Phi, M_i) \leq i.$$

Again Lemma 3.2, shows that  $\text{cd}(\Phi, R/\mathfrak{p}) \leq i$ , for all  $\mathfrak{p} \in \text{Ass}_R M_i/M_{i-1}$ . Hence

$$\text{Ass}_R M_i/M_{i-1} \subseteq \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = i\}.$$

Now, let  $\mathfrak{p} \in \text{Ass}_R M$  and  $\text{cd}(\Phi, R/\mathfrak{p}) = i$ . By virtue of part (i),  $\mathfrak{p} \in \text{Ass}_R M_i$ , and  $\mathfrak{p} \notin \text{Ass}_R M_{i-1}$ . Now the exact sequence,

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

yields  $\mathfrak{p} \in \text{Ass}_R M_i/M_{i-1}$ . Thus

$$\text{Ass}_R M_i/M_{i-1} = \{\mathfrak{p} \in \text{Ass}_R M \mid \text{cd}(\Phi, R/\mathfrak{p}) = i\}.$$

□

One of the main aims of this section is to establish the following theorem, which gives a characterization of the cd-filtration of  $M$  with respect to  $\Phi$ , in terms of associated prime ideals of its factors.

**Theorem 3.8.** *Let  $\mathcal{M} = \{M_i\}_{i=0}^c$  be a filtration of the finite  $R$ -module  $M$  and  $\Phi$  be a system of ideals of  $R$  such that  $\text{cd}(\Phi, M_0) = 0$ . The following conditions are equivalent:*

- (i)  $\text{Ass}_R(M_i/M_{i-1}) = \text{Ass}_R^i(M)$ , for all  $1 \leq i \leq c$ .
- (ii)  $\mathcal{M}$  is the cd-filtration of  $M$  with respect to  $\Phi$ .

PROOF. By applying Proposition 3.7 (iii), (ii  $\Rightarrow$  i) is clear. Thus it is enough to prove (i  $\Rightarrow$  ii). Considering the short exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0.$$

First, we claim that

$$\text{Ass}_R(M_{i-1}) \cap \text{Ass}_R(M_i/M_{i-1}) = \emptyset, \text{ for all } 1 \leq i \leq c.$$

Suppose that, contrarily, for some  $1 \leq i \leq c$ , then there exists  $\mathfrak{p} \in \text{Ass}_R(M_{i-1}) \cap \text{Ass}_R(M_i/M_{i-1})$ . Therefore,  $\text{cd}(\Phi, M_{i-1}) \geq i$  by (i). By the assumption,  $\text{Ass}_R^{i-1}(M) = \text{Ass}_R(M_{i-1}/M_{i-2})$  so  $\mathfrak{p} \notin \text{Ass}_R(M_{i-1}/M_{i-2})$ . The short exact sequence

$$0 \longrightarrow M_{i-2} \longrightarrow M_{i-1} \longrightarrow M_{i-1}/M_{i-2} \longrightarrow 0,$$

yields  $\mathfrak{p} \in \text{Ass}_R(M_{i-2})$ . As  $\text{cd}(\Phi, R/\mathfrak{p}) = i$ , thus  $\text{cd}(\Phi, M_{i-2}) \geq i$ . By the continuation of this descending process, we have  $\text{cd}(\Phi, M_0) \geq i > 0$ , which is a contradiction. Now consider the exact sequence

$$0 \longrightarrow M_{c-1} \longrightarrow M \longrightarrow M/M_{c-1} \longrightarrow 0.$$

Thus,  $\text{cd}(\Phi, M_{c-1}) \leq c-1$  as  $\text{Ass}_R(M_{c-1}) \cap \text{Ass}_R(M/M_{c-1}) = \emptyset$ . Now, suppose that the largest submodule of  $M$  is denoted by  $N$  such that  $\text{cd}(\Phi, N) \leq c-1$  and  $\mathfrak{p} \in$

$\text{Ass}_R(N/M_{c-1})$ . Because of  $\text{Ass}_R(N/M_{c-1}) \subseteq \text{Ass}_R^c(M)$ , we have  $\text{cd}(\Phi, R/\mathfrak{p}) = c$ . But  $\mathfrak{p} \in \text{Supp}_R(N/M_{c-1}) \subseteq \text{Supp}_R(N)$  and therefore  $\text{cd}(\Phi, R/\mathfrak{p}) \leq \text{cd}(\Phi, N) \leq c - 1$  which is impossible. Hence,  $\text{Ass}_R(N/M_{c-1}) = \emptyset$  and  $M_{c-1}$  is the largest submodule of  $M$  such that  $\text{cd}(\Phi, M_{c-1}) \leq c - 1$ . Now descendingly, we proceed with this method to prove that  $\mathcal{M}$  is the cd-filtration of  $M$  with respect to  $\Phi$ .  $\square$

**Corollary 3.9.** Let  $\bigcap_{j=1}^n N_j$  be a reduced primary decomposition of the zero submodule 0 in  $M$ , where  $N_i$  is  $\mathfrak{p}_i$ -primary. Let  $\Phi$  be a system of ideals of  $R$  and  $M_i = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > i} N_j$  for all  $0 \leq i \leq c = \text{cd}(\Phi, M)$ . If  $\text{cd}(\Phi, \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) > 0} N_j) = 0$ , then  $\{M_i\}_{i=0}^c$  is the cd-filtration of  $M$  with respect to  $\Phi$ .

PROOF. Let  $L_i = \bigcap_{\text{cd}(\Phi, R/\mathfrak{p}_j) = i} N_j$  for all  $0 \leq i \leq c$ . Obviously,  $M_{i-1} = M_i \cap L_i$ . By rewriting the indices, let  $L_i = N_1 \cap \dots \cap N_m$ . By Theorem 3.8, it is enough to show that  $\text{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ . We know that  $\text{Ass}_R(M_i/M_{i-1}) = \text{Ass}_R(M_i + L_i/L_i) \subseteq \text{Ass}_R(M/L_i)$ . Also,  $\text{Ass}_R(M/L_i) = \text{Ass}_R(\bigoplus_{j=1}^n M/N_j) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ , and so  $\text{Ass}_R(M_i/M_{i-1}) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ . We will show that  $\mathfrak{p}_r \in \text{Ass}_R(M_i/M_{i-1})$  for all  $1 \leq r \leq m$ . Since  $0 = \bigcap_{j=1}^n N_j$  is a reduced primary decomposition of zero submodule, it yields

$$M_{i-1} \subsetneq (N_1 \cap \dots \cap \widehat{N_r} \cap \dots \cap N_m) \cap L_{i+1} \cap \dots \cap L_c.$$

Let  $A := (N_1 \cap \dots \cap \widehat{N_r} \cap \dots \cap N_m) \cap L_{i+1} \cap \dots \cap L_c$ . So there exists  $x \in A$  such that  $x \notin M_{i-1}$ .

Consequently, we deduce that  $(M_{i-1} : x) = (N_r : x)$ . Since  $N_r$  is  $\mathfrak{p}_r$ -primary, there exists  $t > 0$  such that  $\mathfrak{p}_r^t M \subseteq N_r$ . Hence  $\mathfrak{p}_r^t M \subseteq M_{i-1}$ . Suppose that  $s \geq 0$  is the least integer such that  $\mathfrak{p}_r^{s+1} x \notin M_{i-1}$  and  $\mathfrak{p}_r^s x \in M_{i-1}$ . This implies that there exists  $y \in \mathfrak{p}_r^s x$  such that  $y \notin M_{i-1}$ . Now, it is clear to see that  $\mathfrak{p}_r = (M_{i-1} : y)$ , i.e,  $\mathfrak{p}_r \in \text{Ass}_R(M_i/M_{i-1})$ . This completes the proof.  $\square$

## Acknowledgement

## References

1. A. Atazadeh, M. Sedghi and R. Naghipour, Cohomological dimension filtration and annihilators of top local cohomology modules, *Colloquium Mathematicum*, **139** (2015), 25–35.
2. M. H. Bijan-Zadeh, Torsion theories and local cohomology over commutative and Noetherian rings, *J. London Math. Soc.* (2) **19** (1979), no. 3, 402-410.
3. M. P. Brodmann and R. Y. Sharp, *Local cohomology; An algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.
4. S.O. Faramarzi and H.valadbeigi, On the cd-filtration of modules with respect to a system of ideals, *Journal of Algebra and Related Topics*, Vol. 12, No 1, (2024), 127-136.
5. P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, *Commutative algebra and algebraic geometry*, Lect. Notes in Pure and Appl. Math. Dekker, New York, 206(1999), 245–264.



## Topics in 1-Absorbing Prime Substructures

Mohammad Ali Dehghanizadeh<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences Technical and Vocational University (TVU), Tehran, IRAN.

Email: Mdehghanizadeh@tvu.ac.ir

---

**ABSTRACT.** Structure  $\mathcal{S}$  within algebraic, topological, and combinatorial settings. A rigorous notion of stability for transformations  $T : \mathcal{S} \rightarrow \mathcal{S}$  is formulated, subsuming classical invariance and robustness concepts and extending them to nonlinear and discrete contexts. The framework naturally recovers standard results for normed vector spaces while generalizing to broader structural environments. Combinatorial connectivity is examined through the introduction of maximally connected subsets, defined via an intrinsic adjacency relation. These subsets facilitate a structural decomposition of  $\mathcal{S}$  and reveal essential interactions between stability and combinatorial organization. Several open problems are identified concerning invariant transformations, classification of connected components, and criteria for joint stability under multiple mappings. The proposed formulation positions  $\mathcal{S}$  as a unifying construct linking algebraic and discrete methodologies, providing a foundation for further theoretical refinement and potential applied extensions.

**Keywords:** Structural Stability, Combinatorial Connectivity, Invariant Transformation.

**AMS Mathematics Subject Classification [2020]:** 05C40, 54H20, 15A03

---

### 1. Introduction

Mathematical structures provide a foundation for understanding both theoretical and applied problems, highlighting properties such as resilience and internal organization [4,6]. In particular, two perspectives have been increasingly studied: *stability* and *combinatorial organization*. Stability describes the ability of a structure to maintain its properties under transformations, while combinatorial approaches reveal how elements are interconnected within the system [5]. Integrating these viewpoints allows a comprehensive analysis of robustness and structural coherence [1].

Formally, let  $\mathcal{S}$  be a mathematical object and  $T : \mathcal{S} \rightarrow \mathcal{S}$  a transformation. We say that  $\mathcal{S}$  is *stable* under  $T$  if

$$T(x) \in \mathcal{S} \quad \text{for all } x \in \mathcal{S}.$$

Combinatorial structures can be introduced by defining adjacency relations or connectivity graphs among the elements of  $\mathcal{S}$ , leading to the study of maximally connected subsets and coset partitions [4].

---

\*Speaker.

## 2. Semihyperrings and Hypermodules

We now consider *semihyperrings* and *hypermodules*, which generalize classical algebraic structures by allowing operations to produce sets of elements rather than single outputs [2]. This framework naturally accommodates the study of stability and combinatorial properties in a unified algebraic context.

DEFINITION 2.1. A *semihyperring*  $R$  is a set equipped with two hyperoperations,  $+$  and  $\cdot$ , such that

- (1)  $(R, +)$  is a commutative hypermonoid with identity  $0$ ,
- (2)  $(R, \cdot)$  is a semigroup with identity  $1$ ,
- (3) Multiplication distributes over hyperaddition:

$$a \cdot (b + c) \subseteq a \cdot b + a \cdot c, \quad (a + b) \cdot c \subseteq a \cdot c + b \cdot c,$$

- (4)  $0$  is absorbing:  $a \cdot 0 = 0 \cdot a = 0$ .

DEFINITION 2.2. A *left  $R$ -hypermodule*  $M$  is a set with a commutative hypermonoid  $(M, +)$  and a scalar hyperaction

$$\cdot : R \times M \rightarrow \mathcal{P}^*(M),$$

satisfying distributivity, associativity, and identity properties similar to classical modules [3].

DEFINITION 2.3. A subset  $N \subseteq M$  is *stable* if

$$r \cdot x \subseteq N \quad \text{for all } r \in R, x \in N.$$

THEOREM 2.4. Let  $M$  be a left  $R$ -hypermodule and  $N \subseteq M$  a nonempty subset. Then  $N$  is stable under the scalar action if and only if  $N$  is a subhypermodule of  $M$ .

DEFINITION 2.5. For a subhypermodule  $W \subseteq M$ , define

$$x \approx y \iff x - y \in W.$$

This relation partitions  $M$  into cosets of  $W$ , forming an adjacency graph composed of complete subgraphs. Endomorphisms preserving  $W$  permute these cosets, reflecting the combinatorial structure of the hypermodule.

LEMMA 2.6. If  $\varphi : M \rightarrow M$  is an  $R$ -hypermodule endomorphism with  $\varphi(W) \subseteq W$ , then  $\varphi$  is adjacency-preserving.

EXAMPLE 2.7. Let  $R = \mathbb{Z}_n$  with  $M = R$  as a hypermodule over itself, and  $W = d\mathbb{Z}_n$ . Then elements in the same coset modulo  $d$  are adjacent, and maps  $\varphi(x) = ax$  with  $\gcd(a, n) = 1$  permute the cosets while preserving adjacency.

THEOREM 2.8. Let  $M$  be a hypermodule and  $W$  a subhypermodule. Let  $\varphi : M \rightarrow M$  be an endomorphism preserving  $W$ . Then  $\varphi$  induces a permutation action on the set of cosets  $M/W$ , hence adjacency components (cosets) are mapped to cosets.

## 3. 1-Absorbing Prime Subhypermodules

A subhypermodule  $N \subsetneq M$  is called *1-absorbing prime* if for all  $a, b \in R$  and  $x \in M$ ,

$$ab \cdot x \subseteq N \implies a \cdot x \subseteq N \text{ or } b \cdot x \subseteq N \text{ or } x \in N.$$

If the above condition holds only when  $ab \cdot x \neq 0$ ,  $N$  is *weakly 1-absorbing prime*. Every 1-absorbing prime subhypermodule is weakly 1-absorbing prime, but the converse does not hold in general.

EXAMPLE 3.1. For  $R = \mathbb{Z}_6$ ,  $M = R$ , the subset  $N = \{0, 3\}$  is weakly 1-absorbing prime but not 1-absorbing prime.

EXAMPLE 3.2. For prime  $p$ ,  $R = \mathbb{Z}_p$ , the only proper subhypermodule  $N = \{0\}$  is trivially 1-absorbing prime.

THEOREM 3.3. *Every 1-absorbing prime subhypermodule is weakly 1-absorbing prime, but not conversely.*

THEOREM 3.4. *Let  $M$  be an  $R$ -hypermodule,  $N \subsetneq M$  a 1-absorbing prime subhypermodule, and  $\varphi : M \rightarrow M$  an  $R$ -endomorphism such that  $\varphi(N) \subseteq N$ . Then  $\varphi^{-1}(N)$  is also a 1-absorbing prime subhypermodule of  $M$ .*

#### 4. Conclusion

This paper introduces a unified framework for studying semihyperrings and hypermodules, integrating algebraic stability and combinatorial properties. By formalizing stable subsets, adjacency relations, and (weakly) 1-absorbing prime subhypermodules, we provide tools for analyzing invariant structures and their internal connectivity.

#### References

- [1] Bousquet, O. and Elisseeff, A. (2002) *Stability and generalization*, Journal of Machine Learning Research, vol. 2, pp. 499–526.
- [2] Corsini P. and Leoreanu, V. (2003) *Applications of Hyperstructure Theory*, Kluwer Academic Publishers.
- [3] Davvaz, B. (2015) *Hyperring Theory and Applications*, Springer.
- [4] Diestel, R. (2017) *Graph Theory*, 5th ed., Springer.
- [5] Guckenheimer, J. and Holmes, P. (2013) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer.
- [6] Rudin, W. (1991) *Functional Analysis*, 2nd ed., McGraw-Hill.



## The index of the $n$ -center subgroup of an $n$ -abelian group

Azam Hokmabadi<sup>1,\*</sup> and Fatemeh Mirzaei<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: ahokmabadi@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: fmirzaei@pnu.ac.ir

**ABSTRACT.** Let  $n$  be a positive integer. A group  $G$  is said to be  $n$ -abelian, if  $(xy)^n = x^n y^n$ , for any  $x, y \in G$ . In 1979, Fay and Waals introduced the  $n$ -center subgroup of a group  $G$ , as  $Z^n(G) = \{x \in G \mid xy^n = y^n x, \forall y \in G\}$ . In this paper, we give some conditions under which the index of the  $n$ -center subgroup of an  $n$ -abelian group is finite. Moreover, we give an upper bound for the index of the  $n$ -center subgroup of an  $n$ -abelian group  $G$ , when the  $n$ -center factor subgroup is finitely generated.

**Keywords:**  $n$ -center subgroup,  $n$ -abelian group, upper bound.

**AMS Mathematics Subject Classification [2020]:** 20F14, 20D99

### 1. Introduction

Because of the importance of abelian groups in group theory, many generalizations have been considered. One of these generalizations is the concept of  $n$ -abelian group which has been presented in 1944 by Levi [2], for the first time. If  $n$  is an integer and  $n \geq 1$ , then a group  $G$  is called  $n$ -abelian whenever  $(xy)^n = x^n y^n$ , for all elements  $x$  and  $y$  in  $G$ , from which it follows that  $[x^n, y] = [x, y]^n = [x^n, y^n]$ , where  $[x, y] = x^{-1}y^{-1}xy$ . Here we use two other concepts, the  $n$ -potent and the  $n$ -center subgroups of a group  $G$ , that have been introduced by Fay and Waals [1]. For a positive integer  $n$ , the  $n$ -potent and the  $n$ -center subgroups of a group  $G$  are defined respectively, as follows:

$$G_n = \langle [x, y^n] \mid x, y \in G \rangle, \quad Z^n(G) = \{x \in G \mid xy^n = y^n x, \forall y \in G\}.$$

It is easy to see that  $G_n$  is a fully invariant subgroup and  $Z^n(G)$  is a characteristic subgroup of group  $G$ . The concepts of  $n$ -potent and  $n$ -center subgroups are extensions of the important subgroups  $G'$  and  $Z(G)$ , respectively.

A famous classical theorem due to I. Schur [6] states that for a group  $G$  the finiteness of  $|G : Z(G)|$  implies the finiteness of the order of  $G'$ . Infinite extra special groups show that the converse of Schur's theorem does not hold, in general. Therefore, many attempts have been made to provide conditions under which the converse of the Schur's theorem

\*Speaker.

holds. (For example see [3, 4].)

Pourmirzaei et al. [5] extended Schur's theorem in the variety of  $n$ -abelian groups and proved that the finiteness of the index of  $n$ -center subgroup implies the finiteness of the order of  $n$ -potent subgroup, for an  $n$ -abelian group  $G$ . In this paper we give some conditions under which the index of the  $n$ -center subgroup of an  $n$ -abelian group is finite. Moreover, we give an upper bound for the index of the  $n$ -center subgroup of an  $n$ -abelian group  $G$ , when the  $n$ -center factor subgroup is finitely generated.

## 2. Main results

**Lemma 2.1.** *Let  $G$  be an  $n$ -abelian group. Suppose that  $A$  is a normal subgroup of  $G$  such that  $|G : C_G(A^n)|$  is finite and  $G/A$  is finitely generated by  $g_1A, g_2A, \dots, g_rA$  where  $|(g_i^n)^G|$  is finite for all  $1 \leq i \leq r$ . Then  $|G : Z^n(G)|$  is finite.*

PROOF. Let  $X = \{g_1, g_2, \dots, g_r\}$  and  $A$  be generated by a set  $Y$ . Then  $G = \langle X \cup Y \rangle$  and  $Z^n(G) = C_G(X^n) \cap C_G(Y^n)$ . On the other hand,  $C_G(A^n) = C_G(Y^n)$  and so  $|G : C_G(Y^n)|$  is finite. Also, since  $|(g_i^n)^G|$  is finite, for all  $1 \leq i \leq r$ ,  $|G : C_G(X^n)|$  is finite. Hence  $|G : Z^n(G)|$  is finite. □

**Theorem 2.2.** *Let  $G$  be an  $n$ -abelian group such that  $G/Z^n(G)$  is finitely generated by  $x_1Z^n(G), x_2Z^n(G), \dots, x_tZ^n(G)$  and the conjugacy class of  $x_i^n$  in  $G$  is of finite length, for all  $1 \leq i \leq t$ . Then  $|G : Z^n(G)|$  is finite. Moreover  $|G : Z^n(G)| \leq \prod_{i=1}^t |(x_i^n)^G|$ , where  $(x_i^n)^G$  denotes the conjugacy class of  $x_i^n$  in  $G$ .*

PROOF. Using Lemma 2.1, with  $A = Z^n(G)$ , we have  $|G : Z^n(G)|$  is finite. Also, we have

$$|G : Z^n(G)| = |G : \bigcap_{i=1}^t C_G(x_i^n)| \leq \prod_{i=1}^t |G : C_G(x_i^n)| = \prod_{i=1}^t |(x_i^n)^G|.$$
□

We recall that the second  $n$ -center subgroup of a group  $G$  is defined as follows,

$$Z_2^n(G)/Z^n(G) = Z^n(G/Z^n(G)).$$

**Theorem 2.3.** *Let  $G$  be an  $n$ -abelian group such that  $G_n$  is finite. Then  $G/Z^n(G)$  is finite if one of the following holds.*

- i) *There exists a normal subgroup  $N$  of  $G$  such that  $G/N$  is finite and  $N/Z^n(N)$  or  $G/Z^n(N)$  is finitely generated.*
- ii) *There exists a normal subgroup  $N$  of  $G$  such that  $G/N$  is finite and  $N_n = 1$ .*
- iii)  *$G/Z_2^n(G)$  is finite and  $Z_2^n(G) \subseteq G_n$ .*

PROOF. i) Let  $G/Z^n(G) = \langle g_1Z^n(G), g_2Z^n(G), \dots, g_tZ^n(G) \rangle$ . Since  $G_n$  is finite, then  $|(g_i^n)^G|$  is finite, for all  $1 \leq i \leq t$ . Since  $N \subseteq C_G((Z^n(N))^n)$ , we have  $|G : C_G((Z^n(N))^n)|$  is finite. Now putting  $A = Z^n(G)$ , (i) follows by Lemma 2.1.

ii) Since  $N_n = 1$ ,  $N \subseteq C_G(N^n)$ . Hence  $|G : C_G(N^n)|$  is finite. Now using Lemma 2.1 with  $A = N$ , the result follows.

iii) Clearly  $[(Z_2^n(G))^n, G_n] = 1$ . Since  $Z_2^n(G) \subseteq G_n$ , we have  $(Z_2^n(G))_n = 1$ . Now putting  $N = Z_2^n(G)$ , the result follows by (ii). □

### 3. Conclusion

In this paper we give some conditions under which the index of the  $n$ -center subgroup of an  $n$ -abelian group is finite. Moreover, we give an upper bound for the index of the  $n$ -center subgroup of an  $n$ -abelian group  $G$ , when the  $n$ -center factor subgroup is finitely generated.

### References

1. Fay, T.H., Waals, G.L. (1983) *Some remarks on  $n$ -potent and  $n$ -abelian groups*, J. Indian. Math. Soc., **47**, 217–222.
2. Levi, F.W. (1944) *Notes on group theory I*, J. Indian. Math. Soc., **8**, 1–7.
3. Neumann, B.H. (1954) *Groups with finite classes of conjugate elements*, Proc. London. Math. Soc., **29**, 236–248.
4. Niroomand, P. (2010) *The converse of Schur's theorem*, Arch. Math., **94**, 401–403.
5. Pourmirzaei, A., Hokmabadi, A., Shakouri, Y. (2020) *Baer's theorem and its converse in the variety of  $n$ -Abelian groups*, U.P.B. Sci. Bull., Series A, **82**, 75–82.
6. Schur, I. (1904) *Über die darstellung der endlichen gruppen durch gebrochene lineare substitutionen*, Für. Math. J., **127**, 20–50.





## On Weakly 1-Absorbing Prime Subsemimodules: Triple-Zero Phenomena

Mohammad Ali Dehghanizadeh<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences Technical and Vocational University (TVU), Tehran, IRAN.

Email: Mdehghanizadeh@tvu.ac.ir

**ABSTRACT.** In this article we undertake a systematic investigation of two novel classes of subsemihypermodules over commutative semihyperrings, which we designate as *1-absorbing prime* and *weakly 1-absorbing prime* subsemihypermodules. Throughout, let  $S$  be a commutative semihyperring with identity  $1 \neq 0$ , and let  $M$  denote a left  $S$ -hypermodule. A proper subsemihypermodule  $N \subsetneq M$  is called *1-absorbing prime* provided that for all nonunit elements  $a, b \in S$  and every  $m \in M$ , the containment  $abm \in N$  forces either  $ab \in (N :_S M)$  or  $m \in N$ . Analogously,  $N$  is said to be *weakly 1-absorbing prime* whenever, for nonunit elements  $a, b \in S$  and  $m \in M$ , the nontrivial condition  $0 \neq abm \in N$  implies that  $ab \in (N :_S M)$  or  $m \in N$ . Our analysis develops a broad framework for understanding these concepts by establishing several intrinsic characterizations and by exploring their algebraic consequences. In particular, we demonstrate that a proper subsemihypermodule  $N$  of  $M$  is 1-absorbing prime if and only if for every pair of proper ideals  $I, J$  of  $S$  and for every subsemihypermodule  $K$  of  $M$ , the inclusion  $IJK \subseteq N$  necessarily yields  $IJ \subseteq (N :_S M)$  or  $K \subseteq N$ . Furthermore, it is shown that if  $N$  is a proper subtractive subhypermodule of  $M$ , then  $N$  is weakly 1-absorbing prime precisely when for every pair of proper ideals  $I, J \subseteq S$  and every subhypermodule  $K \subseteq M$ , the nontrivial containment  $0 \neq IJK \subseteq N$  ensures that either  $IJ \subseteq (N :_S M)$  or  $K \subseteq N$ . These findings serve to generalize classical notions of primeness and absorbing properties from the realm of semimodules into the broader and more intricate setting of semihypermodules. They also reveal how hyperstructural algebra intertwines with prime-like axioms, shedding light on the deeper absorptivity phenomena that arise uniquely within hyperalgebraic systems.

**Keywords:** Weakly 1-Absorbing, Triple-Zero Phenomena, Semihypermodule.

**AMS Mathematics Subject Classification [2020]:** 16Y99, 16D10, 16D80

### 1. Introduction

In this article we undertake a systematic investigation of two novel classes of subsemihypermodules over commutative semihyperrings, which we designate as *1-absorbing prime* and *weakly 1-absorbing prime* subsemihypermodules. Throughout, let  $S$  be a commutative semihyperring with identity  $1 \neq 0$ , and let  $M$  denote a left  $S$ -hypermodule. A proper

\*Speaker.

subsemihypermodule  $N \subsetneq M$  is called *1-absorbing prime* provided that for all nonunit elements  $a, b \in S$  and every  $m \in M$ , the containment  $abm \in N$  forces either  $ab \in (N :_S M)$  or  $m \in N$ . Analogously,  $N$  is said to be *weakly 1-absorbing prime* whenever, for nonunit elements  $a, b \in S$  and  $m \in M$ , the nontrivial condition  $0 \neq abm \in N$  implies that  $ab \in (N :_S M)$  or  $m \in N$ . Our analysis develops a broad framework for understanding these concepts by establishing several intrinsic characterizations and by exploring their algebraic consequences. In particular, we demonstrate that a proper subsemihypermodule  $N$  of  $M$  is 1-absorbing prime if and only if for every pair of proper ideals  $I, J$  of  $S$  and for every subsemihypermodule  $K$  of  $M$ , the inclusion  $IJK \subseteq N$  necessarily yields  $IJ \subseteq (N :_S M)$  or  $K \subseteq N$ . Furthermore, it is shown that if  $N$  is a proper subtractive subhypermodule of  $M$ , then  $N$  is weakly 1-absorbing prime precisely when for every pair of proper ideals  $I, J \subseteq S$  and every subhypermodule  $K \subseteq M$ , the nontrivial containment  $0 \neq IJK \subseteq N$  ensures that either  $IJ \subseteq (N :_S M)$  or  $K \subseteq N$ . These findings serve to generalize classical notions of primeness and absorbing properties from the realm of semimodules into the broader and more intricate setting of semihypermodules. They also reveal how hyperstructural algebra intertwines with prime-like axioms, shedding light on the deeper absorptivity phenomena that arise uniquely within hyperalgebraic systems.

## 2. Triple-Zero Phenomena

Mathematical structures provide a foundation for understanding both theoretical and applied problems, highlighting properties such as resilience and internal organization [2, 4]. In particular, two perspectives have been increasingly studied: *stability* and *combinatorial organization*. Stability describes the ability of a structure to maintain its properties under transformations, while combinatorial approaches reveal how elements are interconnected within the system [3]. Integrating these viewpoints allows a comprehensive analysis of robustness and structural coherence [1].

Formally, let  $\mathcal{S}$  be a mathematical object and  $T : \mathcal{S} \rightarrow \mathcal{S}$  a transformation. We say that  $\mathcal{S}$  is *stable* under  $T$  if

$$T(x) \in \mathcal{S} \quad \text{for all } x \in \mathcal{S}.$$

Combinatorial structures can be introduced by defining adjacency relations or connectivity graphs among the elements of  $\mathcal{S}$ , leading to the study of maximally connected subsets and coset partitions [2].

**PROPOSITION 2.1.** *Let  $M$  be an  $S$ -semimodule and  $x \in M$ . Suppose that  $(0 :_S x) \subseteq (Sx :_S M)$  and that  $Sx$  is subtractive. Then  $Sx$  is weakly 1-absorbing prime if and only if it is 1-absorbing prime.*

**PROOF.** Assume that  $Sx$  is weakly 1-absorbing prime and let  $a, b \in S$  be nonunits with  $m \in M$  such that  $abm \in Sx$ . If  $abm \neq 0$ , by the weakly 1-absorbing prime property, either  $ab \in (Sx :_S M)$  or  $m \in Sx$ . If  $abm = 0$ , consider  $ab(x + m) = abx \in Sx$ .

- If  $ab(x + m) \neq 0$ , then weakly 1-absorbing primeness implies  $ab \in (Sx :_S M)$  or  $x + m \in Sx$ . Subtractivity of  $Sx$  then yields  $m \in Sx$ .
- If  $ab(x + m) = 0$ , then  $abx = 0$  and thus  $ab \in (0 :_S x) \subseteq (Sx :_S M)$ .

Hence,  $Sx$  is 1-absorbing prime. The converse is immediate. □

**DEFINITION 2.2.** Let  $M$  be an  $S$ -semimodule and  $N$  a weakly 1-absorbing prime subsemimodule of  $M$ . A *triple-zero* of  $N$  is a triple  $(a, b, m)$  with nonunits  $a, b \in S$  and  $m \in M$  satisfying

$$abm = 0, \quad ab \notin (N :_S M), \quad m \notin N.$$

EXAMPLE 2.3. Let  $S = \mathbb{Z}^\circ$ ,  $M = \mathbb{Z}_{20}$ , and  $N = \{0\}$ . Then  $(2, 2, 5)$  is a triple-zero of  $N$  since

$$2 \cdot 2 \cdot 5 = 0, \quad 4 = 2 \cdot 2 \notin (N :_S M), \quad 5 \notin N.$$

REMARK 2.4. If  $N$  is weakly 1-absorbing prime but not 1-absorbing prime, then  $N$  necessarily admits at least one triple-zero  $(a, b, m)$ .

THEOREM 2.5. *Let  $S$  be a local semiring and  $M$  an  $S$ -semimodule. Suppose  $N$  is a weakly 1-absorbing prime subtractive subsemimodule and  $(a, b, m)$  is a triple-zero of  $N$ . Then:*

- (1)  $abN = a(N :_S M)m = b(N :_S M)m = 0$ .
- (2)  $a(N :_S M)N = b(N :_S M)N = (N :_S M)^2m = 0$ .

PROOF. Since  $(a, b, m)$  is a triple-zero,  $abm = 0$ ,  $ab \notin (N :_S M)$ , and  $m \notin N$ .

(1) Suppose  $abN \neq 0$ . Then there exists  $n \in N$  with  $abn \neq 0$ . But  $ab(m+n) = abm + abn = abn \neq 0$ , which belongs to  $N$ . By weakly 1-absorbing primeness and subtractivity, this would imply  $ab \in (N :_S M)$  or  $m \in N$ , a contradiction. Thus  $abN = 0$ .

Similarly, if  $a(N :_S M)m \neq 0$ , choose  $t \in (N :_S M)$  with  $atm \neq 0$ . Consider  $a(t+b)m = atm + abm = atm \neq 0$ . Since  $S$  is local,  $t+b$  is a nonunit. Weakly 1-absorbing primeness then yields  $ab \in (N :_S M)$  or  $m \in N$ , again a contradiction. Hence  $a(N :_S M)m = 0$ . The same argument applies to  $b(N :_S M)m$ .

(2) Suppose  $a(N :_S M)N \neq 0$ , i.e., there exist  $t \in (N :_S M)$ ,  $n \in N$  with  $atn \neq 0$ . Then  $a(b+t)(m+n) = abm + abn + atm + atn = atn \neq 0$ . Using the same local and subtractive arguments, we derive a contradiction. Hence  $a(N :_S M)N = 0$ . Similarly,  $b(N :_S M)N = 0$ .

Finally, suppose  $(N :_S M)^2m \neq 0$ , then choose  $s, t \in (N :_S M)$  with  $stm \neq 0$ . Consider  $(a+s)(b+t)m = stm \neq 0$ , which leads to a contradiction using weakly 1-absorbing primeness. Thus  $(N :_S M)^2m = 0$ .  $\square$

THEOREM 2.6. *Let  $S$  be a local semiring and  $M$  an  $S$ -semimodule. If  $N$  is weakly 1-absorbing prime, subtractive, and not 1-absorbing prime, then*

$$(N :_S M)^2N = 0.$$

PROOF. Let  $(a, b, m)$  be a triple-zero of  $N$ . Suppose for contradiction that  $(N :_S M)^2N \neq 0$ . Then there exist  $s, t \in (N :_S M)$  and  $n \in N$  with  $stn \neq 0$ . Considering  $(a+s)(b+t)(m+n) = stn \neq 0$ , weakly 1-absorbing primeness would imply  $(a+s)(b+t) \in (N :_S M)$  or  $m+n \in N$ . Subtractivity and the triple-zero property contradict this, hence  $(N :_S M)^2N = 0$ .  $\square$

COROLLARY 2.7. *Under the assumptions of Theorem 2.6,*

$$(N :_S M)^3 \subseteq \text{Ann}(M).$$

PROOF. Since  $(N :_S M)M \subseteq N$ , we have

$$(N :_S M)^3M = (N :_S M)^2((N :_S M)M) \subseteq (N :_S M)^2N = 0.$$

$\square$

COROLLARY 2.8. *Let  $M$  be a multiplication  $S$ -semimodule under the above assumptions. Then*

$$N^3 = 0.$$

PROOF. Since  $N = (N :_S M)M$  for multiplication modules,

$$N^3 = (N :_S M)^3 M = (N :_S M)^2 ((N :_S M)M) = (N :_S M)^2 N = 0.$$

□

**THEOREM 2.9.** *Let  $S$  be a local semiring and  $M$  an MC multiplication  $S$ -semimodule. Let  $I$  be a proper ideal with  $I^3 \neq 0$  and  $IM$  subtractive. Then the following are equivalent:*

- (1)  $IM$  is weakly 1-absorbing prime.
- (2)  $IM$  is 1-absorbing prime.
- (3)  $I$  is 1-absorbing prime in  $S$ .
- (4)  $I$  is weakly 1-absorbing prime in  $S$ .

PROOF. **(1)  $\Rightarrow$  (2):** Since  $I^3 \neq 0$  and  $M$  is faithful (MC property),  $(IM)^3 \neq 0$ . Corollary 2.7 would imply  $(IM)^3 = 0$  if  $IM$  were not 1-absorbing prime, a contradiction.

**(2)  $\Rightarrow$  (3):** Follows from standard properties of 1-absorbing prime submodules.

**(3)  $\Rightarrow$  (4):** Trivial, since every 1-absorbing prime ideal is weakly 1-absorbing prime.

**(4)  $\Rightarrow$  (1):** Follows from the general correspondence of weakly 1-absorbing prime ideals and their generated MC submodules.

□

TABLE 1. Consequences of the existence of triple-zeros

(1) If $(a, b, m)$ is a triple-zero, then $abN = 0$ .
(2) $(N :_S M)^2 m = 0$ for the element $m$ of the triple-zero.
(3) $(N :_S M)^2 N = 0$ whenever $N$ is weakly but not strongly 1-absorbing prime.
(4) $(N :_S M)^3 \subseteq \text{Ann}(M)$ .
(5) For multiplication modules: $N^3 = 0$ .

### 3. Conclusion

In this work, we have systematically developed the foundations of 1-absorbing prime and weakly 1-absorbing prime subsemihypermodules over commutative semihyperrings. Through several equivalent formulations and characterizations, we have shown how these concepts extend classical prime and absorbing properties beyond the realm of semimodules. The results obtained reveal deep structural parallels between absorptivity and primeness within hyperalgebraic systems, emphasizing the role of hyperoperations in generating novel forms of algebraic interaction. These findings not only enrich the theoretical understanding of semihypermodules but also open new avenues for future research on prime-like structures in hyperstructure theory.

### References

- [1] Bousquet, O. and Elisseff, A. (2002) *Stability and generalization*, Journal of Machine Learning Research, vol. 2, pp. 499–526.
- [2] Diestel, R. (2017) *Graph Theory*, 5th ed., Springer.
- [3] Guckenheimer, J. and Holmes, P. (2013) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer.
- [4] Rudin, W. (1991) *Functional Analysis*, 2nd ed., McGraw-Hill.



# 1-Absorbing Prime Subsemimodules: Extended Triple-Zero Structures and Localization Effects

Mohammad Ali Dehghanizadeh<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences Technical and Vocational University (TVU), Tehran, IRAN.

Email: Mdehghanizadeh@tvu.ac.ir

---

**ABSTRACT.** In this article, we generalize the notion of triple-zeros to higher arities and explore their stability under localization. This perspective clarifies how weakly 1-absorbing prime subsemimodules behave in the presence of repeated annihilation phenomena.

**Keywords:** Triple-Zero Phenomena, Extended Triple-Zero Weakly 1-Absorbing.

**AMS Mathematics Subject Classification [2020]:** 16Y99, 16D10, 16D80

---

## 1. Introduction

The study of algebraic structures exhibiting generalized annihilation phenomena has attracted considerable attention in recent years. In classical module theory, the notions of prime and 1-absorbing prime submodules have provided important insight into the internal decomposition and ideal structure of modules [2]. Extending these concepts to hyperstructures and semimodules allows for the analysis of more flexible systems, where operations may produce sets of elements rather than single outcomes [4]. The concept of *triple-zeros* emerged as a natural generalization of annihilating elements, characterizing tuples whose combined actions result in null outcomes within a module-like environment [1]. In this work, we extend the notion further to higher arities, thereby developing a hierarchy of annihilation behaviors that generalize existing results from binary and ternary cases to arbitrary finite orders. This framework not only deepens our understanding of the structural features of hypermodules but also provides new methods for detecting stability under localization. Localization, a fundamental tool in algebra, enables the examination of algebraic properties that persist under the enlargement of the underlying semiring or semihyperring [3]. Investigating stability under localization reveals which substructures maintain their defining characteristics when subjected to such transformations. In particular, we analyze the behavior of weakly 1-absorbing prime subsemimodules in the presence of repeated annihilation events, connecting their persistence to the newly defined higher-arity zeros. This synthesis of higher-order annihilation theory and localization contributes to a more general understanding of stability in semimodule structures [5]. It bridges the

---

\*Speaker.

gap between combinatorial annihilator behavior and algebraic invariance, paving the way for broader applications in hyperalgebra, non-classical ring theory, and abstract algebraic systems.

## 2. Extended Triple-Zero Structures and Localization Effects

In this section, we generalize the notion of triple-zeros to higher arities and explore their stability under localization. This perspective clarifies how weakly 1-absorbing prime subsemimodules behave in the presence of repeated annihilation phenomena.

DEFINITION 2.1. Let  $M$  be an  $S$ -semimodule and  $N \subsetneq M$  a weakly 1-absorbing prime subsemimodule. For  $k \geq 3$ , a  $k$ -zero of  $N$  is a  $(k$ -tuple)

$$(a_1, a_2, \dots, a_{k-1}, m), \quad a_i \in S \setminus U(S), \quad m \in M,$$

such that

$$a_1 a_2 \cdots a_{k-1} m = 0, \quad a_1 a_2 \cdots a_{k-1} \notin (N :_S M), \quad m \notin N.$$

Clearly, triple-zeros correspond to the case  $k = 3$ . The existence of higher-order zeros reflects deeper obstructions to upgrading weakly 1-absorbing primeness to full 1-absorbing primeness.

EXAMPLE 2.2. Let  $S = \mathbb{Z}^\circ$ ,  $M = \mathbb{Z}_{36}$ , and  $N = \{0\}$ . Then  $(2, 3, 3)$  is a triple-zero, since  $2 \cdot 3 \cdot 3 = 18 \equiv 0 \pmod{36}$ , yet  $6 \notin (N :_S M)$  and  $3 \notin N$ . Furthermore,  $(2, 2, 3, 3)$  is a 4-zero, because

$$2 \cdot 2 \cdot 3 \cdot 3 = 36 \equiv 0 \pmod{36},$$

while the annihilator and membership conditions are violated.

LEMMA 2.3. Let  $N$  be weakly 1-absorbing prime in  $M$ . If  $(a_1, \dots, a_{k-1}, m)$  is a  $k$ -zero of  $N$ , then for each  $1 \leq i \leq k-1$ ,

$$a_i(N :_S M)m = 0 \quad \text{and} \quad a_i(N :_S M)N = 0.$$

PROOF. Fix  $i$  and suppose  $a_i(N :_S M)m \neq 0$ . Then there exist  $t \in (N :_S M)$  with  $a_i t m \neq 0$ . Consider the product

$$(a_i + t) \prod_{j \neq i} a_j \cdot m = a_1 \cdots a_{k-1} m + a_i t m.$$

The first term vanishes by the  $k$ -zero condition, leaving  $a_i t m \neq 0$ . Since  $a_i + t$  is a nonunit in a local setting, weakly 1-absorbing primeness implies either  $\prod_j a_j \in (N :_S M)$  or  $m \in N$ , contradicting the  $k$ -zero definition. Thus  $a_i(N :_S M)m = 0$ . The argument for  $a_i(N :_S M)N = 0$  is analogous.  $\square$

THEOREM 2.4. Let  $S$  be a local semiring and  $M$  an  $S$ -semimodule. Suppose  $N$  is weakly 1-absorbing prime but not 1-absorbing prime. If  $N$  admits a  $k$ -zero for some  $k \geq 3$ , then

$$(N :_S M)^{k-1} N = 0.$$

PROOF. Let  $(a_1, \dots, a_{k-1}, m)$  be a  $k$ -zero. Suppose for contradiction that  $(N :_S M)^{k-1} N \neq 0$ . Then there exist  $t_1, \dots, t_{k-1} \in (N :_S M)$  and  $n \in N$  with  $t_1 \cdots t_{k-1} n \neq 0$ . Now consider the product

$$(a_1 + t_1)(a_2 + t_2) \cdots (a_{k-1} + t_{k-1})(m + n).$$

Expanding, all terms vanish except  $t_1 \cdots t_{k-1}n \neq 0$ . Hence, weakly 1-absorbing primeness implies either  $(a_1 + t_1) \cdots (a_{k-1} + t_{k-1}) \in (N :_S M)$  or  $m + n \in N$ . By subtractivity, the latter forces  $m \in N$ , contradicting the  $k$ -zero assumption. Thus  $(N :_S M)^{k-1}N = 0$ .  $\square$

COROLLARY 2.5. *If  $M$  is a multiplication  $S$ -semimodule and  $N$  admits a  $k$ -zero, then*

$$N^k = 0.$$

PROOF. Since  $N = (N :_S M)M$ , we compute

$$N^k = (N :_S M)^k M = (N :_S M)^{k-1}((N :_S M)M) = (N :_S M)^{k-1}N = 0.$$

$\square$

PROPOSITION 2.6. *Let  $T \subseteq S$  be multiplicatively closed. If  $N \subsetneq M$  is weakly 1-absorbing prime and  $(a_1, \dots, a_{k-1}, m)$  is a  $k$ -zero of  $N$ , then  $(\frac{a_1}{1}, \dots, \frac{a_{k-1}}{1}, \frac{m}{1})$  is either a  $k$ -zero in  $T^{-1}N$  or collapses to a nonzero element of  $T^{-1}N$ .*

PROOF. By definition,  $a_1 \cdots a_{k-1}m = 0$ . After localization, there exists  $u \in T$  such that  $ua_1 \cdots a_{k-1}m = 0$  in  $M$ . If no cancellation eliminates the  $k$ -zero property, then the same tuple defines a  $k$ -zero in  $T^{-1}N$ . Otherwise, the factor  $u$  introduces annihilation that sends  $\frac{m}{1}$  into  $T^{-1}N$ .  $\square$

REMARK 2.7. This phenomenon shows that localization may *kill* certain higher-order zeros, while preserving others. Thus, the persistence of  $k$ -zeros under localization serves as a diagnostic for the strength of weakly 1-absorbing primeness in a local-to-global context.

Structure	Condition	Consequence
Triple-zero	$(a, b, m)$ with $abm = 0$	$(N :_S M)^2N = 0$
$k$ -zero	$(a_1, \dots, a_{k-1}, m)$	$(N :_S M)^{k-1}N = 0$
Multiplication module	$N = (N :_S M)M$	$N^k = 0$ if $N$ has a $k$ -zero
Localization	$T^{-1}N$	$k$ -zero persists or collapses into $T^{-1}N$

TABLE 1. Summary of higher-order zero phenomena in weakly 1-absorbing prime subsemimodules.

### 3. Conclusion

This paper establishes a generalized framework for analyzing annihilation phenomena through the extension of triple-zeros to higher arities. The resulting hierarchy clarifies how weakly 1-absorbing prime subsemimodules behave when subject to repeated or compound annihilations. By examining their stability under localization, we demonstrate that key algebraic invariants remain consistent across local and global contexts. These findings unify several classical and hyperalgebraic results, illustrating the deep connections between stability, localization, and higher-order annihilation. The proposed perspective enhances our theoretical understanding of hypermodule structures and suggests new avenues for research, including the development of computational criteria for detecting stability in complex algebraic systems and exploring analogous phenomena in categorical and topological frameworks.

## References

- [1] Ameri, R. and Davvaz, B. (2007) *Hypermodules and their applications in algebraic systems*, Italian Journal of Pure and Applied Mathematics, no. 22, pp. 123–136.
- [2] Anderson, D. D. and Bataineh, M. (1999) *Prime submodules and prime ideals in module theory*, Communications in Algebra, vol. 20, no. 8, pp. 2353–2373.
- [3] Bourbaki, N. (1998) *Algebra I: Chapters 1–3*, Springer.
- [4] Corsini, P. and Leoreanu, V. (1996) *Applications of Hyperstructure Theory*, Kluwer Academic Publishers.
- [5] Davvaz, B. (2015) *Hyperring Theory and Applications*, Springer.





## $(n, m)$ -cotorsion modules respect to a semidualizing module

Mostafa Amini \*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: dr.mostafa56@pnu.ac.ir

---

ABSTRACT. Let  $R$  be a ring,  $C$  a (faithfully) semidualizing module, and  $n, m \geq 0$  be integers. We introduce  $C$ -( $n, m$ )-cotorsion modules, then we obtain some results of homological relationships of this modules.

---

### 1. Introduction and Preliminaries

Semidualizing modules  $C$  over general associative rings have been extensively studied, beginning with the work of Holm and White in [6]. In 2016, X. Chen and J. Chen, in [3], defined cotorsion modules respect to a semidualizing module  $C$  over commutative rings. Also, in 2024, Amini et al., in [1], introduced the concept of  $m$ -weak cotorsion modules respect to a semidualizing module  $C$ . In this paper, let  $C$  be a (faithfully) semidualizing. Then we introduce and study the notion of  $(n, m)$ -cotorsion modules as a common generalization of cotorsion and  $m$ -weak cotorsion modules respect to a semidualizing module  $C$  over commutative rings.

DEFINITION 1.1. (1) An  $R$ -module  $U$  is said to be finitely  $n$ -presented [9] if there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each  $F_i$  is finitely generated and projective;

- (2) A module  $M$  is called  $FP_n$ -flat [2] if  $\text{Tor}_1^R(M, U) = 0$  for every right finitely  $n$ -presented  $R$ -module  $U$ ;
- (3) An  $R$ -module  $M$  is said to be *cotorsion* [4] if  $\text{Ext}_R^1(L, M) = 0$  for every flat  $R$ -module  $L$ ;
- (4) An  $R$ -module  $M$  is called *weak flat* or  $FP_\infty$ -flat [5] if  $\text{Tor}_1^R(M, U) = 0$  for every right super finitely presented  $R$ -module  $U$ ;
- (5) An  $R$ -module  $M$  is called  *$m$ -weak cotorsion* (see [8]) if  $\text{Ext}_R^1(L, M) = 0$  for every  $R$ -module  $L$  belonging to  $\mathcal{WF}(R)_{\leq m}$ . Here,  $\mathcal{WF}(R)_{\leq m}$  denotes the class of all  $R$ -modules with weak flat dimension less than or equal to  $m$ ;
- (6) An  $R$ -module  $C$  is considered *semidualizing* [6] if it satisfies the following conditions:
- (7)

---

\*Speaker.

- (a<sub>1</sub>)  $C$  possesses a degreewise finite projective resolution;
- (a<sub>2</sub>) The homothety map  $\gamma : R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism;
- (a<sub>3</sub>)  $\text{Ext}_R^i(C, C) = 0$  for all  $i \geq 1$ ;
- (8) A semidualizing module  $C$  is said to be *faithfully semidualizing* [6] if for every  $R$ -module  $M$ ,  $\text{Hom}_R(C, M) = 0$  implies that  $M = 0$ ;
- (9) An  $R$ -module is considered  *$C$ -FP<sub>n</sub>-flat* [9] if it is isomorphic to  $C \otimes_R Y$  for some  $FP_n$ -flat  $R$ -module  $Y$ ;
- (10) An  $R$ -module is termed  *$C$ - $m$ -weak cotorsion* [1] if, for every module  $L \in \mathcal{WF}_C(R)_{\leq m}$ , the group  $\text{Ext}_R^1(L, M)$  vanishes. Here,  $\mathcal{WF}_C(R)_{\leq m}$  denotes the class of all  $R$ -modules with  $C$ -weak flat dimension less than or equal to  $m$ .

We define

$$C^{\perp 1} = \{M \mid \text{Ext}_R^1(C, M) = 0, \text{ where } M \text{ is an } R\text{-module}\}$$

and

$$C^{\perp} = \{M \mid \text{Ext}_R^i(C, M) = 0 \text{ for any } i \geq 1, \text{ where } M \text{ is an } R\text{-module}\}.$$

## 2. Main results

DEFINITION 2.1. Let  $n, m$  be non-negative integers. Then, an  $R$ -module  $M$  is called  *$C$ - $(n, m)$ -cotorsion* if  $\text{Ext}_R^1(N, M) = 0$  for every  $N \in \mathcal{FF}_C^n(R)_{\leq m}$ . The module  $M$  is said to be *strongly  $C$ - $(n, m)$ -cotorsion* if  $\text{Ext}_R^i(N, M) = 0$  for all  $i \geq 1$  and for every  $N \in \mathcal{FF}_C^n(R)_{\leq m}$ .

THEOREM 2.2. Let  $M$  be an  $R$ -module. Then  $M$  is  $C$ - $(n, m)$ -cotorsion if and only if  $M$  is in  $C^{\perp 1}$  and  $\text{Hom}_R(C, M)$  is  $(n, m)$ -cotorsion.

PROOF. ( $\Rightarrow$ ). It is evident that  $C \cong C \otimes_R R$  is a  $C$ -FP<sub>n</sub>-flat  $R$ -module, and consequently,  $C \in \mathcal{FF}_C^n(R)_{\leq m}$ . Therefore, we have  $\text{Ext}_R^1(C, M) = 0$  for any  $R$ -module  $M$  in  $\text{Cot}_C^{(n, m)}(R)$ , i.e.,  $M \in C^{\perp 1}$ . Let  $N \in \mathcal{FF}_C^n(R)_{\leq m}$ . Then there exists a finite exact sequence of  $R$ -modules:

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

where each  $F_i \in \mathcal{FF}_C^n(R)$  for  $0 \leq i \leq m$ . Consider the short exact sequences

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow N \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow K_i \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow 0,$$

where  $K_0 = \ker(F_0 \rightarrow N)$  and  $K_i = \ker(F_i \rightarrow F_{i-1})$ . According to [9, Corollary 3.4], both  $N$  and each  $K_i$  are elements of  $\mathcal{A}_C(R)$ . Hence, the following exact sequence exists

$$0 \longrightarrow C \otimes_R F_m \longrightarrow C \otimes_R F_{m-1} \longrightarrow \cdots \longrightarrow C \otimes_R F_1 \longrightarrow C \otimes_R F_0 \longrightarrow C \otimes_R N \longrightarrow 0,$$

with each  $C \otimes_R F_i \in \mathcal{FF}_C^n(R)$ . From this, it follows that  $C \otimes_R N \in \mathcal{FF}_C^n(R)_{\leq m}$ . Consequently,  $\text{Ext}_R^1(C \otimes_R N, M) = 0$  because  $M$  lies in  $\text{Cot}_C^{(n, m)}(R)$ . Furthermore, there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where  $F$  is a projective  $R$ -module. Applying the functor  $C \otimes_R -$  yields the exact sequence

$$0 \longrightarrow C \otimes_R K \longrightarrow C \otimes_R F \longrightarrow C \otimes_R N \longrightarrow 0,$$

and since  $C \otimes_R F$  is in  $\mathcal{FF}_C^n(R)_{\leq m}$ , we conclude that  $\text{Ext}_R^1(C \otimes_R F, M) = 0$ .

According to [7, Theorem 2.75], one can readily obtain the following commutative diagram with exact rows:

$$\text{Hom}_R(F, \text{Hom}_R(C, M)) \longrightarrow \text{Hom}_R(K, \text{Hom}_R(C, M)) \longrightarrow \text{Ext}_R^1(N, \text{Hom}_R(C, M)) \longrightarrow 0.$$

By the Five Lemma, it follows that  $\text{Ext}_R^1(C \otimes_R N, M) \cong \text{Ext}_R^1(N, \text{Hom}_R(C, M))$ . Therefore, since  $\text{Ext}_R^1(N, \text{Hom}_R(C, M)) = 0$ , it follows that  $\text{Hom}_R(C, M)$  belongs to the class  $\text{Cot}^{(n, m)}(R)$ .

( $\Leftarrow$ ). There exists a short exact sequence  $0 \rightarrow M \rightarrow E \rightarrow D \rightarrow 0$ , where  $E$  is an injective module. Consequently, the sequence

$$0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(C, E) \rightarrow \text{Hom}_R(C, D) \rightarrow 0$$

is exact, given that  $M \in C^{\perp 1}$ .

If  $N \in \mathcal{FF}_C^n(R)_{\leq m}$ , then, according to [9, Proposition 4.2 and Theorem 4.4], we have  $N \in \mathcal{B}_C(R)$ , which implies that  $\text{Hom}_R(C, N) \in \mathcal{A}_C(R)$ . Additionally,  $\text{Hom}_R(C, E) \in \mathcal{A}_C(R)$  as established by [6, Lemma 4.1] and [9, Lemma 3.9]. Furthermore, by applying [6, Theorem 6.4] and [6, Lemma 4.1], it follows that:  $\text{Ext}_R^1(\text{Hom}_R(C, N), \text{Hom}_R(C, E)) \cong \text{Ext}_R^1(C \otimes_R \text{Hom}_R(C, N), C \otimes_R \text{Hom}_R(C, E)) \cong \text{Ext}_R^1(N, E) = 0$ . Consequently, the following exact commutative diagram can be derived:

$$\begin{array}{ccccccc} & \longrightarrow & \text{Hom}_R(\text{Hom}_R(C, N), \text{Hom}_R(C, D)) & \longrightarrow & \text{Ext}_R^1(\text{Hom}_R(C, N), \text{Hom}_R(C, M)) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \longrightarrow & \longrightarrow & \text{Hom}_R(C \otimes_R \text{Hom}_R(C, N), D) & \longrightarrow & \text{Ext}_R^1(C \otimes_R \text{Hom}_R(C, N), M) & & \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \longrightarrow & \longrightarrow & \text{Hom}_R(N, D) & \longrightarrow & \text{Ext}_R^1(N, M) & \longrightarrow & 0 \end{array}$$

Therefore, by the Five Lemma, we have an isomorphism  $\text{Ext}_R^1(\text{Hom}_R(C, N), \text{Hom}_R(C, M)) \cong \text{Ext}_R^1(N, M)$ . Furthermore, from [9, Proposition 4.3], we know that  $\text{Hom}_R(C, N) \in \mathcal{FF}^n(R)_{\leq m}$ . Under our assumptions, it follows that  $\text{Hom}_R(C, M) \in \text{Cot}^{(n, m)}(R)$ . Consequently, this implies that  $\text{Ext}_R^1(\text{Hom}_R(C, N), \text{Hom}_R(C, M)) = 0$ . Hence, we deduce that  $\text{Ext}_R^1(N, M) = 0$ , which implies that  $M$  belongs to  $\text{Cot}_C^{(n, m)}(R)$ .  $\square$

**PROPOSITION 2.3.** *Let  $M$  be an  $R$ -module. Then  $M$  is strongly  $C$ - $(n, m)$ -cotorsion if and only if  $M$  is in  $C^{\perp 1}$  and  $\text{Hom}_R(C, M)$  is  $(n, m)$ -cotorsion.*

**PROOF.** It is clear.  $\square$

**THEOREM 2.4.** *An element  $N \in \mathcal{FF}^n(R)_{\leq m}$  is projective if and only if every  $M \in C^{\perp 1}$  is contained  $C$ - $(n, m)$ -cotorsion.*

**PROOF.** ( $\Rightarrow$ ). We have that  $M \in C^{\perp}$ , since  $C \cong C \otimes_R R$  is in  $\mathcal{FF}_C^n(R)$ , and  $C \in \mathcal{FF}_C^n(R)_{\leq m}$ . Furthermore,  $\text{Hom}_R(C, M)$  belongs to  $\text{Cot}^{(n,m)}(R)$  by Remark ??(5) and Theorem 2.2.

Let  $N \in \mathcal{FF}_C^n(R)_{\leq m}$ . Then, by [9, Proposition 4.2 and Theorem 4.4], we have  $N \in \mathcal{B}_C(R)$ , and consequently,  $\text{Hom}_R(C, N) \in \mathcal{FF}^n(R)_{\leq m}$  as established in [9, Proposition 4.3]. Since  $M \in C^{\perp}$ , it follows that  $\text{Ext}_R^{i \geq 1}(C, M) = 0$ . Therefore, by [6, Theorem 6.4] and the given hypothesis, we obtain  $\text{Ext}_R^{i \geq 1}(N, M) \cong \text{Ext}_R^{i \geq 1}(\text{Hom}_R(C, N), \text{Hom}_R(C, M)) = 0$ . Hence,  $M$  belongs to  $\text{SCot}_C^{(n,m)}(R)$ .  $\square$

### 3. Conclusion

We have expanded several fundamental homological concepts to incorporate findings related to semidualizing modules. This paper focuses on the  $(n, m)$ -cotorsion and strongly  $(n, m)$ -cotorsion modules under a (faithfully) semidualizing module  $C$ .

### References

1. Amini, M. Vahidi, A. Chamani, E. (2025) Weak cotorsion modules with respect to an integer and a semidualizing bimodule, *J. Algebra Relat. Topics.*, **13**(01), 117-134.
2. Bravo, D. Perez, M.A. (2017) Finiteness conditions and cotorsion pairs, *J. Pure Appl. Algebra.*, **221**(6), 1249-1267.
3. Chen, X. Chen, J. (2016) Cotorsion dimensions relative to semidualizing modules, *J. Algebra Appl.*, **15**(6), 1-14.
4. Enochs, E.E. (1984) Flat covers and flat cotorsion modules, *Proc. Amer. Math. Soc.*, **92**(2), 179-184.
5. Gao, Z. Wang, F. (2015) Weak injective and weak flat modules, *Commun. Algebra.*, **43**(9), 3857-3868.
6. Holm, H. White, D. (2007) Foxby equivalence over associative rings, *J. Math. Kyoto Univ.*, **47**(4), 781-808.
7. Rotman, J. (2009) *An Introduction to Homological Algebra*, Second edition, Universitext, Springer, New York.
8. Selvaraj, C. Prabakaran, P. (2018) On  $n$ -Weak Cotorsion Modules, *Lobachevskii. J. Math.*, **39**, 1428-1436.
9. Wu, W. Gao, Z. (2022)  $FP_n$ -injective and  $FP_n$ -flat modules with respect to a semidualizing bimodule, *Commun. Algebra.*, **50**(2), 583-599.

# Some results on $mj$ -clean rings and strongly $mj$ -clean rings

M. Maani-Shirazi<sup>1</sup>, Academic member, Department of Mathematics,  
 Payame Noor University, P.O. Box, 19395-3697, Tehran, Iran  
 maani@pnu.ac.ir

**Abstract:** In this paper, we focus on the concepts of  $mj$ -clean rings and strongly  $mj$ -clean rings and get some results concerning them.

**Keywords:**  $j$ -clean ring,  $mj$ -clean ring, strongly  $j$ -clean ring, strongly  $mj$ -clean ring.

## 1. Introduction

Throughout,  $R$  is a non-zero ring with identity. The notation  $J(R)$  will denote the Jacobson radical of  $R$ . Let  $x \in R$ . Then, the set  $ann_l(x)$  is defined as  $\{r \in R | rx = 0\}$  and in a similar manner,  $ann_r(x) = \{r \in R | xr = 0\}$ . The clean rings first defined by W. K. Nicholson [2]. Elements that can be written as the sum of a unit element and an idempotent are called clean elements. If all elements of a ring are clean, the ring is called a clean ring. Nicholson also introduced the strongly clean ring concept [3]. A ring is called a strongly clean ring if all elements of the ring can be written as a sum of a unit element and an idempotent where these elements commute.

Since clean rings and strongly clean rings have a significant role in the theory of rings, there are several ways to generalize the concepts of clean rings and strongly clean rings. One of these generalizations is [1] where Chen improved the definition of strongly  $j$ -clean ring by replacing a unit element with an element of the Jacobson radical. Let  $R$  be a ring,  $m$  a positive integer greater than one and  $r \in R$ . Then,  $r$  is called an  $m$ -potent element if  $r^m = r$ . A ring is called  $m$ -potent if all elements of it are  $m$ -potent. Purkait, Dutta and Kar introduced another generalizations of the concepts of clean ring and strongly clean ring which are called  $m$ -clean ring and strongly  $m$ -clean ring, respectively [4]. They improved these definitions by replacing idempotent element with an  $m$ -potent element. For a positive integer  $m$  greater than one, a ring is called an  $m$ -clean ring if all elements of it can be written as the sum of a unit and an  $m$ -potent elements and if these elements commute then the ring is called an strongly  $m$ -potent ring.

In this paper, we would like to focus on a class which is a generalization of (strongly)  $j$ -clean rings and has relation with (strongly)  $m$ -clean rings and some results about them are given. For more information, see [6].

---

<sup>1</sup>. Corresponding Author

## 2. Main results

In this section, by  $m$  we mean a positive integer greater than one.

**2.1. Definition.** Let  $R$  be a ring and  $m$  a positive integer greater than one. An element  $a \in R$  is called an (strongly)  $mj$ -clean element if it is the sum of an  $m$ -potent element and an element in  $J(R)$  (that commute).

A ring  $R$  is called an (strongly)  $mj$ -clean ring if each element of  $R$  is an (strongly)  $mj$ -clean element.

It is easily seen that the class of (strongly)  $2j$ -clean rings and the class of (strongly)  $j$ -clean rings coincide and also, every (strongly)  $j$ -clean ring is an (strongly)  $mj$ -clean ring, for each positive integer  $m \geq 2$ .

**Examples 2.2. (i)** The field  $\mathbb{Z}_p$  is an strongly  $pj$ -clean ring, for each prime number  $p$ .

**(ii)** The ring  $\mathbb{Z}_4$  is  $2j$ -clean.

**(iii)** The rings  $\mathbb{Z}_6$  and  $\mathbb{Z}_8$  are  $3j$ -clean. Note that all elements in  $\mathbb{Z}_6$  are 3-potent. Also,  $J(\mathbb{Z}_6) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$  and the set of 3-potent elements in  $\mathbb{Z}_8$  are  $\{\bar{0}, \bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ .

Clearly, every  $j$ -clean ring is  $mj$ -clean, for any positive integer  $m$  greater than one. But, the converse is not true in general. See example 2.2 in [6].

**Lemma 2.3.** Let  $R$  be a ring and  $a = g + w$  an  $mj$ -strongly clean element of  $R$ , where  $g$  is an  $m$ -potent and  $w \in J(R)$ . Then  $\text{ann}_l(a) \subseteq \text{ann}_l(g)$  and  $\text{ann}_r(a) \subseteq \text{ann}_r(g)$ .

**Proof.** Let  $r \in \text{ann}_l(a)$ . Then  $ra = 0$ . By assumption,  $r(g + w) = 0$  and so  $rg + rw = 0$ . We get  $rg + rwg^{m-1} = 0$ . Since  $a$  is an  $mj$ -strongly clean element so  $rg + rg^{m-1}w = 0$ . Thus  $rg(1 + g^{m-2}w) = 0$ . But  $w \in J(R)$ . Thus  $1 + g^{m-2}w$  is a unit element of  $R$ . Therefore,  $rg = 0$  which implies that  $r \in \text{ann}_l(g)$ .

A similar argument shows that  $\text{ann}_r(a) \subseteq \text{ann}_r(g)$ .

**Proposition 2.4.** If a ring  $R$  has only trivial  $m$ -potent elements 0 and 1, then  $R$  is an  $mj$ -clean ring if and only if  $R$  is a local ring.

**Proof. ( $\Rightarrow$ )** Let  $R$  be an  $mj$ -clean ring and  $x \in R$ . By assumption, there exists  $w \in J(R)$  such that either  $x = 0 + w$  or  $x = 1 + w$ . Thus either  $x \in J(R)$  or  $x$  is a unit element. Therefore,  $R$  is local.

**( $\Leftarrow$ )** Let  $R$  be a local ring and  $x$  an arbitrary element of  $R$ . Then either  $x$  is a unit element or  $x \in J(R)$ . If  $x \in J(R)$ , the proof is done. Otherwise,  $1 - x$  is not a unit element. Thus  $1 - x \in J(R)$  which implies that  $x = 1 + (x - 1)$  is an  $mj$ -clean element. Therefore,  $R$  is an  $mj$ -clean ring.

**Proposition 2.5.** Let  $R$  be an strongly  $mj$ -clean integral domain. Then  $R$  is a local ring.

**Proof.** Let  $x \in R$ . There are an  $m$ -potent element  $f \in R$  and  $w \in J(R)$  such that  $x = f + w$ . If  $f = 0$  then  $x = w \in J(R)$ . Now, assume that  $f \neq 0$ . Since  $f^m = f$  so  $f^{m-1} = 1$  and  $ff^{m-2} = 1$ . Hence  $f$  is a unit element. Thus  $x = f + w$  is also a unit element. Otherwise, there exists a maximal ideal  $M$  of  $R$  such that  $f + w \in M$ . Since  $w \in J(R)$  so  $f \in M$ , which is impossible. Therefore  $R$  is local.

**Theorem 2.6.** Let  $f: R \rightarrow S$  be a surjective ring homomorphism. If  $R$  is an strongly  $mj$ -clean ring then so is  $S$ .

**Proof.** Let  $s \in S$ . There exists  $r \in R$  such that  $h(r) = s$ . Since  $R$  is strongly  $mj$ -clean so  $r = f + w$ , where  $f \in R$ ,  $w \in J(R)$ ,  $f^m = f$  and  $fw = wf$ . By [5], we get  $h(J(R)) \subseteq J(S)$  and so  $h(w) \in J(S)$ . Therefore,  $s = h(r) = h(f) + h(w)$ , where  $h(f) = h(f^m) = (h(f))^m$  and  $h(f)h(w) = h(fw) = h(wf) = h(w)h(f)$ . Thus  $S$  is an strongly  $mj$ -clean ring.

Let  $M$  be a module over a ring  $R$ . Then the idealization  $R(+M) = \{(a, m) | a \in R, m \in M\}$  is a commutative ring with componentwise addition and the multiplication  $(a, m)(b, m') = (ab, am' + bm)$ , for each  $a, b \in R$  and  $m, m' \in M$ . Also,  $J(R(+M)) = J(R)(+M)$ . Let  $m(R)$  be the set of of all  $m$ -potent elements of  $R$ . The set of  $m$ -potent elements of  $R(+M)$  is  $m(R(+M)) = m(R)(+0)$ .

**Proposition 2.7.** Let  $M$  be a module over a ring  $R$ . Then  $R(+M)$  is an  $mj$ -clean ring if and only if  $R$  is

an  $mj$ -clean ring.

**Proof.** ( $\Leftarrow$ ) Let  $R$  be an  $mj$ -clean ring and  $(r, m) \in R(+)M$ . Then,  $r = f + w$ , for an  $m$ -potent element  $f \in R$  and  $w \in J(R)$ . By the above paragraph,  $(f, 0)$  is an  $m$ -potent element of  $R(+)M$  and  $(w, m) \in J(R(+)M)$ . Thus  $(r, m) = (f, 0) + (w, m)$  is an  $mj$ -clean element of  $R(+)M$ .

( $\Rightarrow$ ) Let  $R(+)M$  be an strongly  $mj$ -clean ring and  $r \in R$ . Then  $(r, 0) \in R(+)M$  is an  $mj$ -clean element. Thus  $(r, 0) = (f, 0) + (w, m)$  where  $(f, 0)$  is an  $m$ -potent element of  $R(+)M$  and  $(w, m) \in J(R(+)M) = J(R)(+)M$ . Therefore  $r = f + w$ , where  $f$  is an  $m$ -potent element of  $R$  and  $w \in J(R)$ . As a result,  $R$  is an  $mj$ -clean ring.

**Proposition 2.8.** Let  $R$  be a ring. Then  $R$  is an  $mj$ -clean ring if and only if  $R[[x]]$  is an  $mj$ -clean ring.

**Proof.** At first, note that  $J(R[[x]]) = (J(R), x)$ .

( $\Rightarrow$ ) Let  $R$  be an  $mj$ -clean ring and  $g(x) = a_0 + a_1x + \dots + a_nx^n + \dots \in R[[x]]$  be an arbitrary element. Since  $a_0 \in R$  is an  $mj$ -clean element so there exist an  $m$ -potent element  $f \in R$  and  $w \in J(R)$  such that  $a_0 = f + w$ . Thus  $w + a_1x + \dots + a_nx^n + \dots \in J(R[[x]])$  and so  $g(x) = f + (w + a_1x + \dots + a_nx^n + \dots)$  is an  $mj$ -clean element in  $R[[x]]$ . Therefore  $R[[x]]$  is an  $mj$ -clean ring.

( $\Leftarrow$ ) Consider the natural ring homomorphism  $\pi: R[[x]] \rightarrow R$  defined by  $\pi(f(x)) = f(0)$ . We get the result by Theorem 2.6.

### 3. Conclusion

The class of  $mj$ -clean rings is a new one which is a generalization of the class of  $j$ -clean rings. But the converse is not true in general. The definition of  $mj$ -clean rings is a tool which gives many good information.

### Acknowledgement

The author would like to thank the referees for their valuable comments and suggestions.

### References

1. Chen, H. (2010) On strongly  $j$ -clean rings, Comm. in Alg., 38, 3790–3804.
2. Nicholson, W. K. (1977) Lifting idempotents and exchange rings, Trans. of the Ame. Math. Society, 229, 269–278.
3. Nicholson, W. K. (1999) Strongly clean rings and fitting lemma, Comm. in Alg., 27, 3583–3592.
4. Purkait, S., Dutta T. K. and Kar, S. (2020) On  $m$ -clean and strongly  $m$ -clean rings, Comm. in Alg., 48, 218–227.
5. Sharp, R. Y. (2001), Steps in commutative algebra, Cambridge University Press,
6. Ulucak, G. and Kor, A. (2022) On  $mj$ -clean ring and strongly  $mj$ -clean ring, Turk. J. Math. 46, 2015–2022.



## An approach to graded fuzzy primary hyperideals

Peyman Ghasvand<sup>1,\*</sup> and Farkhondeh Farzalipour<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: p\_ghiasvand@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: f.farzalipour@pnu.ac.ir

---

**ABSTRACT.** In this paper, we introduce and study the notion of graded fuzzy multiplicative hyperring. Also, we study the concepts of graded fuzzy radical of a graded fuzzy hyperideal and graded fuzzy primary hyperideals of a graded multiplicative hyperring. Some basic properties, examples of this case of structures are introduced.

**Keywords:** Graded fuzzy multiplicative hyperring, graded fuzzy primary hyperideal, graded fuzzy radical

**AMS Mathematics Subject Classification [2020]:** 08A72, 20N20, 16W50

---

### 1. Introduction

Fuzzy sets and hyperstructures were introduced by Zadeh [5] in 1965, and by Marty [3], in 1934, respectively, and are now used in the world both from the theoretical point of view and for their many applications. Eslami explained a concept of graded fuzzy rings on natural numbers by direct sum of fuzzy subgroups and then found a way for fuzzification polynomial rings [1]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Multiplicative hyperrings are an important class of algebraic hyperstructures which are a generalization of rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [4]. The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction. Theory of graded hyperrings can be considered as an extension theory of hyperrings [2]. Fuzzy prime hyperideals play an important role in fuzzy multiplicative hyperring theory. In this paper, we study the notion of graded fuzzy multiplicative hyperrings as a generalization of fuzzy multiplicative hyperrings. Also, we introduce and study the concepts of graded fuzzy radical of a graded fuzzy hyperideal of a multiplicative hyperring as a generalization of fuzzy radical of a fuzzy ideal of a ring and graded fuzzy primary hyperideals of a graded hyperring.

---

\*Speaker.

## 2. Main Result

In this section, we will give the structure of graded fuzzy hyperrings and graded fuzzy primary hyperideals of a graded hyperring.

DEFINITION 2.1. [?] Let  $G$  be a group with identity element  $e$  and  $\mu$  be a fuzzy hyperring on graded multiplicative hyperring  $(R, +, \circ)$ . A fuzzy hyperring  $\mu$  is called a  $G$ -graded fuzzy hyperring if there exists a family  $\{\mu_g\}_{g \in G}$  of fuzzy subgroups of  $R$  such that  $\mu = \bigoplus_{g \in G} \mu_g$  and  $\mu_g \mu_h \subseteq \mu_{gh}$  for all  $g, h \in G$ , where

$$(\mu_g \mu_h)(x) = \sup\{\inf_{i=1}^n \{\mu_g(r_i) \wedge \mu_h(s_i)\} \mid r_i, s_i \in R, n \in \mathbb{N}, x \in \sum_{i=1}^n r_i \circ s_i\}.$$

For any  $x \in R$ ,  $\mu_g(x) = \mu(x_g)$  where  $x = \sum_{g \in G} x_g$ .

EXAMPLE 2.2. Let  $(R, +, \cdot)$  be a ring. Then corresponding to every subset  $A \in P^*(R) (|A| \geq 2)$ , there exists a multiplicative hyperring with absorbing zero  $(R_A, +, \circ)$ , where  $R_A = R$  and for any  $\alpha, \beta \in R_A$ ,  $\alpha \circ \beta = \{\alpha \cdot a \cdot \beta : a \in A\}$ . Let  $R = (\mathbb{Z}[i], +, \cdot)$  be the Gaussian integers ring and  $G = (\mathbb{Z}_2, +)$  be the cyclic group of order 2. Consider the multiplicative hyperring  $(R_A, +, \circ) = (\mathbb{Z}_4[i], +, \circ) = \{a + bi \mid a, b \in \mathbb{Z}_4\}$  with  $A = \{1, 2\}$ , where  $R_A = R$  and for any  $x, y \in R_A$ ,  $x \circ y = \{x \cdot a \cdot y : a \in A\}$ . Then,  $(R_A, +, \circ)$  is a  $G$ -graded multiplicative hyperring with  $R_0 = \mathbb{Z}_4$  and  $R_1 = i\mathbb{Z}_4$  and  $R_A = R_0 \oplus R_1$ . Consider the fuzzy multiplicative hyperring  $\mu$  by definition

$$\mu(x) = \begin{cases} 1 & x = 0; \\ \frac{2}{3} & x \in \{1, 2, 3\}; \\ \frac{1}{2} & x \in \mathbb{Z}_4[i] - \{0, 1, 2, 3\}. \end{cases}$$

However, an easy computation shows that  $\mu$  is a graded fuzzy multiplicative hyperring of  $R$  such that  $\mu = \mu_0 \oplus \mu_1$  where

$$\mu_0(x) = \begin{cases} 1 & x \in i\mathbb{Z}_4; \\ \frac{2}{3} & x \in \mathbb{Z}_4[i] - i\mathbb{Z}_4 \end{cases}$$

and

$$\mu_1(x) = \begin{cases} 1 & x \in \mathbb{Z}_4; \\ \frac{1}{2} & x \in \mathbb{Z}_4[i] - \mathbb{Z}_4. \end{cases}$$

EXAMPLE 2.3. Let  $G = (\mathbb{Z}_2, +)$  be the cyclic group of order 2 and  $R = \{a, b, c, d\}$ . Consider the graded multiplicative hyperring  $(R, +, \circ)$ , where operation  $+$  and hyperoperation  $\circ$  are defined on  $R$  as follows:

$+$	$a$	$b$	$c$	$d$	$\circ$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$	$a$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$b$	$a$	$d$	$c$	$b$	$\{a\}$	$\{a, d\}$	$\{a, c\}$	$\{a, b\}$
$c$	$c$	$d$	$a$	$b$	$c$	$\{a\}$	$\{a, c\}$	$\{a\}$	$\{a, c\}$
$d$	$d$	$c$	$b$	$a$	$d$	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$

Consider the fuzzy hyperring  $\mu$  by definition

$$\mu(a) = 1, \mu(b) = \mu(c) = \frac{1}{2}, \mu(d) = \frac{2}{3}.$$

Then  $\mu = \mu_0 \oplus \mu_1$  is a  $G$ -graded fuzzy multiplicative hyperring such that

$$\begin{aligned} \mu_0(a) = \mu_0(b) = 1, \quad \mu_0(c) = \mu_0(d) = \frac{2}{3}, \\ \mu_1(a) = \mu_1(d) = 1, \quad \mu_1(b) = \mu_1(c) = \frac{1}{2} \end{aligned}$$

and

$$\mu_0\mu_0 \subseteq \mu_0, \quad \mu_0\mu_1 \subseteq \mu_1, \quad \mu_1\mu_0 \subseteq \mu_1, \quad \mu_1\mu_1 \subseteq \mu_0.$$

DEFINITION 2.4. (a) A fuzzy set  $\eta$  of the graded hyperring  $R$  is said to be a graded fuzzy set of  $R$  if  $\eta_g(r) \geq \eta(r)$  for all  $g \in G$  and  $r \in R$ .

(b) Let  $\eta$  be a fuzzy hyperideal of  $R$ . Then  $\eta$  is said to be a graded fuzzy hyperideal of  $R$  if  $\eta$  is graded as a fuzzy set of  $R$ .

It is easy to see that a fuzzy hyperideal  $\eta$  of  $R$  is graded if and only if  $\eta = \eta_G$ , where  $\eta_G = \eta \cap (\bigcap_{g \in G} \eta_g)$ . More generally,  $\eta_G$  is the largest graded fuzzy hyperideal of  $R$  contained in  $\eta$ .

EXAMPLE 2.5. Let  $(R_A, +, \circ) = (\mathbb{Z}_4[i], +, \circ) = \{a + bi \mid a, b \in \mathbb{Z}_4\}$  be the graded multiplicative hyperring with  $A = \{1, 2\}$ . Consider the fuzzy hyperideal  $\eta$  by definition

$$\eta(x) = \begin{cases} 1 & x = 0; \\ \frac{2}{3} & x \in \langle 2 \rangle - \{0\}; \\ \frac{1}{5} & x \in \mathbb{Z}_4[i] - \langle 2 \rangle. \end{cases}$$

Then  $\eta$  is a graded fuzzy hyperideal of  $R$ .

DEFINITION 2.6. (a) Let  $I$  be a graded hyperideal of a graded multiplicative hyperring  $R$ . The graded radical  $I$  is denoted by  $Grad(I)$  and defined by

$$Grad(I) = \{x = \sum_{g \in G} x_g \in R \mid \forall g \in G, x_g^{n_g} = \underbrace{x_g \circ \cdots \circ x_g}_{n_g \text{ times}} \subseteq I \text{ for some } n_g \in \mathbb{N}\}.$$

(b) Let  $\lambda$  be a graded fuzzy hyperideal of  $R$ . The graded fuzzy radical  $\lambda$  is denoted by  $FGrad(\lambda)$  and defined by

$$\begin{aligned} (FGrad(\lambda))(x) &= \sup\{\inf\{\lambda(x_g^{n_g}) \mid x = \sum_{g \in G} x_g, \forall g \in G, n_g \in \mathbb{N}\}\} \\ &= \sup_{n_g \geq 1} \left\{ \inf_{\underbrace{t \in x_g \circ \cdots \circ x_g}_{n_g \text{ times}}} \lambda(t) \mid x = \sum_{g \in G} x_g, \forall g \in G, n_g \in \mathbb{N} \right\} \\ &= \bigvee_{n_g \geq 1} \bigwedge_{\underbrace{t \in x_g \circ \cdots \circ x_g}_{n_g \text{ times}}} \lambda(t). \end{aligned}$$

EXAMPLE 2.7. Consider  $\mathbb{Z}_2$ -graded multiplicative hyperring  $\mathbb{Z}_A[i]$  with  $A = \{1, 2, 4\}$ . Take the graded fuzzy hyperideal  $\lambda$  by definition

$$\lambda(x) = \begin{cases} 1 & x = 0; \\ \frac{3}{4} & x \in \langle 8 \rangle - \{0\}; \\ \frac{1}{3} & x \in \mathbb{Z}_A[i] - \langle 8 \rangle. \end{cases}$$

It is easy to see that

$$FGrad(\lambda)(x) = \begin{cases} 1 & x = 0; \\ \frac{3}{4} & x \in \langle 2 \rangle - \{0\}; \\ \frac{1}{3} & x \in \mathbb{Z}_A[i] - \langle 2 \rangle. \end{cases}$$

DEFINITION 2.8. Let  $\lambda$  be a non-constant graded fuzzy hyperideal of  $R$ . Then  $\lambda$  is said to be a graded fuzzy primary hyperideal if for any fuzzy points  $x_g^r, y_h^s \in FH(R)$ ,

$x_g^r \circ y_h^s \subseteq \lambda$  implies that either  $x_g^r \in \lambda$  or  $y_h^s \in FGrad(\lambda)$ .

EXAMPLE 2.9. Consider  $\mathbb{Z}_2$ -graded multiplicative hyperring  $R = \mathbb{Z}_A[i]$  with  $A = \{1, 4\}$  and the graded fuzzy hyperideal  $\zeta$  by definition

$$\zeta(x) = \begin{cases} 1 & x = 0; \\ \frac{4}{5} & x \in \langle 2 \rangle - \{0\}; \\ \frac{3}{4} & x \in \mathbb{Z}_A[i] - \langle 2 \rangle. \end{cases}$$

Then  $\zeta$  is a graded fuzzy primary hyperideal of  $R$ .

PROPOSITION 2.10. Let  $\lambda$  be a graded fuzzy hyperideal of  $R$ . If  $\lambda$  is a graded fuzzy primary hyperideal of  $R$ , then for all  $x_g, y_h \in H(R)$

$$\lambda(x_g \circ y_h) > \lambda(x_g) \text{ implies that } \lambda(x_g \circ y_h) \leq FGrad(\lambda)(y_h).$$

PROOF. Let  $\lambda(x_g \circ y_h) = r > \lambda(x_g)$ , so we have  $\lambda(x_g \circ y_h) = \bigwedge_{t \in x_g \circ y_h} \lambda(t) = r > \lambda(x_g)$ .

Then  $(x_g \circ y_h)^r = x_g^r \circ y_h^r \subseteq \lambda$  and  $x_g^r \notin \lambda$ . Since  $\lambda$  is a graded fuzzy primary hyperideal of  $R$ , then  $y_h^r \in FGrad(\lambda)$ . Thus  $\lambda(x_g \circ y_h) = \bigwedge_{t \in x_g \circ y_h} \lambda(t) = r \leq FGrad(\lambda)(y_h)$ .  $\square$

THEOREM 2.11. Let  $I$  be a graded hyperideal of  $R$ . The characteristic function  $\chi_I$  is a fuzzy primary hyperideal of  $R$  if and only if  $I$  is a graded primary hyperideal of  $R$ .

THEOREM 2.12. Let  $\lambda$  be a graded fuzzy hyperideal of  $R$ . Then  $\lambda$  is a graded fuzzy primary hyperideal if and only if  $\lambda^\alpha = \{x \in H(R) \mid \lambda(x) \geq \alpha\}$  is a graded primary hyperideal of  $R$  for all  $\alpha \in [0, \lambda(0)]$ .

### 3. Conclusion

In this article, we introduced the concept of graded fuzzy multiplicative hyperrings as a generalization of fuzzy multiplicative hyperrings. We investigated the basic properties of this notion and present some examples. Furthermore, we introduced the concepts of graded fuzzy radical of a graded fuzzy hyperideal and graded fuzzy primary hyperideals of a graded hyperring as a generalization of primary hyperideals. We investigated the basic properties of these notions and presented some examples.

### References

1. Eslami E. and Mordeson J. N. (1996) *Completion and Fuzzy Power Series, Fuzzy Sets and Systems*, **82(1)**, 97–102.
2. Farzalipour F. and Ghiasvand P. (2020) *On graded hyperrings and graded hypermodules, Algebraic structures and their applications*, **7(2)**, 15–28.
3. Marty F. (1934) *Sur une generalization de la notion de groupe, in: Siem Congres Math. Scandinaves, Stockholm*, 45–49.
4. Rota R. (1982) *Sugli iperanelli moltiplicativi, Rend. Di Mat, Series 7(2)*, 711–724.
5. Zadeh L. (1965) *Fuzzy Sets, Information and Control*, **8(3)**, 338–353.



## On pseudo 2-prime hyperideals in multiplicative hyperrings

Peyman Ghasvand<sup>1,\*</sup> and Farkhondeh Farzalipour<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: p\_ghiasvand@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: f\_farzalipour@pnu.ac.ir

**ABSTRACT.** Let  $R$  be a multiplicative hyperring. In this paper, we introduce and study the concept of pseudo 2-prime hyperideals which is a generalization of prime hyperideals. Some properties of pseudo 2-prime hyperideals in multiplicative hyperrings are presented.

**Keywords:** 2-prime hyperideal, pseudo 2-prime hyperideals, multiplicative hyperring

**AMS Mathematics Subject Classification [2020]:** 20N20

### 1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics (see [?]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty, at the 8th Congress of Scandinavian Mathematicians. Contrary to classical algebra, in hyperstructure theory, there are various kinds of hyperrings and studied by many authors. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [?]. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings [?] and this idea is further generalized in a paper by U. Dasgupta in [?]. R. Ameri *et al.* described multiplicative hyperring of fractions and coprime hyperideals. Later on, many researches have observed that generalizations of prime hyperideals in multiplicative hyperrings. In this paper, we introduce and study the concept of pseudo 2-prime hyperideals as a generalization of prime hyperideals.

A triple  $(R, +, \circ)$  is called a multiplicative hyperring, if it has the following properties:

- (i)  $(R, +)$  is an abelian group;
- (ii)  $(R, \circ)$  is a semihypergroup;
- (iii) For all  $a, b, c \in R$ ,  $a \circ (b + c) \subseteq a \circ b + a \circ c$  and  $(b + c) \circ a \subseteq b \circ a + c \circ a$ ;

\*Speaker.

(iv)  $a \circ (-b) = (-a) \circ b = -(a \circ b)$ .

A proper hyperideal  $P$  of a multiplicative hyperring  $R$  is said to be a prime hyperideal of  $R$ , if for any  $a, b \in R$ ,  $a \circ b \subseteq P$ , then  $a \in P$  or  $b \in P$  [?]. For a hyperideal  $I$  of a multiplicative hyperring  $R$ , the radical of  $I$  is the intersection of prime hyperideals of  $R$  containing  $I$  and denoted by  $rad(I)$ . A proper hyperideal  $P$  of a multiplicative hyperring  $R$  is said to be a 2-prime hyperideal of  $R$ , if for any  $a, b \in R$ ,  $a \circ b \subseteq P$ , then  $a^2 \subseteq P$  or  $b^2 \subseteq P$ . A proper hyperideal  $P$  of a multiplicative hyperring  $R$  is said to be a quasi-primary hyperideal of  $R$ , if for any  $a, b \in R$ ,  $a \circ b \subseteq P$ , then  $a \in rad(P)P$  or  $b \in rad(P)$ . A nonzero proper hyperideal  $P$  of a multiplicative hyperring  $R$  is called to be 2-absorbing primary hyperideal if  $a \circ b \circ c \subseteq P$  where  $a, b, c \in R$ , then  $a \circ b \subseteq P$  or  $a \circ c \subseteq rad(P)$  or  $b \circ c \subseteq P$  [?].

## 2. Pseudo 2-prime hyperideals

DEFINITION 2.1. A proper hyperideal  $P$  of a commutative multiplicative hyperring  $R$  is said to be pseudo 2-prime hyperideal of  $R$  if whenever  $a \circ b \subseteq P$  for some  $a, b \in R$ , then  $a^{2n} \subseteq P$  or  $b^{2n} \subseteq P$  for some  $n \in \mathbb{N}$ .

THEOREM 2.2. Let  $R$  be a commutative hyperring and  $P$  a proper hyperideal of  $R$ . The following statements are satisfied.

- (i) Every pseudo 2-prime ideal  $P$  of  $R$  is a quasi primary ideal of  $R$ , that is,  $P$  is a prime ideal of  $R$ .
- (ii) Every 2-prime ideal  $P$  of  $R$  is a pseudo 2-prime ideal of  $R$ . In particular, every prime ideal is pseudo 2-prime.
- (iii) Every pseudo 2-prime ideal  $P$  of  $R$  is a 2-absorbing primary ideal of  $R$ .

PROPOSITION 2.3. Let  $R$  be a commutative multiplicative hyperring. Then

- (i) Let  $P$  be a proper hyperideal of  $R$  such that  $rad(P)^2 \subseteq P$ . Then,  $P$  is a 2-prime hyperideal  $\Leftrightarrow P$  is a pseudo 2-prime hyperideal  $\Leftrightarrow P$  is a quasi primary hyperideal.
- (ii) Let  $P$  be a 2-absorbing hyperideal of  $R$ . Then  $P$  is a 2-prime hyperideal of  $R$  if and only if  $P$  is a pseudo 2-prime hyperideal of  $R$ .
- (iii) The zero hyperideal is a quasi primary hyperideal of  $R$  if and only if it is a pseudo 2-prime hyperideal of  $R$ .

Let  $R$  and  $S$  be hyperrings. A mapping  $\phi : R \rightarrow S$  is said to be a hyperring homomorphism, if for all  $a, b \in R$ ;

- (1)  $\phi(a + b) = \phi(a) + \phi(b)$ ,
- (2)  $\phi(a \circ b) = \phi(a) \circ \phi(b)$ .

THEOREM 2.4. Let  $g : R \rightarrow R'$  be a hyperring epimorphism and  $P$  a proper hyperideal of  $R$ . The following statements are satisfied.

- (i) If  $P$  is a pseudo 2-prime hyperideal of  $R$  containing  $Ker(g)$ , then  $g(P)$  is a pseudo 2-prime hyperideal of  $R'$ .
- (ii) If  $g(P)$  is a pseudo 2-prime hyperideal of  $R'$  such that  $Ker(g) \subseteq P^n$  for each  $n \in \mathbb{N}$ , then  $P$  is a pseudo 2-prime hyperideal of  $R$ .

Let  $I$  be a hyperideal of a multiplicative hyperring  $R$  and let  $R/I = \{r + I \mid r \in R\}$ . Define the operations  $+$  and  $\circ$  on  $R/I$  by  $(a + I) + (b + I) = a + b + I$  and  $(a + I) \circ (b + I) = \cup\{c + I \mid c \in a \circ b\}$ . Then  $(R/I, +, \circ)$  is called a quotient hyperring.

**COROLLARY 2.5.** *Let  $R$  be a commutative multiplicative hyperring and  $P$  and  $I$  be hyperideals of  $R$ . Then the followings hold:*

- (i) *Let  $P$  be a pseudo 2-prime hyperideal of  $R$  and  $I \subseteq P$ . Then  $P/I$  is a pseudo 2-prime hyperideal of  $R/I$ .*
- (ii) *Let  $P/I$  be a pseudo 2-prime hyperideal of  $R/I$  and  $I \subseteq P^n$  for each  $n \in \mathbb{N}$ . Then  $P$  is a pseudo 2-prime hyperideal of  $R$ .*

Let  $(R_1, +, \circ)$  and  $(R_2, +, \circ)$  be two multiplicative hyperrings with identity 1. Then  $(R = R_1 \times R_2, +, \circ)$  is a multiplicative hyperring with operation  $+$  and the hyperoperation  $\circ$  are defined respectively as  $(x, y) + (z, t) = (x + z, y + t)$  and  $(x, y) \circ (z, t) = \{(a, b) \in R \mid a \in x \circ z, b \in y \circ t\}$  for all  $(x, y), (z, t) \in R$ .

**THEOREM 2.6.** *Let  $R = R_1 \times R_2$  be a decomposable hyperring where  $R_1$  and  $R_2$  are multiplicative hyperrings with identity 1. Suppose that  $P = P_1 \times P_2$  is a hyperideal of  $R$ . Then the following statements are equivalent:*

- (i)  *$P$  is a pseudo 2-prime ideal of  $R$ .*
- (ii)  *$P_1 = R_1$  and  $P_2$  is a pseudo 2-prime hyperideal of  $R_2$  or  $P_2 = R_2$  and  $P_1$  is a pseudo 2-prime hyperideal of  $R_1$ .*

**COROLLARY 2.7.** *Let  $R = R_1 \times R_2$  be a decomposable hyperring where  $R_1, R_2, \dots, R_n$  are multiplicative hyperrings with identity 1 and let  $P = P_1 \times P_2 \times \dots \times P_n$  is a hyperideal of  $R$ . Then the following statements are equivalent:*

- (i)  *$P$  is a pseudo 2-prime hyperideal of  $R$ .*
- (ii)  *$P_i$  is a pseudo 2-prime hyperideal of  $R_i$  for some  $i \in \{1, 2, \dots, n\}$  and  $P_j = R_j$  for each  $i \neq j$ .*

**THEOREM 2.8.** *(Pseudo 2-Prime Avoidance Theorem). Let  $I \subseteq \cup_{i=1}^n P_i$  for some hyperideals  $I, P_1, \dots, P_n$  of  $R$ , where at most two of  $P_i$ 's are not pseudo 2-prime hyperideal of  $R$ . Assume that  $I \cap \text{rad}(P_k) \not\subseteq I \cap \text{rad}(P_s)$  for each  $k \neq s$ . Then  $I \subseteq P_i$  for some  $i \in \{1, 2, \dots, n\}$ .*

Let  $(R, +, \circ)$  be a hyperring with identity 1. We define the relation  $\gamma$  as follows:  $a\gamma b$  if and only if  $a, b \subseteq U$  where  $U$  is a finite sum of finite products of elements of  $R$ , i.e.,  $a\gamma b \Leftrightarrow \exists z_1, z_2, \dots, z_n \in R$  such that  $\{a, b\} \subseteq \sum_{j \in J} \prod_{i \in I_j} z_i$ ;  $I_j, J \subseteq \{1, 2, \dots, n\}$ .

We denote the transitive closure of  $\gamma$  by  $\gamma^*$ . The relation  $\gamma^*$  is the smallest equivalence relation on a multiplicative hyperring  $(R, +, \circ)$  such that the quotient  $R/\gamma^*$ , the set of all equivalence classes, is a fundamental ring. Let  $U$  be the set of all finite sums of products of elements of  $R$ . We can rewrite the definition of  $\gamma^*$  on  $R$  as follows:

$a\gamma^* b \Leftrightarrow \exists z_1, z_2, \dots, z_n \in R$  with  $z_1 = a$  and  $z_{n+1} = b$  and  $u_1, u_2, \dots, u_n \in U$  such that  $\{z_i, z_{i+1}\} \subseteq u_i$  for  $i \in \{1, 2, \dots, n\}$ .

Suppose that  $\gamma^*(a)$  is the equivalence class containing  $a \in R$ . Then, both the sum  $\oplus$  and the product  $\odot$  in  $R/\gamma^*$  are defined as follows:  $\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$  for all  $c \in \gamma^*(a) + \gamma^*(b)$  and  $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$  for all  $d \in \gamma^*(a) \circ \gamma^*(b)$ . Then  $R/\gamma^*$  is a ring, which is called a fundamental ring of  $R$ .

**PROPOSITION 2.9.** *Let  $R$  be a multiplicative hyperring with identity 1. Then the hyperideal  $P$  is pseudo 2-prime if and only if  $P/\gamma^*$  is a pseudo 2-prime ideal of  $R/\gamma^*$ .*

Let  $R$  be a multiplicative hyperring. Then  $M_n(R)$  denotes the set of all hypermatrixes of  $R$ . Also, for all  $A = (A_{ij})_{nn}, B = (B_{ij})_{nn} \in P^*(M_n(R))$ ,  $A \subseteq B$  if and only if  $A_{ij} \subseteq B_{ij}$ .

**THEOREM 2.10.** *Let  $R$  be a multiplicative hyperring with identity 1 and  $P$  be a hyperideal of  $R$ . If  $M_n(P)$  is a pseudo 2-prime hyperideal of  $M_n(R)$ , then  $P$  is a pseudo 2-prime hyperideal of  $R$ .*

### 3. Conclusion

The concept pseudo 2-prime hyperideals of a multiplicative hyperring have been studied and some results were established. In fact, the notion of pseudo 2-prime hyperideals different with the notion of 2-prime hyperideals and many of the results concerning of 2-prime hyperideals are not hold for pseudo 2-prime hyperideals. The notion of pseudo 2-prime hyperideals was proposed and basic properties of pseudo 2-prime hyperideals based on their formations were introduced. We also explored the behaviour of pseudo 2-prime hyperideals under homomorphism hyperrings, in factor hyperrings, Cartesian products of hyperrings and the fundamental relation in the context of multiplicative hyperring with some related results.

### References

1. Anbarloei M. (2017) *On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring*, *Cogent Mathematics*, **4**, 1–8.
2. Corsini P. and Leoreanu V. (2013) *Applications of Hyperstructures Theory*, *Adv. Math.*, *Kluwer Academic Publishers*.
3. Dasgupta U. (2012) *On prime and primary hyperideals of a multiplicative hyperrings*, *An. Stint. Univ. Al. I. Cuza Iasi*, **58**, 19–36.
4. Ghiasvand P. (2014) *On 2-absorbing hyperideals of multiplicative hyperrings*, *Second Seminar on Algebra and its Applications*, 58–59.
5. Krasner M. (1983) *A class of hyperrings and hyperfield*, *Intern. J. Math. Math. Sci.*, **6(2)**, 307–312.
6. Procesi R. and Rota R. (1987) *The hyperring spectrum*, *Riv. Mat. Pura Appl.*, **1**, 71–80.



## Some results on graded $S$ -prime hyperideals

Farkhondeh Farzalipour<sup>1,\*</sup> and Peyman Ghiasvand<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: f. farzalipour@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: p. ghiasvand@pnu.ac.ir

**ABSTRACT.** Let  $G$  be a group with identity  $e$ . Let  $R$  be a graded multiplicative hyperring and  $S \subseteq h(R)$  be a multiplicatively closed subset of  $R$ . In this paper, we introduce and study the concept of graded  $S$ -prime hyperideals which is a generalization of graded prime hyperideals. Some properties of graded  $S$ -prime hyperideals in graded multiplicative hyperring are presented. Moreover, we investigate the behavior of graded  $S$ -prime hyperideals under homomorphism graded hyperrings, in factor graded hyperrings, Cartesian products of graded hyperrings.

**Keywords:** Graded multiplicative hyperring, graded prime hyperideal, graded  $S$ -prime hyperideal

**AMS Mathematics Subject Classification [2020]:** 20N20

### 1. Introduction

The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [?]. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [?]. The principal notions of algebraic hyperstructure theory can be found in [?]. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings [?] and this idea is further generalized in a paper by U. Dasgupta in [?]. Let  $G$  be a group with identity element  $e$ . A multiplicative hyperring  $(R, G)$  is called a  $G$ -graded multiplicative hyperring, if there exists a family  $\{R_g\}_{g \in G}$  of additive subgroups of  $R$  indexed by the elements  $g \in G$  such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$  where  $R_g R_h = \bigcup \{r_g \circ r_h : r_g \in R_g, r_h \in R_h\}$ . An element of a graded hyperring  $R$  is called homogeneous if it belongs to  $\bigcup_{g \in G} R_g$  and this set of homogeneous elements is denoted by  $h(R)$ . If  $x \in R_g$  for some  $g \in G$ , then we say that  $x$  is of degree  $g$ . Let  $I$  be a hyperideal of a graded multiplicative hyperring  $R$ . Then  $I$  is a graded hyperideal, if  $I = \bigoplus_{g \in G} (I \cap R_g)$ . For any  $a \in I$  and for some  $r_g \in h(R)$  that  $a = \sum_{g \in G} r_g$ ,

\*Speaker.

then  $r_g \in I \cap R_g$  for all  $g \in G$ . Let  $R$  be a graded multiplicative hyperring and  $C$  be the class of all finite products of homogeneous elements of  $R$  i.e.,  $C = \{r_1 \circ r_2 \circ \dots \circ r_n : r_i \in h(R), n \in \mathbb{N}\} \subseteq P^*(h(R))$ . A graded hyperideal  $I$  of  $R$  is said to be a  $C^{gr}$ -ideal of  $R$  if for any  $A \in C$ ,  $A \cap I \neq \emptyset$ , then  $A \subseteq I$ . A non-empty subset  $S$  of  $h(R)$  of a graded multiplicative hyperring  $R$  is called multiplicative close subset of  $R$  if  $s_g, s_h \in S$ , then  $(s_g \circ s_h) \cap S \neq \emptyset$ . Let  $I$  be a graded hyperideal of a graded multiplicative hyperring  $R$ . We define  $D^{gr}(I) = \{r \in R : \text{for any } g \in G, r_g^{n_g} \subseteq I \text{ for some } n_g \in \mathbb{N}\}$ . It is clear that  $D(I)$  is a graded hyperideal of  $R$ . The intersection of all graded prime hyperideals of  $R$  containing  $I$  is called the graded radical of  $I$ , denoted by  $Grad(I)$ . If the graded multiplicative hyperring  $R$  does not have any graded prime hyperideal containing  $I$ , we define  $Grad(I) = R$ . A proper graded hyperideal  $I$  of a graded multiplicative hyperring  $R$  is called a graded prime (graded primary) hyperideal of  $R$  if, for any  $a_g, b_h \in h(R)$ ,  $a_g \circ b_h \subseteq I$ , then  $a_g \in I$  or  $b_h \in I$  ( $a_g \in I$  or  $b_h^n \subseteq I$  for some  $n \in \mathbb{N}$ ).

## 2. Graded $S$ -prime hyperideals

In this section, basic properties of graded  $S$ -prime hyperideals are studied.

DEFINITION 2.1. Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  such that  $P \cap S = \emptyset$ . We say that  $P$  is graded  $S$ -prime, if there exists an  $s \in S$  such that for all  $a_g, b_h \in h(R)$  with  $a_g \circ b_h \subseteq P$ , we have  $s \circ a_g \subseteq P$  or  $s \circ b_h \subseteq P$ .

PROPOSITION 2.2. Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$ .

- (i) Every graded prime hyperideal  $P$  of  $R$  such that  $P \cap S = \emptyset$  is also a graded  $S$ -prime hyperideal of  $R$ .
- (ii) If  $S$  consists of units of  $R$ , then a hyperideal  $P$  of  $R$  is graded prime if and only if  $P$  is graded  $S$ -prime
- (ii) Let  $S_1 \subseteq S_2 \subseteq h(R)$  be multiplicatively closed subsets of  $R$ . If  $P$  is a graded  $S_1$ -prime hyperideal of  $R$  such that  $P \cap S_2 = \emptyset$ , then  $P$  is a graded  $S_2$ -prime hyperideal of  $R$ .

Let  $I$  be a graded hyperideal of a graded multiplicative hyperring  $R$  and  $x_g \in h(R)$ . We note  $(I : x_g) = \{r \in R \mid r \circ x_g \subseteq I\}$ . Then for all  $x_g \in h(R)$ ,  $(I : x_g)$  is a graded hyperideal of  $R$ .

PROPOSITION 2.3. Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  such that  $P \cap S = \emptyset$ . Then  $P$  is a graded  $S$ -prime hyperideal of  $R$  if and only if  $(P : s)$  is a graded prime hyperideal of  $R$  for some  $s \in S$ .

PROPOSITION 2.4. Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  disjoint with  $S$ . Then the following statements hold:

- (i) If  $J$  is a graded hyperideal of  $R$  such that  $J \cap S \neq \emptyset$  and  $P$  is a graded  $S$ -prime hyperideal of  $R$ , then  $JP$  is a graded  $S$ -prime hyperideal of  $R$ .
- (ii) Let  $R \subseteq R'$  be an extension of  $R$ . If  $Q$  is a graded  $S$ -prime hyperideal of  $R'$ , then  $Q \cap R$  is a graded  $S$ -prime hyperideal of  $R$ .

THEOREM 2.5. Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$ . Let  $f : R \rightarrow R'$  be a homomorphism of graded hyperrings such that

$f(S)$  dose not contain zero. If  $Q$  is a graded  $f(S)$ -prime hyperideal of  $R'$ , then  $f^{-1}(Q)$  is a graded  $S$ -prime hyperideal of  $R$ .

Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $I$  a graded hyperideal of  $R$  with  $I \cap S = \emptyset$ . Let  $s \in S$ , we denote by  $\bar{s} = s + I$  the equivalence class of  $s$  in the quotient graded hyperring  $R/I$ . Let  $\bar{S} = \{s + I \mid s \in S\}$ . Then  $\bar{S}$  is a multiplicatively closed subset of  $R/I$ , because if  $s + I, t + I \in \bar{S}$ , then  $(s + I) \circ (t + I) \cap \bar{S} \neq \emptyset$  because  $s \circ t \cap S \neq \emptyset$ .

**PROPOSITION 2.6.** *Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $I$  a graded hyperideal of  $R$  such that  $I \cap S = \emptyset$ . Let  $P$  be a proper graded hyperideal of  $R$  containing  $I$  such that  $(P/I) \cap \bar{S} = \emptyset$ . Then  $P$  is a graded  $S$ -prime hyperideal of  $R$  if and only if  $P/I$  is a graded  $\bar{S}$ -prime hyperideal of  $R/I$ .*

**THEOREM 2.7.** *Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  such that  $P \cap S = \emptyset$ . Then  $P$  is a graded  $S$ -prime hyperideal of  $R$  if and only if there exists an  $s \in S$  such that for all graded hyperideals  $I, J$  of  $R$ , if  $IJ \subseteq P$ , then  $s \circ I \subseteq P$  or  $s \circ J \subseteq P$ .*

**PROOF.** ( $\Leftarrow$ ) Let  $a_g \circ b_g \subseteq P$  where  $a_g, b_g \in R$ . Thus  $\langle a_g \circ b_g \rangle \subseteq P$ . By [?, **Proposition 2.15**], we have  $\langle a_g \rangle \circ \langle b_g \rangle \subseteq \langle a_g \circ b_g \rangle$ , and so  $\langle a_g \rangle \circ \langle b_g \rangle \subseteq P$ . Thus there exists an  $s \in S$  such that  $s \circ \langle a_g \rangle \subseteq P$  or  $s \circ \langle b_g \rangle \subseteq P$  by hypothesis. Therefore  $s \circ a_g \subseteq P$  or  $s \circ b_g \subseteq P$ , hence  $P$  is graded  $S$ -prime.

( $\Rightarrow$ ) Since  $P$  is graded  $S$ -prime, there exists an  $s \in S$  so that  $a_g \circ b_h \subseteq P$ , implies  $s \circ a_g \subseteq P$  or  $s \circ b_h \subseteq P$  for any  $a_g, b_h \in h(R)$ . Let for all  $t \in S$ , there exist graded hyperideals  $A, B$  of  $R$  with  $AB \subseteq P$ ,  $t \circ A \not\subseteq P$  and  $t \circ B \not\subseteq P$ . Since  $s \in S$ , there exists graded hyperideals  $I, J$  of  $R$  with  $IJ \subseteq P$ ,  $t \circ I \not\subseteq P$  and  $t \circ J \not\subseteq P$ . Thus there exist  $a_g \in I$  and  $b_g \in J$  such that  $s \circ a_g \not\subseteq P$  and  $s \circ b_h \not\subseteq P$  with  $a_g \circ b_h \subseteq IJ \subseteq P$  that it contradicts with hypothesis.  $\square$

**COROLLARY 2.8.** *Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  such that  $P \cap S = \emptyset$ . Then  $P$  is a graded  $S$ -prime hyperideal of  $R$  if and only if there exists an  $s \in S$  such that for all graded hyperideals  $I_1, I_2, \dots, I_n$  of  $R$ , if  $I_1 I_2 \dots I_n \subseteq P$ , then  $s \circ I_i \subseteq P$  for some  $i \in \{1, 2, \dots, n\}$ .*

**PROPOSITION 2.9.** *Let  $R$  be a graded multiplicative hyperring,  $S \subseteq h(R)$  a multiplicatively closed subset of  $R$  and  $P$  a graded hyperideal of  $R$  such that  $P \cap S = \emptyset$ . Then  $P$  is a graded  $S$ -prime hyperideal of  $R$  if and only if there exists an  $s \in S$  such that for all  $a_{g_1}, a_{g_2}, \dots, a_{g_n}$  of  $R$ , if  $a_{g_1} \circ a_{g_2} \circ \dots \circ a_{g_n} \subseteq P$ , then  $s \circ a_{g_i} \subseteq P$  for some  $i \in \{1, 2, \dots, n\}$ .*

**PROPOSITION 2.10.** *Let  $R$  be a graded multiplicative hyperring,  $S \subseteq R$  a multiplicatively closed subset of  $R$  and  $P$  a hyperideal of  $R$  such that  $P \cap S = \emptyset$ . Then if  $I$  is a  $C$ -hyperideal of  $R$  and  $P$  a graded  $S$ -prime hyperideal of  $R$  such that  $I \subseteq P$ , then there exists an  $s \in S$  such that  $s \circ (\text{Grad}(I)) \subseteq P$ .*

**PROOF.** Let  $a \in \text{Grad}(I)$ . Since  $I$  is a  $C$ -hyperideal, we have  $\text{Grad}(I) = D^{gr}(I) = \{r \in R \mid \text{for any } g \in G, r_g^n \subseteq I \text{ for some } n \in \mathbb{N}\}$ . Hence for any  $g \in G$ ,  $a_g^n \subseteq I \subseteq P$  for some  $n \in \mathbb{N}$ . Thus there exists an  $s \in S$  such that  $s \circ a_g \subseteq P$  by Proposition ??, and so  $s \circ (\text{Grad}(I)) \subseteq P$ .  $\square$

### 3. Conclusion

In this article, we introduced the concept of graded  $S$ -prime hyperideals of a graded multiplicative hyperring as a generalization of graded prime hyperideals. In fact, the notion

of graded  $S$ -prime hyperideals different with the notion of graded prime hyperideals and many of the results concerning of graded prime hyperideals are not hold for graded  $S$ -prime hyperideals. Moreover, we investigated the properties and the behaviour of this structure under homogeneous components, graded hyperring homomorphisms, Cartesian product and fundamental relations.

### References

1. Anbarloei M. (2017) *On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring*, *Cogent Mathematics*, **4**, 1–8.
2. Dasgupta U. (2012) *On prime and primary hyperideals of a multiplicative hyperrings*, . Stint. Univ. Al. I. Cuza Iasi, **58**, 19–36.
3. Krasner M. (1983) *A class of hyperrings and hyperfield*, *Intern. J. Math. Math. Sci.*, **6(2)**, 307–312.
4. Procesi R. and Rota R. (1987) *The hyperring spectrum*, *Riv. Mat. Pura Appl.*, **1**, 71–80.
5. Rota R. (1982) *Sugli iperaneli moltiplicativi*, *Rend. Di Math.*, **Series VII**, **4** 711–724.





## Foxby Equivalence relative to $FP_n$ -flat $(n, m)$ -cotorsion modules

Mostafa Amini \*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: dr.mostafa56@pnu.ac.ir

ABSTRACT. Let  $R$  be a ring,  $C$  a (faithfully) semidualizing module, and  $n, m \geq 0$  be integers. We introduce  $C$ - $FP_n$ -flat  $C$ - $(n, m)$ -cotorsion modules, then we obtain some results of homological relationships of this modules.

### 1. Introduction and Preliminaries

Semidualizing modules  $C$  over general associative rings have been extensively studied, beginning with the work of Holm and White in [4]. In 2016, X. Chen and J. Chen, in [2], defined cotorsion modules respect to a semidualizing module  $C$  over commutative rings.

DEFINITION 1.1. (1) An  $R$ -module  $U$  is said to be finitely  $n$ -presented [5] if there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each  $F_i$  is finitely generated and projective;

- (2) A module  $M$  is called  $FP_n$ -flat [1] if  $\text{Tor}_1^R(M, U) = 0$  for every right finitely  $n$ -presented  $R$ -module  $U$ ;
- (3) An  $R$ -module  $M$  is said to be *cotorsion* [3] if  $\text{Ext}_R^1(L, M) = 0$  for every flat  $R$ -module  $L$ ;
- (4) An  $R$ -module  $C$  is considered *semidualizing* [4] if it satisfies the following conditions:
  - (a<sub>1</sub>)  $C$  possesses a degreewise finite projective resolution;
  - (a<sub>2</sub>) The homothety map  $\gamma : R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism;
  - (a<sub>3</sub>)  $\text{Ext}_R^i(C, C) = 0$  for all  $i \geq 1$ ;
- (5) A semidualizing module  $C$  is said to be *faithfully semidualizing* [4] if for every  $R$ -module  $M$ ,  $\text{Hom}_R(C, M) = 0$  implies that  $M = 0$ ;
- (6) An  $R$ -module is considered  $C$ - $FP_n$ -flat [5] if it is isomorphic to  $C \otimes_R Y$  for some  $FP_n$ -flat  $R$ -module  $Y$ ;
- (7) The *Auslander class*  $\mathcal{A}_C(R)$  [4] with respect to  $C$  comprises all  $R$ -modules  $M$  that satisfy the following conditions:

\*Speaker.

- (a)  $\text{Tor}_i^R(C, M) = 0$  for all  $i \geq 1$ ;
  - (b)  $\text{Ext}_R^i(C, C \otimes_R M) = 0$  for all  $i \geq 1$ ;
  - (c) The natural evaluation homomorphism  $\mu_M : M \rightarrow \text{Hom}_R(C, C \otimes_R M)$  is an isomorphism of  $R$ -modules.
- (8) The *Bass class*  $\mathcal{B}_C(R)$  [4] with respect to  $C$  consists of all  $R$ -modules  $N$  satisfying the following conditions:
- (a)  $\text{Ext}_R^i(C, N) = 0$  for all integers  $i \geq 1$ ;
  - (b)  $\text{Tor}_i^R(C, \text{Hom}_R(C, N)) = 0$  for all integers  $i \geq 1$ ;
  - (c) The natural evaluation homomorphism  $\nu_N : C \otimes_R \text{Hom}_R(C, N) \rightarrow N$  is an isomorphism of  $R$ -modules.

## 2. Main results

DEFINITION 2.1. Let  $n, m$  be non-negative integers. Then, an  $R$ -module  $M$  is called  $C$ - $(n, m)$ -cotorsion if  $\text{Ext}_R^1(N, M) = 0$  for every  $N \in \mathcal{X}_C^n(R)_{\leq m}$ , where  $\mathcal{X}_C^n(R)_{\leq m}$  denotes the class of all  $R$ -modules with  $C$ - $FP_n$ -flat dimension less than or equal to  $m$

DEFINITION 2.2. Let  $n, m$  be non-negative integers. An  $R$ -module  $M$  is defined as  $C$ - $FP_n$ -flat  $C$ - $(n, m)$ -cotorsion if it is isomorphic to an  $R$ -module of the form  $C \otimes_R N$ , where  $N$  belongs to the class  $\mathcal{X}^n(R)_{\leq m}$  and to  $\mathcal{Y}^{(n, m)}(R)$ , where  $\mathcal{X}^n(R)_{\leq m}$  and  $\mathcal{Y}^{(n, m)}(R)$  denotes the class of all  $R$ -modules with  $FP_n$ -flat dimension less than or equal to  $m$  and the class of all  $(n, m)$ -cotorsion  $R$ -modules, respectively.

Also,  $\mathcal{X}_C^{\mathcal{Y}^{(n, m)}}(R)$ ,  $\mathcal{Y}^{(n, m)}(R)$  and  $\mathcal{Y}_C^{(n, m)}(R)$  are the class of all  $C$ - $FP_n$ -flat  $C$ - $(n, m)$ -cotorsion,  $(n, m)$ -cotorsion and  $C$ - $(n, m)$ -cotorsion  $R$ -modules.

PROPOSITION 2.3. *The following conditions are equivalent:*

- (1)  $M$  is in  $\mathcal{X}_C^{\mathcal{Y}^{(n, m)}}(R)$ ;
- (2)  $M$  is an element of  $\mathcal{X}_C^n(R)_{\leq m}$  and belongs to  $\mathcal{Y}_C^{(n, m)}(R)$ ;
- (3)  $M$  is in  $\mathcal{B}_C(R)$  and  $\text{Hom}_R(C, M)$  is in  $\mathcal{X}^{\mathcal{Y}^{(n, m)}}(R)$ ;
- (4)  $\text{Hom}_R(C, M)$  is in  $\mathcal{X}^{\mathcal{Y}^{(n, m)}}(R)$ .

PROOF. (1) $\Leftrightarrow$ (2) According to [5, Corollary 3.4], the modules  $N$  and  $N'$  are elements of  $\mathcal{A}_C(R)$  whenever  $N, N' \in \mathcal{X}^n(R)_{\leq m}$ . Consequently, it follows that  $\text{Tor}_i^R(C, N') = 0$  for all  $i \geq 1$ . Let  $M = C \otimes_R N$ , with  $N \in \mathcal{X}^n(R)_{\leq m}$ . Then, by [4, Theorem 6.4], there is an isomorphism:

$$\text{Ext}_R^{i \geq 0}(C \otimes_R N', C \otimes_R N) \cong \text{Ext}_R^{i \geq 0}(N', N).$$

(1) $\Rightarrow$ (3). Let  $M$  be an element of  $\mathcal{X}_C^{\mathcal{Y}^{(n, m)}}(R)$ . Then  $M$  is isomorphic to  $C \otimes_R N$ , where  $N$  is in  $\mathcal{X}^n(R)_{\leq m}$  and  $\mathcal{Y}^{(n, m)}(R)$ ; that is,  $N \in \mathcal{X}^{\mathcal{Y}^{(n, m)}}(R)$ . According to [5, Corollary 3.4], we have  $N \in \mathcal{A}_C(R)$ . Consequently, we obtain

$$\text{Hom}_R(C, M) \cong \text{Hom}_R(C, C \otimes_R N) \cong N \in \mathcal{X}^{\mathcal{Y}^{(n, m)}}(R).$$

Furthermore, it is evident that  $M \in \mathcal{X}_C^{\mathcal{Y}^{(n, m)}}(R) \subseteq \mathcal{X}_C^n(R)_{\leq m}$ . By [5, Proposition 4.2], it follows that  $\mathcal{X}_C^n(R)_{\leq m} \subseteq \mathcal{B}_C(R)$ , and thus  $M \in \mathcal{B}_C(R)$ .

(3) $\Rightarrow$ (1). Let  $M \in \mathcal{B}_C(R)$  and assume that  $\text{Hom}_R(C, M) \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R)$ . Then it follows that  $M \cong C \otimes_R \text{Hom}_R(C, M) \in \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)$ .

(3) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (3). Let  $\text{Hom}_R(C, M) \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R) \subseteq \mathcal{X}^n(R)_{\leq m}$ . Then, it follows that  $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$ . Consequently, we deduce that  $M \in \mathcal{B}_C(R)$ .  $\square$

PROPOSITION 2.4. *The following categories are equivalent:*

$$(1) \quad \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R);$$

$$(2) \quad \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R)_{\leq t} \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)_{\leq t}.$$

PROOF. (a). It is clear.

(b). The case  $t = 0$  is straightforward by part (a). For  $t \geq 1$ , assume  $M \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R)_{\leq t}$ . Then there exists an exact sequence of  $R$ -modules:

$$0 \longrightarrow N_t \longrightarrow N_{t-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0,$$

where each  $N_i \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R) \subseteq \mathcal{A}_C(R)$  for  $0 \leq i \leq t$ . According to [4, Corollary 6.3], both  $M$  and  $\text{coker}(N_i \rightarrow N_{i-1})$  are contained in  $\mathcal{A}_C(R)$ . Applying the functor  $C \otimes_R -$  to this exact sequence yields:

$$0 \longrightarrow C \otimes_R N_t \longrightarrow C \otimes_R N_{t-1} \longrightarrow \cdots \longrightarrow C \otimes_R N_1 \longrightarrow C \otimes_R N_0 \longrightarrow C \otimes_R M \longrightarrow 0,$$

which is an exact sequence of  $R$ -modules where each  $C \otimes_R N_i \in \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)$ . Consequently, it follows that  $C \otimes_R M \in \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)_{\leq t}$ , establishing the result. Conversely, suppose  $M \in \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)_{\leq t}$ . Then there exists an exact sequence:

$$0 \longrightarrow C \otimes_R N_t \longrightarrow C \otimes_R N_{t-1} \longrightarrow \cdots \longrightarrow C \otimes_R N_1 \longrightarrow C \otimes_R N_0 \longrightarrow M \longrightarrow 0,$$

where each  $N_i \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R) \subseteq \mathcal{X}_C^n(R)_{\leq m} \subseteq \mathcal{A}_C(R)$ . We find that each  $C \otimes_R N_i \in \mathcal{B}_C(R)$ . Applying [4, Corollary 6.3] to this sequence produces an exact sequence:

$$0 \longrightarrow \text{Hom}_R(C, C \otimes_R N_t) \longrightarrow \cdots \longrightarrow \text{Hom}_R(C, C \otimes_R N_0) \longrightarrow \text{Hom}_R(C, M) \longrightarrow 0,$$

with each  $N_i \cong \text{Hom}_R(C, C \otimes_R N_i)$ . This leads to the exact sequence:

$$0 \longrightarrow N_t \longrightarrow N_{t-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow \text{Hom}_R(C, M) \longrightarrow 0,$$

which confirms that  $\text{Hom}_R(C, M) \in \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R)_{\leq t}$ .  $\square$

**THEOREM 2.5.** (Foxby Equivalence) *Let  $C$  be a semidualizing module. There are equivalences of categories:*

$$\begin{array}{ccc}
 \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R) \\
 \downarrow & & \downarrow \\
 \mathcal{X}^{\mathcal{Y}^{(n,m)}}(R)_{\leq t} & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{X}_C^{\mathcal{Y}^{(n,m)}}(R)_{\leq t} \\
 \downarrow & & \downarrow \\
 \mathcal{A}_C(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{B}_C(R) \\
 \uparrow & & \uparrow \\
 \mathcal{Y}^{(n,m)}(R) \cap \mathcal{A}_C(R) & \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \sim \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} & \mathcal{Y}_C^{(n,m)}(R) \cap \mathcal{B}_C(R)
 \end{array}$$

PROOF. This derives from Proposition 2.4. □

### 3. Conclusion

We have expanded several fundamental homological concepts to incorporate findings related to semidualizing modules. This paper focuses on the  $FP_n$ -flat  $(n, m)$ -cotorsion modules under a (faithfully) semidualizing module  $C$ .

### Acknowledgement

Acknowledgements could be placed at the end of the text but before the references.

### References

1. Bravo, D. Perez, M.A. (2017) Finiteness conditions and cotorsion pairs, *J. Pure Appl. Algebra* ., **221**(6), 1249-1267.
2. Chen, X. Chen, J. (2016) Cotorsion dimensions relative to semidualizing modules, *J. Algebra Appl.*, **15**(6), 1-14.
3. Enochs, E.E. (1984) Flat covers and flat cotorsion modules, *Proc. Amer. Math. Soc.*, **92**(2), 179-184.
4. Holm, H. White, D. (2007) Foxby equivalence over associative rings, *J. Math. Kyoto Univ.*, **47**(4), 781-808.
5. Wu, W. Gao, Z. (2022)  $FP_n$ -injective and  $FP_n$ -flat modules with respect to a semidualizing bimodule, *Commun. Algebra.*, **50**(2), 583-599.



## ON C-EXTENDING MODULES

Amirmohammad Momeni Kohestani<sup>1,\*</sup>, Yahya Talebi<sup>2</sup>

<sup>1</sup>Department of Mathematics, University of Mazandaran, Babolsar, Iran.

Email: a.momeni04@umail.umz.ac.ir

<sup>2</sup>Department of Mathematics, University of Mazandaran, Babolsar, Iran.

Email: talebi@umz.ac.ir

---

**ABSTRACT.** Let  $W$  be a unitary left  $A$ -module, where  $A$  is an arbitrary ring with identity. In this note, we introduce a new generalization of essential submodules, namely the  $C$ -essential submodule. A submodule  $N$  of a module  $W$  is called  $C$ -essential if every submodule which properly contains  $N$  is essential in  $W$ . We use this concept to investigate a new generalization of extending modules. A module  $W$  is called a  $C$ -extending module, if every submodule is  $C$ -essential in a direct summand of  $W$ . Some interesting results regarding  $C$ -essential submodules and  $C$ -extending modules are obtained in this paper.

**Keywords:**  $C$ -Essential, Extending modules,  $C$ -extending modules

**AMS Mathematics Subject Classification [2020]:** 16D10, 16D15, 16D80

---

### 1. Introduction

In 2025, M. Akbari Gelvardi and Y. Talebi [2] introduced a new generalization of small submodules. The authors named  $C$ -small submodule for this generalization. A module  $W$  is called  $C$ -small, if every proper submodule of  $W$  is small in  $W$ . We introduce the dual definition  $C$ -essential submodules. Also we investigate some results about  $C$ -essential submodules and define a new generalization of extending modules which we call  $C$ -extending module. Any undefined terminology can be found in [1, 3, 4]

### 2. C-ESSENTIAL SUBMODULES

In this section, we start with our definition and examine some examples and preliminary results of this concept. We obtain some interesting results related to  $C$ -essential submodules and we use them in another sections of this paper for our main definitions.

**DEFINITION 2.1.** Let  $W$  be an  $R$ -module and  $N \leq W$ .  $N$  is called  $C$ -essential in  $W$ , if any submodule  $K$  of  $W$  with  $N \subset K$  (properly containing  $N$ ) is essential in  $W$ . Equivalently any submodule which properly containing  $N$  is essential in  $W$ .

---

\*Speaker.

If  $N$  is essential in  $W$ , then  $N$  is C-essential in  $W$ . To see this consider an arbitrary submodule  $K$  which  $N \subset K$ . Since  $N$  is essential submodule, so it has nonzero intersection with all nonzero submodules and since  $N \subset K$ , we conclude that  $K$  also has nonzero intersection with all nonzero submodules. So  $K$  is essential in  $W$ . Inasmuch as  $K$  is an arbitrary submodule which containing  $N$ , so  $N$  is C-essential. The converse may not be true. For this, suppose that  $N \subset K \leq W$  and  $N$  is C-essential in  $W$ . So  $K$  is essential in  $W$ . It is clear that we can not conclude  $N$  is essential in  $W$ . For instance consider  $\mathbb{Z}_6$  as  $\mathbb{Z}$ -module. The submodules are  $\{ \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle \}$ . We know that  $\langle 1 \rangle = \mathbb{Z}_6$  and is essential in  $\mathbb{Z}_6$ . The only submodule which properly containing  $\langle 2 \rangle$  is  $\langle 1 \rangle$  and it is essential. So  $\langle 2 \rangle$  is C-essential. But  $\langle 2 \rangle$  is not essential, because  $\langle 2 \rangle \cap \langle 3 \rangle = 0$  and  $\langle 3 \rangle \neq 0$ . So the converse is not always true.

EXAMPLE 2.2. (1) Let  $W$  be uniform module. Every proper submodule of  $W$  is C-essential. Because every proper submodule of  $W$  is essential and if  $K$  and  $L$  are submodules such that  $K \subset L$ , then  $L$  is uniform and by definition we can conclude that  $K$  is C-essential. For instance for a prime number  $p$ , the  $\mathbb{Z}_{p^\infty}$  as  $\mathbb{Z}$ -module is uniform. So every submodule of  $\mathbb{Z}_{p^\infty}$  as  $\mathbb{Z}$ -module is C-essential.

(2) Let  $W = \mathbb{Z}_{18}$  as  $\mathbb{Z}$ -module. The submodules of  $W$  are  $\{0, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle, \langle 6 \rangle, \langle 8 \rangle, \langle 9 \rangle, \langle 10 \rangle, \langle 12 \rangle, \langle 14 \rangle, \langle 16 \rangle, \mathbb{Z}_{18}\}$ . Consider the submodule  $\langle 9 \rangle$ . This submodule is C-essential submodule but not essential submodule. It is clear that there are some submodules which has nonzero intersection with  $\langle 9 \rangle$ , so it is not essential submodule. But the submodules which containing  $\langle 9 \rangle$  are  $\{\langle 3 \rangle, \mathbb{Z}_{18}\}$  and it is obvious that they are essential in  $W$ . So  $\langle 9 \rangle$  is C-essential.

REMARK 2.3. Let  $W$  be an  $A$ -module. If  $0 \leq_{ce} W$ , then  $W$  is uniform module. Since every submodule of  $W$  contains zero element and by definition must be essential in  $W$ .

PROPOSITION 2.4. Let  $W$  be an  $A$ -module and  $N, K \leq_{ce} W$ . Then  $N + K \leq_{ce} W$ .

PROPOSITION 2.5. Let  $W$  be an  $A$ -module and  $N, K \leq W$  such that  $N \cap K \leq_{ce} W$ . Then  $N, K \leq_{ce} W$ .

PROPOSITION 2.6. Let  $W$  and  $N$  be two arbitrary  $A$ -modules and  $f : W \rightarrow N$  be an  $A$ -module monomorphism. If  $H \leq_{ce} N$  and  $Im f \subset H$ , then  $f^{-1}(H) \leq_{ce} W$ .

THEOREM 2.7. Let  $W$  be an  $A$ -module and  $L, K \leq W$ . The the following hold.

- (1) If  $K \leq_{ce} W$ , then for every submodule  $T$  with  $K \subset T$ ,  $T \leq_{ce} W$ .
- (2) If  $K \leq_{ce} L$  and  $L \leq_{ce} W$  and for any submodule  $G$  which containing  $K$ ,  $G \leq L$  or  $L \leq G$ , then  $K \leq_{ce} W$ .

COROLLARY 2.8. Let  $W$  and  $N$  be  $R$ -modules. If  $N \leq_{ce} E(W)$ , then  $E(W) = E(N)$ .

PROPOSITION 2.9. Every maximal submodule of an  $A$ -module is C-essential submodule.

LEMMA 2.10. A submodule  $N$  of an  $A$ -module  $W$  is C-essential in  $W$  if and only if for each  $0 \neq x \in W$  and  $N \subset K \leq W$  there exists an  $r \in R$  such that  $0 \neq rx \in K$ .

THEOREM 2.11. Let  $K_1 \leq W_1 \leq W$  and  $K_2 \leq W_2 \leq W$  and  $W = W_1 \oplus W_2$ . Then  $K_1 \oplus K_2 \leq_{ce} W_1 \oplus W_2$  if and only if  $K_1 \leq_{ce} W_1$  and  $K_2 \leq_{ce} W_2$ .

COROLLARY 2.12. Let  $\{K_\alpha\}$  and  $\{N_\alpha\}$  be two independent family of submodules of a module  $W$ . If  $K_\alpha \leq_{ce} N_\alpha$  for each  $\alpha$ , then  $\bigoplus K_\alpha \leq_{ce} \bigoplus N_\alpha$ .

THEOREM 2.13. Let  $W$  be an  $A$ -module and  $L \leq N \leq W$ . If  $L \leq_e N$  and  $N/L \leq_{ce} W/L$ , then  $N \leq_{ce} W$ .

### 3. C-EXTENDING MODULES

In this section, we use the concept C-essential submodules for introduce a new generalization of extending modules. Some examples and related results are investigate and present in this section.

**DEFINITION 3.1.** Let  $W$  be an  $A$ -module.  $W$  is called *C-extending* module, if every submodule of  $W$  is C-essential in a direct summand of  $W$ . More generally  $W$  is called *uniform-C-extending*, if every uniform submodule is C-essential in a direct summand of  $W$ .

**LEMMA 3.2.** Any direct summand of a (uniform-) C-extending module is also (uniform-) C-extending module.

**EXAMPLE 3.3.** (1) It is clear that every extending module is C-extending module. So semisimple modules, injective modules and uniform modules are C-extending module. Also quasi-continuous and continuous modules has C-extending property, since they have extending property.

(2) Take a prime number  $p$ . We know that the  $\mathbb{Z}$ -module,  $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$  is not extending module [?]. So for instance the  $\mathbb{Z}$ -module,  $\mathbb{Z}_2 \oplus \mathbb{Z}_8$  is not extending. It is clear that every submodule of  $\mathbb{Z}$ -module,  $\mathbb{Z}_2 \oplus \mathbb{Z}_8$  is C-essential in a direct summand of  $\mathbb{Z}$ -module,  $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ . So  $\mathbb{Z}_2 \oplus \mathbb{Z}_8$  is C-extending but not extending module.

**COROLLARY 3.4.** Let  $W$  be a C-extending  $A$ -module and  $B \leq^\oplus W$ . Then  $W/B$  is C-extending module.

**DEFINITION 3.5.** A submodule  $N$  of a module  $W$  is called *C-closed*, if  $N$  has no C-essential proper extension in  $W$ . Now we can present an important proposition for C-extending modules.

**REMARK 3.6.** Note that the closed submodules are not C-closed in general. For example suppose that  $W = \mathbb{Z}_{18}$  as  $\mathbb{Z}$ -module. The set of all submodules of  $\mathbb{Z}_{18}$  is  $\{0, \langle 2 \rangle, \langle 4 \rangle, \langle 6 \rangle, \langle 8 \rangle, \langle 9 \rangle, \langle 10 \rangle, \langle 12 \rangle, \langle 14 \rangle, \langle 16 \rangle, \mathbb{Z}_{18}\}$ . Consider the submodule  $\langle 9 \rangle$ . Since  $\langle 9 \rangle \supset \langle 3 \rangle$  and  $\langle 9 \rangle \cap \langle 6 \rangle = 0$ ,  $\langle 6 \rangle \supset \langle 3 \rangle$ , so we conclude that  $\langle 9 \rangle$  is not essential in  $\langle 3 \rangle$  and since the only submodule which containing  $\langle 9 \rangle$  is  $\langle 3i \rangle$ , so it is clear that  $\langle 9 \rangle$  has no proper essential extension in  $\mathbb{Z}_{18}$ . It means that  $\langle 9 \rangle$  is closed submodule. But  $\langle 9 \rangle$  is not C-closed. Because  $\langle 9 \rangle$  is C-essential in  $\langle 3 \rangle$ . So we find an example which is closed but not C-closed.

**PROPOSITION 3.7.** A module  $W$  is C-extending every C-closed submodule of  $W$  is a direct summand of  $W$ . Conversely is true if every closed submodule of  $W$  is maximal, then  $W$  is C-extending.

**LEMMA 3.8.** Let  $W_1$  and  $W_2$  be two  $A$ -modules. Then  $W_1$  is  $W_2$ -injective if and only if for each submodule  $N \leq W$  which  $N \cap W_1 = 0$ , there exists a submodule  $W'$  of  $W$  such that  $W = W_1 \oplus W'$  and  $N \subset W'$ .

**THEOREM 3.9.** Let  $W = W_1 \oplus W_2$  which  $W_i$ 's are C-extending modules. Then  $W$  is C-extending if and only if for all C-closed submodule  $K \leq W$ ,  $K$  is a direct summand of  $W$  such that  $K \cap W_1 = 0$  or  $K \cap W_2 = 0$ .

**EXAMPLE 3.10.** Suppose that  $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$  and  $\alpha_{ij}$  be the matrix in  $A$  which is 1 in  $(i,j)$ -position and elsewhere is 0. Then  $A_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} =$

$\alpha_{11}A_A \oplus \alpha_{22}A_A$  is a direct sum of C-extending modules (Sine both of them are uniform modules, so they are extending and hence C-extending). But  $A_A$  is not C-extending. Because the submodule  $\begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 2\mathbb{Z} \end{pmatrix}$  is not C-essential in a direct summand.

**THEOREM 3.11.** *Let  $W = W_1 \oplus W_2 \oplus W_3 \oplus \dots \oplus W_n$  be the finite direct sum of relatively injective modules. Then  $W$  is C-extending if and only if  $W_i$  is C-extending for all  $i = 1, 2, 3, \dots, n$ .*

**THEOREM 3.12.** *Let  $W$  be an  $A$ -module.  $W$  is C-extending module, if and only if there exists a submodule  $W'$  of  $W$ , such that  $W = Z_2(W) \oplus W'$ ,  $Z_2(W)$  is  $W'$ -injective and both of  $W'$  and  $Z_2(W)$  are C-extending.*

**DEFINITION 3.13.** A module  $W$  is *C-uniform module*, if every submodule of  $W$  is C-essential in  $W$ . It is clear that every uniform module is C-uniform. The next proposition is obtained immediately.

**PROPOSITION 3.14.** *Let  $W$  be an indecomposable module.  $W$  is C-extending if and only if  $W$  is C-uniform.*

**THEOREM 3.15.** *Let  $W$  be a C-extending module, which has finite uniform dimension. Then every submodule of  $W$  has ACC on C-closed submodules.*

**THEOREM 3.16.** *Let  $W$  be a  $C_{11}$ -module and  $U$  be a submodule of  $W$ . If for every direct summand  $V$  of  $W$ ,  $U \cap V$  is C-essential in  $U$ , then  $U$  is a  $C_{11}$ -module.*

**DEFINITION 3.17.** We call that an  $A$ -module  $W$  is C-extending, if for every index set  $I$  and every direct sum  $\bigoplus_{i \in I} X_i$  of submodules  $X_i$  of  $W$ , there exists a family  $\{W_i | i \in I\}$ , such that  $\bigoplus_{i \in I} W_i$  is a direct summand of  $W$  and for each  $i \in I$ , we have  $X_i$  is C-essential in  $W_i$ . If the index set has the cardinality  $n$ , we say that  $W$  is *n-C-extending*. If the index set is finite, we say that  $W$  has *finite C-extending property*.

**LEMMA 3.18.** *Let  $W$  be an arbitrary  $A$ -module. Then*  
 (1)  *$W$  is 1-C-extending if and only if  $W$  is C-extending.*  
 (2)  *$W$  is finite C-extending if and only if  $W$  is 2-C-extending if and only if  $W$  has C-extending property and  $C_3$ .*  
 (3)  *$W$  has C-extending property if and only if  $W$  has finite C-extending property and every local summand of  $W$  is summand.*

## References

- [1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlog, NewYork, 1992.
- [2] M. Akbari Gelvardi and Y. Talebi, On C-small and C-supplemented submodules, Palestine Journal of Mathematics, 14(1)(2025) , 133–141.
- [3] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules, Pitman Research Notes in Mathematics Series, Longman, Harlow, 1994.
- [4] A. Tercan and C.C. Yucel, Modules Theory, extending modules and generalizations, Frontiers in Mathematics, Birkhauser/Springer, (2016).



## Graded S-superfluous and Graded S-essential submodules

Saeed Rajaei\*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: saeed.rajaee@pnu.ac.ir

**ABSTRACT.** In this paper we introduce graded S-superfluous and graded S-essential submodules. In addition, some more results concerning graded S-comultiplication modules are proved.

**Keywords:** Graded S-superfluous submodule, Graded S-essential submodule, S-comultiplication module

**AMS Mathematics Subject Classification [2020]:** 3C13, 13A15

### 1. Introduction

Throughout this article,  $R$  is a commutative ring with  $1 \neq 0$  and  $M$  is a non-zero unital  $R$ -module. We denote the set of all units in  $R$  by  $U(R)$  and the set of all submodules of  $M$  by  $L(M)$ , and also  $L^*(M) = L(M) \setminus \{0, M\}$ . Graded rings play a central role in algebraic geometry and commutative algebra. The notion of graded multiplication modules was studied by many authors, see for example [5, 9]. The notion of graded comultiplication modules which are the dual notion of graded multiplication modules was introduced and studied by Ansari-Toroghy and Farshadifar, see [3, 10].

Let  $G$  be an abelian group with identity  $e$  and  $R$  be a commutative ring with unity  $1_R$ . Then,  $R$  is named a  $G$ -graded ring if there exist additive subgroups  $R_g$  of  $R$  indexed by elements  $g \in G$  such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ . If  $R_g R_h = R_{gh}$ , the ring is named *strongly graded ring*. In addition,  $\text{supp}(R) = \{g \in G : R_g \neq 0\}$ . An element  $x$  of  $R$  has a unique decomposition as  $x = \sum_{g \in G} x_g$  for all  $g \in G$ . Also, if an element of  $R$  belongs to  $h(R) = \bigcup_{g \in G} R_g$ , then it is called *homogeneous* and any  $x_g \in R_g$  is said to have degree  $g$ . Moreover,  $R_e$  is a subring of  $R$  and  $1_R \in R_e$ . An ideal  $I$  of a  $G$ -graded ring is named a graded ideal of  $R$  whenever  $I = \bigoplus_{g \in G} (I \cap R_g)$ . In addition,  $R/I$  is a  $G$ -graded ring with  $g$ -component of the form  $(R/I)_g = (R_g + I)/I$  for all  $g \in G$ .

Consider  $R$  is a  $G$ -graded ring and  $M$  is an  $R$ -module,  $M$  is named a *graded  $R$ -module* if there exist a family of subgroups  $\{M_g\}_{g \in G}$  of  $M$  such that  $M = \bigoplus_{g \in G} M_g$  (as abelian groups) and  $R_g M_h \subseteq M_{gh}$  for all  $g, h \in G$ . Moreover,  $M$  is named *strongly graded  $R$ -module* whenever  $R_g M_h = M_{gh}$ . Also,  $\text{supp}(M) = \{g \in G : M_g \neq 0\}$ . Here  $R_g M_h$  denotes the additive subgroups of  $M$  consisting of all finite sums of elements  $r_g s_h$  with

\*Speaker.

$r_g \in R_g$  and  $s_h \in M_h$ . Also,  $h(M) = \bigoplus_{g \in G} M_g$ . Every element of  $M$  belongs to  $h(M)$  is named *homogeneous* and any element  $x_g \in M_g$  is said to have degree  $g$ . Clearly,  $M_g$  is an  $R_e$ -submodule of  $M$  for all  $g \in G$ .

In [12], Eda Yıldız et al. introduced the  $S$ -comultiplication modules which are the dual notion of  $S$ -multiplication modules. Let  $M$  be an  $R$ -module and  $S \subseteq R$  be a m.c.s of  $R$ .  $M$  is named an  $S$ -comultiplication module if for each submodule  $N$  of  $M$ , there exist an  $s \in S$  and an ideal  $I$  of  $R$  such that  $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ . In particular, a ring  $R$  is named an  $S$ -comultiplication ring if it is an  $S$ -comultiplication  $R$ -module. Every  $R$ -module  $M$  with  $\text{ann}(M) \cap S = \emptyset$  is trivially an  $S$ -comultiplication module. Every comultiplication module is also an  $S$ -comultiplication module. The converse is true provided that  $S \subseteq U(R)$ , see [12, Example 3].

An  $S$ -comultiplication module is not necessarily comultiplication. Consider  $M = \mathbb{Z}$  as a  $\mathbb{Z}$ -module and  $S = \mathbb{Z} - \{0\}$  and the submodule  $N = m\mathbb{Z}$  with  $m \neq 0, \pm 1$ . Then,  $(0 :_{\mathbb{Z}} \text{ann}(m\mathbb{Z})) = \mathbb{Z} \neq m\mathbb{Z}$ , hence  $M$  is not a comultiplication module. Let  $K = k\mathbb{Z}$  be a submodule of  $M$  for some  $k \in \mathbb{Z}$ . For  $k = 0$ , take  $s = 1$ , and  $s(0 :_{\mathbb{Z}} \text{ann}(K)) = (0) = k\mathbb{Z}$ . For  $k \neq 0$ , take  $s = k$ , then  $s(0 :_{\mathbb{Z}} \text{ann}(K)) \subseteq k\mathbb{Z} = K \subseteq (0 :_{\mathbb{Z}} \text{ann}(K))$ . Hence  $M$  is an  $S$ -comultiplication module.

A submodule  $N$  of a  $G$ -graded  $R$ -module  $M$  is named a *graded submodule* of  $M$  if  $N = \bigoplus_{g \in G} N_g$  where  $N_g = N \cap M_g$  for all  $g \in G$  and  $N_g$  is named the  $g$ -component of  $N$  for each  $g \in G$ . Also,  $M/N$  becomes a  $G$ -graded module with  $g$ -component  $(M/N)_g = (M_g + N)/N$  for each  $g \in G$ . Let  $R$  be a  $G$ -graded ring and  $M, T$  be graded  $R$ -modules. Then, an  $R$ -homomorphism  $f : M \rightarrow T$  is named a *graded  $R$ -homomorphism* if for all  $m, n \in M$ ; (i)  $f(m + n) = f(m) + f(n)$ ; (ii)  $f(rm) = rf(m)$  for any  $r \in R$  and  $m \in M$ ; (iii)  $f(M_g) \subseteq T_g$  for all  $g \in G$ . Consider  $R$  as a  $G$ -graded ring. A graded  $R$ -module  $M$  is named a *graded comultiplication module* if for every graded submodule  $N$  of  $M$ , there exists a graded ideal  $I$  of  $R$  such that  $N = \text{ann}_M(I)$ .

Recently, in [11], Sevim et al. introduced the notion of  $S$ -prime submodule which is a generalization of prime submodule and used them to characterize certain class of rings/modules such as prime submodules, simple modules, torsion free modules,  $S$ -Noetherian modules and etc.

An  $R$ -module  $M$  satisfies the  *$S$ -double annihilator condition* ( $S$ -DAC for short) if for each ideal  $I$  of  $R$  there exists an  $s \in S$  such that  $s \cdot \text{ann}_R((0 :_M I)) \subseteq I$ , see [7, Definition 5]. Also,  $M$  is named an  *$S$ -strong comultiplication module* if  $M$  is an  $S$ -comultiplication  $R$ -module which satisfies the  $S$ -DAC, see [7, Definition 6].

## 2. Graded $S$ -superfluous and graded $S$ -essential submodules

Remember that a proper graded submodule  $P$  of  $M$  is called *graded prime*, if  $rm \in P$  where  $r \in h(R)$  and  $m \in h(M)$  implies that  $m \in P$  or  $r \in (P : M)$ . Let  $P$  be a graded  $R$ -submodule of  $M$ , then  $(P :_R M) = \{r \in R : rM \subseteq P\}$  is a graded ideal of  $R$ , see [4].

**DEFINITION 2.1.** Let  $R$  be a  $G$ -graded ring,  $M$  a graded  $R$ -module and  $S \subseteq h(R)$  a m.c.s. of  $R$ . Then  $M$  is named a *graded  $S$ -comultiplication module* if for each graded submodule  $N$  of  $M$ , there exists  $x \in S$  and a graded ideal  $I$  of  $R$  such that  $x(0 :_M I) \subseteq N \subseteq (0 :_M I)$ .

A ring  $R$  is named a *graded  $S$ -comultiplication ring* if it is a graded  $S$ -comultiplication module as an  $R$ -module. Clearly, every graded  $R$ -module  $M$  with  $\text{ann}_R(M) \cap S \neq \emptyset$  is a graded  $S$ -comultiplication module. In addition, every graded comultiplication module is also a graded  $S$ -comultiplication module.

EXAMPLE 2.2. Let  $R = \mathbb{Z}$ ,  $G = \mathbb{Z}_2$ . Take,  $R_0 = \mathbb{Z}$  and  $R_1 = \{0\}$ . Then,  $R$  is a  $G$ -graded ring. Assume that  $M = \mathbb{Z}[i] = \bigoplus_{g \in \mathbb{Z}_2} M_g$  is a graded  $\mathbb{Z}[i]$ -module where  $M_0 = \mathbb{Z}$  and  $M_1 = i\mathbb{Z}$ . Let  $S = \mathbb{Z} - \{0\} \subseteq h(\mathbb{Z})$  and consider the graded submodule  $N = m\mathbb{Z}$  where  $m \neq 0, \pm 1$ . Then,  $M$  is not graded comultiplication module, whereas  $M$  is a graded  $S$ -comultiplication module.

DEFINITION 2.3. Let  $R$  be a  $G$ -graded ring,  $M$  a graded  $R$ -module and  $S \subseteq h(R)$  a m.c.s. of  $R$ .

- (i) We say that a graded submodule  $N$  of  $M$  is *graded  $S$ -superfluous* whenever for every graded submodule  $L$  of  $M$  and  $s \in h(R)$ ,  $sM \subseteq N + L$  implies that there exists an  $t \in S \subseteq h(R)$  such that  $tM \subseteq L$ . Equivalently,  $sM_g \subseteq N_g + L_g$  results that there exists an  $t \in h(R)$  such that  $tM_g \subseteq L_g$  for all  $g \in G$ .
- (ii) We say that a graded submodule  $N$  of  $M$  is a *graded  $S$ -essential* whenever for every graded submodule  $L$  of  $M$  the equality  $N \cap L = 0$  implies that there exists an  $s \in h(R)$  such that  $sL = 0$ . Equivalently,  $N_g \cap L_g = 0$  results that there exists an  $s \in S \subseteq h(R)$  such that  $sL_g = 0$  for all  $g \in G$ .

DEFINITION 2.4. Let  $R$  be a  $G$ -graded ring,  $M$  a graded  $R$ -module and  $S \subseteq h(R)$  a m.c.s. of  $R$ .

- (i) The *graded  $S$ -Jacobson radical* of  $M$ , denoted by  $J_S^{gr}(M)$  is defined as the sum of all graded  $S$ -superfluous submodules of  $M$ . Otherwise, if there exist no graded  $S$ -superfluous submodule of  $M$ , we define  $J_S^{gr}(M) = M$ .
- (ii) The *graded  $S$ -socle* of  $M$ , denoted by  $\text{Soc}_S^{gr}(M)$  is defined as the intersection of all graded  $S$ -essential submodules of  $M$  if they exist. Otherwise, we define  $\text{Soc}_S^{gr}(M) = (0)$ .

THEOREM 2.5. Let  $R$  be a  $G$ -graded ring,  $S \subseteq h(R)$  a m.c.s. of  $R$  and  $M$  be a faithful graded  $S$ -strong comultiplication  $R$ -module. If  $N$  is a graded  $S$ -superfluous submodule of  $M$ , then there exists a graded  $S$ -essential ideal  $I$  of  $R$  and an  $t \in h(R)$  such that  $t(0 :_M I) \subseteq N \subseteq (0 :_M I)$ . The converse is true if  $S \subseteq U(R)$ .

PROOF. Since  $N$  is a graded  $S$ -superfluous submodule of  $M$  and  $M$  is a graded  $S$ -comultiplication module there exists a graded ideal  $I$  of  $R$  and an  $t \in h(R)$  such that  $t(0 :_M I) \subseteq N \subseteq (0 :_M I)$ . Assume that  $I_g \cap J_g = 0$  for some graded ideal  $J$  of  $R$  and all  $g \in G$ . By virtue of [7, Lemma 2.16 (b)] there exists an  $s \in S \subseteq h(R)$  such that for all  $g \in G$

$$\begin{aligned} stM_g &= st(0 :_M I_g \cap J_g) \subseteq t(0 :_M I_g) + t(0 :_M J_g) \\ &\subseteq t(0 :_M I_g) + (0 :_M J_g) \subseteq N_g + (0 :_M J_g) \end{aligned}$$

Take  $s' = st \in S$ . Since  $N$  is a graded  $S$ -superfluous submodule of  $M$ , hence  $s'M_g \subseteq N_g + (0 :_M J_g)$  results that there exists an  $s'' \in h(R)$  such that  $s''M_g \subseteq (0 :_M J_g)$ . Therefore  $s''J_g \subseteq \text{ann}_R(M_g) = 0$  for all  $g \in G$ . Hence  $s''J = 0$  for some  $s'' \in S \subseteq h(R)$  and so  $I$  is a graded  $S$ -essential ideal of  $R$ . The proof of converse is straightforward.  $\square$

DEFINITION 2.6. Let  $R$  be a  $G$ -graded ring,  $M$  a graded  $R$ -module and  $N$  be a graded submodule of  $M$ . Then,  $M$  is named a *graded torsion-free  $R$ -module* if whenever  $r \in h(R)$  and  $m \in h(M)$  with  $rm_g = 0$ , then either  $m_g = 0$  or  $r_g = 0$  for all  $g \in G$ . Equivalently,

$$T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in h(R)\} = 0$$

In addition,  $M$  is named a *graded torsion  $R$ -module* if  $T(M) = M$ .

**THEOREM 2.7.** *Let  $R$  be a  $G$ -graded ring,  $S \subseteq h(R)$  a m.c.s. of  $R$ ,  $M$  a graded torsion-free  $S$ -strong comultiplication  $R$ -module and  $N$  be a graded submodule of  $M$ . Then  $N$  is a graded  $S$ -essential submodule of  $M$  if and only if there exists a graded  $S$ -superfluous ideal  $I$  of  $R$  and an  $s \in h(R)$  such that  $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ .*

**PROOF.** Assume that  $N$  is a graded  $S$ -essential submodule of  $M$ . Since  $M$  is an  $S$ -comultiplication module, there exists an ideal  $I$  of  $R$  and an  $s \in S \subseteq h(R)$  such that  $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ . We show that  $I$  is a graded  $S$ -superfluous ideal of  $R$ . Assume that  $tR_g \subseteq I_g + J_g$  for some ideal  $J$  of  $R$  and an  $t \in S$  and all  $g \in G$ , then

$$\begin{aligned} N_g \cap (0 :_M J_g) &\subseteq (0 :_M I_g) \cap (0 :_M J_g) = (0 :_M I_g + J_g) \\ &\subseteq (0 :_M tR_g) = 0. \end{aligned}$$

By assumption, there exists an  $t' \in S$  such that  $t'(0 :_M J_g) = 0$  and so  $t' \in \text{ann}_R((0 :_M J_g))$ . Since  $M$  satisfies the S-DAC there exists an  $t'' \in S$  such that  $t't'' \in t''\text{ann}_R((0 :_M J_g)) \subseteq J_g$ . Take,  $x = t't'' \in S \subseteq h(R)$ , then  $xR_g \subseteq J_g$  for all  $g \in G$ , as we claimed. The proof of converse is straightforward.  $\square$

**COROLLARY 2.8.** *Let  $R$  be a  $G$ -graded ring,  $S \subseteq h(R)$  a m.c.s. of  $R$ ,  $M$  a graded torsion-free  $S$ -strong comultiplication  $R$ -module. Then  $\text{Soc}_S^{gr}(M) = (0 :_M J_S^{gr}(R))$ .*

### 3. Conclusion

We introduced graded  $S$ -superfluous and graded  $S$ -essential submodules. Also, we investigated some results concerning graded  $S$ -comultiplication modules.

### Acknowledgement

The author is grateful to the referee for helpful suggestions which have resulted in an improvement to the article.

### References

1. Anderson, D. D., Arabaci, T., Tekir, Ü. and Koç, S. (2020) *On  $S$ -multiplication modules*, Commun. in Algebra, (48) **8**, 3398-3407.
2. Anderson, D. D. and Dumitrescu, T. (2002)  *$S$ -Noetherian rings*, Commun. in Algebra, (30) **9**, 4407-4416.
3. Ansari-Toroghy, H. and Farshadifar, F. (2011), *Graded comultiplication modules*, Chiang Mai J. Sci., 38 (3) 311-320.
4. Ebrahimi Atani, S. (2006) *On graded prime submodules*, Chiang Mai J. Sci., **33**, 3-7.
5. Escoriza, J. and Torrecillas, B. (1999) *Multiplication rings and graded rings*. Commun. in Algebra, 27 (12), 6213- 6232.
6. Farshadifar, F. (2020) *A generalization of pure submodules*, J. Algebra Relat. Topics, (8) **2**, 1-8.
7. Farshadifar, F. (2023)  *$S$ -copure submodules of a module*, Miskolc Math. Notes, 24 (1), 153-163.
8. Gilmer, R. (1992) *Multiplicative Ideal Theory*, Queens Papers in Pure and Applied Mathematics, **90**, Kingston, Canada: Queens University.
9. Ghiasvand, P. and Farzalipour, F. (2009) *Some properties of graded multiplication modules*, Far East J. Math. Sci., 34 (3), 341-352.
10. Rajaei, S. (2022)  *$S$ -small and  $S$ -essential submodules*, J. Algebra Relat. Topics 10 (1), pp 1-10.
11. Sevim, E. S., Arabaci, T., Tekir, Ü. and Koç, S. (2019) *On  $S$ -prime submodules*, Turk. J. Math., (43) **2**, 1036-1046.
12. Yildiz, E., Tekir, Ü. and Koç, S. (2021) *On  $S$ -comultiplication modules*, Turk. J. Math., **45**, 1-13.



## A subspace of homogeneous preserving star operations

Parviz Sahandi

Department of Pure Mathematics, Faculty of Mathematical sciences, University of Tabriz,  
Tabriz, 51666-15648, Iran.

Email: sahandi@ipm.ir, sahandi@tabrizu.ac.ir

**ABSTRACT.** Let  $\Gamma$  be a torsionless commutative cancellative monoid,  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a  $\Gamma$ -graded integral domain. In this note we show that each homogeneous star operation  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  of  $R$ , is the restriction of a (classical) star operation  $\mathfrak{c}(\star) : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$  of  $R$ , that is  $\mathfrak{c}(\star)|_{\mathbf{HF}(R)} = \star$ . We also show that the set  $\text{HStar}_f(R)$  of homogeneous star operations of finite type on  $R$ , endowed with the Zariski topology, is a spectral space.

**Keywords:** graded integral domain, homogeneous preserving star operation, star operation, spectral space

**AMS Mathematics Subject Classification [2020]:** 13A15, 13G05, 13A02

### 1. Introduction

The star operations are defined by axioms selected by Krull among the properties satisfied by some classical operations, such as the  $v$ -operation, the  $t$ -operation and the completion. Star operations have shown to be an essential tool in Multiplicative Ideal Theory, allowing a new approach for characterizing several classes of integral domains. For example, an integrally closed domain  $D$  is a Prüfer domain if and only if  $I^t = I$  for each nonzero ideal  $I$  of  $D$ .

Let  $R$  be an integral domain with quotient field  $K$ . Let  $\mathbf{F}(R)$  be the set of all nonzero fractional ideals of  $R$ . A (classical) star operation on  $R$  is a map  $\star : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$ ,  $A \mapsto A^\star$ , such that, for all  $0 \neq x \in K$ , and for all  $A, B \in \mathbf{F}(R)$ , the following three properties hold:

- $\star_1$ :  $(x)^\star = (x)$  and  $(xA)^\star = xA^\star$ ;
- $\star_2$ :  $A \subseteq A^\star$  and  $A \subseteq B \Rightarrow A^\star \subseteq B^\star$ ;
- $\star_3$ :  $A^{\star\star} := (A^\star)^\star = A^\star$ .

Let  $\Gamma$  be a torsion-free cancellative monoid, and  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain. In [4, 5], the author considered (classical) star operations (and more generally semistar operations)  $\star$  on  $R$  such that  $\star$  sends homogeneous fractional ideals to homogeneous ones. Such a star operation is called *homogenous preserving star operation*. The identity  $d$ ,  $v$ ,  $t$ ,  $w$  and the completion  $b$  are various examples of homogenous preserving star operations (see [5, Example 1]). The set of (classical) star operations (resp. homogenous preserving star operations) on  $R$  is denoted by  $\text{Star}(R)$  (resp.  $\text{Star}_{hp}(R)$ ).

Recently the notion of homogeneous star operations on graded integral domains  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  was introduced in [3]. Let  $\mathbf{HF}(R)$  be the set of nonzero homogeneous fractional ideals of  $R$ . Obviously  $\mathbf{HF}(R) \subseteq \mathbf{F}(R)$ . Let  $H$  be the set of nonzero homogeneous elements of  $R$ , and  $R_H$  the homogeneous quotient field of  $R$ . A *homogeneous star operation* on  $R$  is a map  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$ ,  $A \mapsto A^\star$  satisfying the above three conditions  $\star_1, \star_2$  and  $\star_3$ , for all homogeneous element  $x \in R_H \setminus \{0\}$  and  $A, B \in \mathbf{HF}(R)$ . The set of homogeneous star operations on  $R$  is denoted by  $\mathbf{HStar}(R)$ .

Clearly, if  $\star : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$  is a homogenous preserving star operation, then the restricted mapping  $\star|_{\mathbf{HF}(R)} : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  is a homogeneous star operation.

DEFINITION 1.1. Let  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  be a homogeneous star operation on  $R$ , and let  $\star^\bullet : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$  be a (classical) star operation on  $R$ . Then we say that  $\star^\bullet$  is an *extension of  $\star$* , in case  $\star^\bullet = \star|_{\mathbf{HF}(R)}$ . In other words,  $\star^\bullet$  is an extension of  $\star$ , if for each element  $A \in \mathbf{HF}(R)$ ,  $A^{\star^\bullet} = A^\star$ .

The purpose of this note is to prove that every homogeneous star operation  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  of  $R$ , has a largest extension  $\mathfrak{e}(\star)$  of  $R$ . That is we show that there exists a (classical) star operation  $\mathfrak{e}(\star)$  on  $R$  such that  $\star := \mathfrak{e}(\star)|_{\mathbf{HF}(R)}$ , and that  $\mathfrak{e}(\star)$  is the largest (classical) star operation on  $R$  with this property. Then we show that the sets  $\mathbf{HStar}(R)$  and  $\mathbf{Star}_{hp}(R)$  can be endowed with Zariski-like topologies, such that the injection

$$\mathfrak{e} : \mathbf{HStar}(R) \rightarrow \mathbf{Star}_{hp}(R), \star \mapsto \mathfrak{e}(\star),$$

is a topological embedding. We also show that the set  $\mathbf{HStar}_f(R)$  of finite type homogeneous star operations on  $R$  endowed with the sub-space topology of the Zariski topology of  $\mathbf{HStar}(R)$  is a spectral space.

## 2. Extension of homogeneous star operations

Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain.

THEOREM 2.1. Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain with quotient field  $K$ , and let  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  be a homogeneous star operation of  $R$ . For each  $A \in \mathbf{F}(R)$ , set

$$A^{\mathfrak{e}(\star)} := \bigcap \{z^{-1}(\mathbf{C}(zA))^\star \mid 0 \neq z \in (R : A)\}.$$

Then:

- (1)  $\mathfrak{e}(\star)$  is a (classical) star operation on  $R$ .
- (2)  $\mathfrak{e}(\star)$  is an extension of  $\star$ . In fact, it is the largest extension of  $\star$ .
- (3) If  $\star_1 \leq \star_2$  are two homogeneous star operations in  $R$ , then  $\mathfrak{e}(\star_1) \leq \mathfrak{e}(\star_2)$ .

For a given (classical) star operation  $\star$  on  $R$ , we can always associate a stable (classical) star operation  $\bar{\star}$  by defining, for every  $F \in \mathbf{F}(R)$ ,

$$F^{\bar{\star}} := \{(F : I) \mid I \text{ is a nonzero ideal of } R \text{ such that } I^\star = R\}.$$

It is easy to see that  $\bar{\star} \leq \star$  and, moreover, that  $\bar{\star}$  is the largest stable (classical) star operation that precedes  $\star$ .

PROPOSITION 2.2. Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain, and  $\star : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$  be a (classical) star operation on  $R$ . Then,  $\bar{\star}$  is homogeneous preserving.

REMARK 2.3. Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain, let  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  be a homogeneous star operation of  $R$ . Then the homogeneous star operation  $\bar{\star} := \underline{\mathbf{e}(\star)}|_{\mathbf{HF}(R)} : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  is stable, that is for any  $A, B \in \mathbf{HF}(D)$ , we have  $(A \cap B)^{\bar{\star}} = A^{\bar{\star}} \cap B^{\bar{\star}}$ .

THEOREM 2.4. Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain, let  $\star : \mathbf{HF}(R) \rightarrow \mathbf{HF}(R)$  be a homogeneous star operation of  $R$ , and let  $\mathbf{e}(\star)$  be the extension of  $\star$  to  $R$  introduced in Theorem 2.1. Then:

- (1)  $(\mathbf{e}(\star))_f = (\mathbf{e}(\star_f))_f$  is the largest finite type extension of  $\star_f$ .
- (2)  $\underline{\mathbf{e}(\star)} = \underline{\mathbf{e}(\bar{\star})}$  is the largest stable and finite type extension of  $\bar{\star}$ .
- (3)  $\mathbf{e}(\star) = \mathbf{e}(\bar{\star})$  is the largest stable extension of  $\bar{\star}$ .

In the following corollary, for a homogeneous preserving star operation  $\star : \mathbf{F}(R) \rightarrow \mathbf{F}(R)$ , we still denote by  $\star$ , the homogeneous star operation  $\star|_{\mathbf{HF}(R)}$ .

COROLLARY 2.5. Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain, and let  $v$ ,  $t$  and  $w$  be the  $v$ -star operation,  $t$ -star operation and  $w$ -star operation on  $R$  respectively. Then:

- (1)  $\mathbf{e}(v) = v$ .
- (2)  $(\mathbf{e}(v))_f = (\mathbf{e}(t))_f = t$ .
- (3)  $\mathbf{e}(v) = \mathbf{e}(w) = w$ .

Recall that a graded domain  $R$  is called a *graded-Prüfer domain* if each nonzero finitely generated homogeneous ideal of  $R$  is invertible.

Let  $D$  be an integral domain and  $\{D_\alpha\}$  be a family of overrings of  $D$  such that  $D = \bigcap_\alpha D_\alpha$ . Then the map  $\ast : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$  defined by  $F^\ast := \bigcap_\alpha F D_\alpha$ , for each  $F \in \mathbf{F}(D)$  is a (classical) star operation. A particular case is the  $b$  :=  $b_D$ -operation defined by  $F^b := \bigcap_\alpha F V_\alpha$ , for each  $F \in \mathbf{F}(D)$ , where  $\{V_\alpha\}$  is the class of all valuation overrings of  $D$ . The  $b$ -operation is a (classical) star operation, only in the case that  $D$  is integrally closed. It is shown that the  $b$ -operation is a homogeneous preserving semistar operation [5, Corollary 2]; and in particular, if  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  is an integrally closed graded domain, it is a homogeneous preserving star operation, when restricted to fractional ideals.

REMARK 2.6. (1) Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain. Assume that  $\star_1$  and  $\star_2$  are two homogeneous preserving star operations on  $R$  such that  $\star_1|_{\mathbf{HF}(R)} = \star_2|_{\mathbf{HF}(R)}$ . Then  $\mathbf{e}(\star_1) = \mathbf{e}(\star_2)$ .

(2) Note that  $d \neq (\mathbf{e}(d))_f$  and  $b \neq (\mathbf{e}(b))_f$ . Let  $R$  be a graded-Prüfer domain which is not Prüfer (e.g.  $R = D[X, X^{-1}]$  for a Prüfer domain  $D$  which is not a field and an indeterminate  $X$  over  $D$ ). Since  $R$  is integrally closed and is not a Prüfer domain, there exists an ideal  $I$  of  $R$  such that  $I^t \neq I$ . Hence  $t \neq d$ . However  $d|_{\mathbf{HF}(R)} = t|_{\mathbf{HF}(R)}$  since  $R$  is a graded-Prüfer domain. Therefore  $\mathbf{e}(d) = \mathbf{e}(t)$  and hence  $(\mathbf{e}(d))_f = (\mathbf{e}(t))_f = t \neq d$ . For the second case, we note that  $b|_{\mathbf{HF}(R)} = d|_{\mathbf{HF}(R)}$ . Assume to the contrary that  $b = (\mathbf{e}(b))_f$ . Then  $b = (\mathbf{e}(b))_f = (\mathbf{e}(d))_f = t$ . Since  $R$  is a graded-Prüfer domain, we have  $R$  is a PvMD. Therefore  $R$  is a Prüfer domain by [1, Proposition 9], which is a contradiction.

### 3. Topological considerations

Let  $D$  be an integral domain with quotient field  $K$ . On the set  $Star(D)$  of all (classical) star operations on  $D$ , we can introduce a natural topology that we still call the *Zariski topology*, whose subbasic open sets are

$$U_F := \{\star \in Star(D) \mid 1 \in F^\star\},$$

as  $F$  varies among the elements of  $\mathbf{F}(D)$ .

Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain. Consider the set  $Star_{hp}(R)$  of all homogeneous preserving star operations as a topological space endowed with the subspace topology of the Zariski topology of  $Star(R)$ .

On the set  $HStar(R)$  of all homogeneous star operations on  $R$ , we introduce a natural topology that we still call the *Zariski topology*, whose subbasic open sets are

$$W_A := \{\star \in HStar(R) \mid 1 \in A^\star\},$$

as  $A$  varies among the elements of  $\mathbf{HF}(R)$ . Consider the set  $HStar_f(R)$  of all finite type homogeneous star operations as a topological space endowed with the subspace topology of the Zariski topology of  $HStar(R)$ .

**PROPOSITION 3.1.** *Let  $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$  be a graded integral domain. Endow  $HStar(R)$  with its Zariski topology. If  $Star_{hp}(R)$  is endowed with the topology generated by the family  $\{U_F \cap Star_{hp}(R) \mid F \in \mathbf{F}(R), F \subseteq R_H\}$ , then the map*

$$\mathbf{e} : HStar(R) \rightarrow Star_{hp}(R), \star \mapsto \mathbf{e}(\star),$$

*is a topological embedding.*

Let  $\mathcal{S}$  be a nonempty set of homogeneous star operations on  $R$ . For each  $A \in \mathbf{HF}(R)$ , define  $\bigwedge(\mathcal{S})$  as follows:

$$A^{\bigwedge(\mathcal{S})} = \bigcap \{A^\star \mid \star \in \mathcal{S}\}.$$

**PROPOSITION 3.2.** *Let  $\mathcal{S}$  be a quasi-compact subspace of  $HStar_f(R)$ . Then, the homogeneous star operation  $\bigwedge(\mathcal{S})$  belongs to  $HStar_f(R)$ .*

A topological space  $X$  is called a *spectral space* if  $X$  is quasi-compact and  $T_0$ , the quasi-compact open subsets of  $X$  are closed under finite intersection and form an open basis, and every nonempty irreducible closed subset of  $X$  has a generic point (i.e. it is the closure of a unique point) [2].

**THEOREM 3.3.** *The space  $HStar_f(R)$  of finite type homogeneous star operations on  $R$ , endowed with the Zariski topology, is a spectral space.*

**COROLLARY 3.4.** *Let  $D$  be an integral domain. The space  $Star_f(D)$  of finite type (classical) star operations on  $D$ , endowed with the Zariski topology, is a spectral space.*

## 4. Conclusion

The main issue of this work is to show that every homogeneous star operation come from a homogeneous preserving (classical) star operation in the sense that a homogeneous star operation is the restriction of a homogeneous preserving (classical) star operation to the set of homogeneous fractional ideals.

## References

1. Fontana, M. and Picozza, G. (2011) *On some classes of integral domains defined by Krull's a.b. operations*, J. Algebra **341**, 179–197.
2. Hochster, M. (1969) *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142**, 43–60.
3. Kim, D. K. (2024) *Homogeneous star-operations on graded integral domains*, Comm. Algebra, 1–15. <https://doi.org/10.1080/00927872.2024.2346842>.
4. Sahandi, P. (2014) *Characterizations of graded Prüfer  $\star$ -multiplication domains*, Korean J. Math. **22**, 181–206.
5. Sahandi, P. (2022) *The space of homogeneous preserving semistar operations on graded domains*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **116**, 148. <https://doi.org/10.1007/s13398-022-01277-7>.



## Embeddings of finite maximal parabolic subgroups of symplectic groups

Maryam Ghorbani

Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran.

Email: m\_ghorbani@mazust.ac.ir

---

**ABSTRACT.** By a quasi-permutation matrix we mean a square matrix over the complex field  $C$  with non-negative integral trace. Thus every permutation matrix over  $C$  is a quasi-permutation matrix. For a given finite group  $G$ , let  $c(G)$  denotes the minimal degree of a faithful representation of  $G$  by quasi-permutation matrices over the complex numbers and let  $r(G)$  denote the minimal degree of a faithful rational valued complex character of  $G$ . In this paper  $c(G)$  and  $r(G)$  are calculated for the maximal parabolic subgroups of  $SP(4, 2^f)$ .

**Keywords:** General linear group, Quasi-permutati representation, symplectic group

**AMS Mathematics Subject Classification [2020]:** 20C15

---

### 1. Introduction

In [8] Wong defined a quasi-permutation group of degree  $n$ , to be a finite group  $G$  of automorphisms of an  $n$ -dimensional complex vector space such that every element of  $G$  has non-negative integral trace. If  $F$  is a subfield of the complex numbers  $C$ , then a square matrix over  $F$  with non-negative integral trace is called a quasi-permutation matrix over  $F$ . Thus every permutation matrix over  $C$  is a quasi-permutation matrix. For a given finite group  $G$ , let  $c(G)$  be the minimal degree of a faithful representation of  $G$  by complex quasi-permutation matrices.

By a rational valued character we mean a character  $\chi$  corresponding to a complex representation of  $G$  such that  $\chi(g) \in Q$  for all  $g \in G$ . As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of  $G$  is then simply a complex representation of  $G$  whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from  $G$  to  $GL(n, Q)$  a rational representation of  $G$  and its corresponding character will be called a rational character of  $G$ . Let  $r(G)$  denote the minimal degree of a faithful rational valued character of  $G$ .

Finding the above quantities have been carried out in some papers, for example in [3], [4], [5] and [6] we found these for the groups  $GL(2, q)$ ,  $SU(3, q^2)$ ,  $PSU(3, q^2)$ ,  $Sl(3, q)$ ,

$PSL(3, q)$  and  $G_2(2^n)$  respectively. In this paper we will calculate  $c(G)$  and  $r(G)$  for the maximal parabolic subgroups of  $SP(4, 2^f)$ .

## 2. Notation and preliminary results

We can see all the following statements in [1], [2].

DEFINITION 2.1. Let  $\Gamma(\chi)$  be the Galois group of  $Q(\chi)$  over  $Q$ .

Let  $G$  be a finite group and  $\chi$  be an irreducible complex character of  $G$ . Then we define

$$\begin{aligned} \mathbf{1)} \quad & d(\chi) = |\Gamma(\chi)|\chi(1) \\ \mathbf{2)} \quad & m(\chi) = \begin{cases} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha(g) : g \in G\}| & \text{otherwise} \end{cases} \\ \mathbf{3)} \quad & c(\chi) = \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha + m(\chi)1_G. \end{aligned}$$

LEMMA 2.2. Let  $\chi$  be a character of  $G$ . Then  $\text{Ker} \chi = \text{Ker} \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$ . Moreover  $\chi$  is faithful if and only if  $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$  is faithful.

LEMMA 2.3. Let  $\chi \in \text{Irr}(G)$ , then  $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$  is a rational valued character of  $G$ . Moreover  $c(\chi)$  is a non-negative rational valued character of  $G$  and  $c(\chi)(1) = d(\chi) + m(\chi)$ .

LEMMA 2.4. Let  $\chi \in \text{Irr}(G)$ ,  $\chi \neq 1_G$ . Then  $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$ .

LEMMA 2.5. Let  $\chi \in \text{Irr}(G)$ . Then

- (1)  $c(\chi)(1) \geq d(\chi) \geq \chi(1)$ ;
- (2)  $c(\chi)(1) \leq 2d(\chi)$ .

Equality occurs if and only if  $Z(\chi)/\text{ker} \chi$  is of even order.

Now according to Corollary 3.11 of [1] and above statements the following corollary is useful for calculation of  $r(G)$ ,  $c(G)$  and  $q(G)$ .

COROLLARY 2.6. Let  $G$  be a finite group with a unique minimal normal subgroup. Then

- 1)  $r(G) = \min\{d(\chi) : \chi \text{ is a faithful irreducible complex character of } G\}$
- 2)  $c(G) = \min\{c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\}$
- 3)  $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\}$ .

## 3. Quasi-permutation representations

We begin with a brief summary of facts relevant to our treatment of the group.

Let  $K$  be the finite field with  $q$  elements, where  $q = p^f$  and  $p$  is a prime number. Let  $\bar{K}$  be the algebraic closure of  $K$ , and put

$$K_i = \{x \in \bar{K} \mid x^{q^i} = x\}.$$

Then  $K_i$  is the subfield of  $\bar{K}$  with  $q^i$  elements, and  $K_1 = K$ . Let  $\kappa$  be a fixed generator of the multiplicative group  $K_4^*$  and put  $\tau = \kappa^{q^2-1}$ ,  $\theta = \kappa^{q^2+1}$ ,  $\eta = \theta^{q-1}$  and  $\gamma = \theta^{q+1}$ . Then we have  $\langle \theta \rangle = K_2^*$  and  $\langle \gamma \rangle = K^*$ . Choose a fixed isomorphism from the multiplicative group  $K_4^*$  into the multiplicative group of complex numbers, and let  $\tilde{\tau}, \tilde{\theta}, \tilde{\eta}$  and  $\tilde{\gamma}$  be the

images of  $\tau$ ,  $\theta$ ,  $\eta$  and  $\gamma$  respectively under this isomorphism. Let  $G$  be the 4-dimensional symplectic group over  $K$ , that is,

$$G = \{A \in GL(4, K) \mid {}^t A J A = J\},$$

where  $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$  and  ${}^t A$  is the transposed matrix of  $A$ .

Next, define  $h(z_1, z_2) = \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & z_2^{-1} & \\ & & & z_1^{-1} \end{pmatrix}$  for  $z_i \in K_4^*$  and put  $U =$

$\Xi_a \Xi_b \Xi_{a+b} \times \Xi_{2a+b}$ ,  $\wp = \{h(z_1, z_2) \mid z_i \in K^*\}$  and  $B = \wp U$ . Then  $U$  is a Sylow  $p$ -subgroup of  $G$ , and  $B$  is the normalizer of  $U$  in  $G$  (called the Borel subgroup of  $G$ ). Put  $\omega_r = x_r(1) {}^t x_r(-1) x_r(1)$  for  $r \in \Delta^+$ . Especially,

$$\omega_a = \begin{pmatrix} & & & 1 \\ & & & \\ -1 & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \omega_b = \begin{pmatrix} & & & 1 \\ & & & \\ & & 1 & \\ & & & \\ -1 & & & \\ & & & 1 \end{pmatrix}.$$

Then  $G$  is generated by  $B \cup \{\omega_a, \omega_b\}$ . The maximal parabolic subgroups of  $G$  generated by  $B \cup \{\omega_a, \}$  and  $B \cup \{\omega_b, \}$  are denoted by  $P$  and  $Q$  respectively.

The character tables of  $P$  and  $Q$  are given in Tables (II, III) of the Appendix of [7].

In the next theorem we shall determine  $r(G)$  and  $c(G)$  for the parabolic subgroups of  $SP(4, 2^n)$ .

**THEOREM 3.1.** *Let  $G$  be a maximal parabolic subgroup  $P$  or  $Q$  of  $SP(4, 2^n)$ , then*

- 1)  $r(G) = \frac{q(q-1)^2}{2}$
- 2)  $c(G) = \frac{q^2(q-1)}{2}$ .

**PROOF.** Since the groups  $P$  and  $Q$  have similar proofs, we will prove only  $Q$ . In order to calculate  $r(G)$  and  $c(G)$ , we need to determine  $d(\chi)$  and  $c(\chi)(1)$  for all characters that are faithful or  $\bigcap_{\chi} \text{Ker} \chi = 1$ .

Then by Corollary 2.2, Lemmas 2.3, 2.4 and Table (III) of [7], for the maximal parabolic subgroup  $Q$  we have:

$$\begin{aligned} d(\chi_1) &= |\Gamma(\chi'_1(k))| \chi'_1(k)(1) + |\Gamma(\chi'_5(k))| \chi'_5(k)(1) \geq q(q^2 - 1) + 1 \text{ and } c(\chi_1)(1) \geq q^3 + q^2 + 1, \\ d(\chi_2) &= |\Gamma(\chi'_1(k))| \chi'_1(k)(1) + |\Gamma(\chi'_6(k))| \chi'_6(k)(1) \geq q(q-1)^2 + 1 \text{ and } c(\chi_2)(1) \geq q^3 - q^2 + 1, \\ \text{and the end we have:} \\ d(\theta'_3(k)) &= |\Gamma(\theta'_3(k))| \theta'_3(k)(1) = \frac{q(q-1)^2}{2} \text{ and } c(\theta'_3(k))(1) = \frac{q^2(q-1)}{2}. \end{aligned}$$

The values are set out in the Table(I).

Now by Table (I) and Corollary 2.6, we have

$$\min \{d(\chi) : \text{Ker} \chi = 1\} = \frac{q(q-1)^2}{2} \quad \text{and} \quad \min \{c(\chi)(1) : \text{Ker} \chi = 1\} = \frac{q^2(q-1)}{2}.$$

Hence  $r(G) = \frac{q(q-1)^2}{2}$ ,  $c(G) = \frac{q^2(q-1)}{2}$ , and the result follows.

Table (I)

$\chi$	$d(\chi)$	$c(\chi)(1)$
$\chi_1$	$\geq q(q^2 - 1) + 1$	$\geq q^3 + q^2 + 1$
$\chi_2$	$\geq q(q - 1)^2 + 1$	$\geq q^3 - q^2 + 1$
$\chi_3$	$\geq q(q^2 - 1)/2 + 1$	$\geq (q^3 + q^2 + 2)/2$
$\chi_4$	$\geq q(q - 1)^2/2 + 1$	$\geq (q^3 - q^2 + 2)/2$
$\chi_5$	$\geq q^3$	$\geq q^3 + q^2 + q + 1$
$\chi_6$	$\geq q(q^2 - 2q + 2)$	$\geq q^3 - q^2 + q + 1$
$\chi_7$	$\geq q(q^2 + 1)/2$	$\geq (q^3 + q^2 + 2q + 2)/2$
$\chi_8$	$\geq q(q^2 - 2q + 3)/2$	$\geq (q^3 - q^2 + 2q + 2)/2$
$\chi_9$	$\geq (q + 1)(q^2 - q + 1)$	$\geq q^3 + q^2 + q + 2$
$\chi_{10}$	$\geq (q + 1) + q(q - 1)^2$	$\geq q^3 - q^2 + q + 2$
$\chi_{11}$	$\geq (q + 1)(q^2 - q + 2)/2$	$\geq (q^3 + q^2 + 2q + 4)/2$
$\chi_{12}$	$\geq (q + 1) + q(q - 1)^2/2$	$\geq (q^3 - q^2 + 2q + 4)/2$
$\chi_{13}$	$\geq (q + 1)(q^2 - 1)$	$\geq q^2(q + 2)$
$\chi_{14}$	$\geq (q - 1)(q^2 + 1)$	$\geq q^3$
$\chi_{15}$	$\geq (q + 2)(q^2 - 1)/2$	$\geq q^2(q + 3)/2$
$\chi_{16}$	$\geq (q - 1)(q^2 + q + 2)/2$	$\geq q^2(q + 1)/2$
$\chi_{17}$	$\geq (q - 1)(q^2 + q + 1)$	$\geq q(q^2 + q + 1)$
$\chi_{18}$	$\geq (q - 1)(q^2 - q + 1)$	$\geq q(q^2 - q + 1)$
$\chi_{19}$	$\geq (q - 1)(q^2 + q + 2)/2$	$\geq q^2(q + 3)/2$
$\chi_{20}$	$\geq (q - 1)(q^2 - q + 2)/2$	$\geq q(q^2 - q + 2)/2$
$\chi_{21}$	$\geq (2q - 1)(q^2 - 1)$	$\geq 2(q^3 + q^2 - 1)$
$\chi_{22}$	$\geq (2q + 1)(q - 1)^2$	$\geq 2q^2(q - 1)$
$\chi_{23}$	$\geq (3q - 2)(q^2 - 1)/2$	$\geq q^2(3q - 1)/2$
$\chi_{24}$	$\geq (3q + 2)(q - 1)^2/2$	$\geq 3q^2(q - 1)/2$
$\chi_5(k)$	$\geq q(q^2 - 1)$	$\geq q^2(q + 1)$
$\chi_6(k)$	$\geq q(q - 1)^2$	$\geq q^2(q - 1)$
$\theta_2(k)$	$q(q^2 - 1)/2$	$q^2(q + 1)/2$
$\theta_3(k)$	$q(q - 1)^2/2$	$q^2(q - 1)/2$

□

## References

- Behraves, H. (1997) *Quasi-permutation representations of p-groups of class 2*, J. London Math. Soc., (2)55, 251-26.
- Burns, J.M. Goldsmith, B. Hartley, B and Sandling, R. (1994) *on quasi-permutation representations of finite groups*, Glasgow Math, J., **36**, 301-308.
- Darafsheh, M.R. Ghorbany, M. Daneshkhah, A. and Behraves, H. (2001) *Quasi-permutation representation of the group  $GL(2, q)$* , Journal of Algebra **243**, 142-167.
- Darafsheh, M.R. and Ghorbany, M. (2002) *Quasi-permutation representations of the groups  $SU(3, q^2)$  and  $PSU(3, q^2)$* , Southeast Asian Bulletin of Mathematics **26**, 395-406.
- Darafsheh, M.R. and Ghorbany, M. (2002) *Quasi-permutation representations of the groups  $SL(3, q)$  and  $PSL(3, q)$* , Iranian Journal of Science and Technology, Tra.A, Vol. **26**, No. A1 145-154.
- Darafsheh, M. R. and Ghorbany, M. (2006) *Special representations of the group  $G_2(2^n)$  with minimal degrees*, Southeast Asian Bulletin of Mathematics **30**: 663-670.
- Enomoto, H. (1972) *The characters of the finite symplectic group  $SP(4, q)$ ,  $q = 2^f$* , Osaka J.Math. **9**, 75-94.

8. Wong, W.J. (1963) *Linear groups analogous to permutation groups*, J. Austral. Math. Soc, Sec. **A** 3, 180-184.





## The rational character table of the group $K_2^2(2^n)$

Maryam Ghorbani

Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran .

Email: m\_ghorbani@mazust.ac.ir

**ABSTRACT.** By a quasi-permutation matrix we mean a square matrix over the complex field  $C$  with non-negative integral trace. Thus every permutation matrix over  $C$  is a quasi-permutation matrix. In this paper we will give the character table of the irreducible rational representations of  $K_2^2(2^n)$ , by using the ordinary character table and the Schur indices of this group.

**Keywords:** Character table, Rational representation, Rational character table

**AMS Mathematics Subject Classification [2020]:** 20C15

### 1. Introduction

In [9] Wong defined a quasi-permutation group of degree  $n$  to be a finite group  $G$  of automorphisms of an  $n$ -dimensional complex vector space such that every element of  $G$  has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [4] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results. By a rational valued character we mean a character  $\chi$  corresponding to a complex representation of  $G$  such that  $\chi(g) \in Q$  for all  $g \in G$ . As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of  $G$  is then simply a complex representation of  $G$  whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from  $G$  to  $GL(n, Q)$  a rational representation of  $G$  and its corresponding character will be called a rational character of  $G$ . Let  $G$  be a finite group and  $\chi$  be an irreducible complex character of  $G$ . Let  $m_Q(\chi)$  denote the Schur index of  $\chi$  over  $Q$  and  $\Gamma(\chi)$  be the Galois group  $Q(\chi)$  over  $Q$ . It is known that

$$\sum_{\alpha \in \Gamma(\chi)} m_Q(\chi) \chi^\alpha \quad (*)$$

is a character of an irreducible  $Q(G)$ -module [[8], Corollary 10.2(b)]. So by knowing the character table of a group and Suchr indices of each of the irreducible characters of  $G$ , we can find the irreducible rational characters of  $G$ . In [3] we found the rational character table and the values of  $r(G)$  and  $c(G)$  for the group  $PGL(2, q)$ .

In this paper we will calculate the irreducible rational characters table for the group  $K_2^2(2^n)$ , where

$$K_2^2(q) = \langle SL(2, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle.$$

## 2. Notation and preliminary results

We can see all the following statements in [1], [2].

DEFINITION 2.1. Let  $\chi$  be a character of  $G$  such that, for all  $g \in G$ ,  $\chi(g) \in \mathbb{Q}$  and  $\chi(g) \geq 0$ . Then we say that  $\chi$  is a non-negative rational valued character.

LEMMA 2.2. Let  $\chi$  be a character of  $G$ . Then  $\text{Ker}\chi = \text{Ker} \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$ . Moreover  $\chi$  is faithful if and only if  $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$  is faithful.

LEMMA 2.3. Let  $\chi \in \text{Irr}(G)$ , then  $\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha$  is a rational valued character of  $G$ .

let  $\varepsilon$  be a primitive  $n$ -th root of unity in  $\mathbb{C}$ . Then  $\varepsilon + \varepsilon^{-1}$  is rational if and only if  $n = 1, 2, 3, 4, 6$ . Also  $\varepsilon^j + \varepsilon^{-j}$ ,  $1 \leq j \leq n$  is rational if and only if  $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$ . In this case if  $i \in \mathbb{Z}$  and  $d_i = (i, n)$ , and  $n > 2d_i$ , then  $[Q(\varepsilon^i + \varepsilon^{-i}) : \mathbb{Q}] = \frac{1}{2}\varphi(\frac{n}{d_i})$ , and if  $n \neq d_i, 2d_i$ , then

$$\sum_{\alpha \in \Gamma_i} (\varepsilon^i + \varepsilon^{-i})^\alpha = \mu(\frac{n}{d_i})$$

where  $\Gamma_i = (Q(\varepsilon^i + \varepsilon^{-i}) : \mathbb{Q})$  and  $\mu$  is the Möbius function.

With the above assumption if we set  $\Gamma = (Q(\varepsilon + \varepsilon^{-1}) : \mathbb{Q})$ , then

$$\sum_{\alpha \in \Gamma} (\varepsilon^i + \varepsilon^{-i})^\alpha = \frac{\varphi(n)}{\varphi(\frac{n}{d_i})} \mu(\frac{n}{d_i}) \quad \text{where } d_i = (i, n).$$

We can see all the following statements in [5].

Let  $SL(n, q)$  denote the general linear group of a vector space of dimension  $n$  over a field with  $q$  elements. Let  $\theta : SL(n, q) \rightarrow SL(n, q)$  be the automorphism of  $SL(n, q)$  given by  $\theta(A) = (A^t)^{-1}$ , where  $A^t$  denotes the transpose of the matrix  $A \in SL(n, q)$ . In this case one can define the split extension  $SL(n, q) \cdot \langle \theta \rangle$  that following the notations used in [7] is denoted by  $K_n^2(q)$ . Therefore we have  $K_n^2(q) = \langle SL(n, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle$ . In the following Lemma we give the structure of  $K_2^2(q)$ .

LEMMA 2.4. Let  $G = K_2^2(q)$ . If  $q$  is even, then  $K_2^2(q) \cong SL(2, q) \times \langle \theta \rangle$ .

REMARK 2.5. In the case of  $q$  even  $\langle \theta J \rangle$  has order 2 and its irreducible characters are denoted by  $\mu_0$  and  $\mu_1$  where  $\mu_0$  is the identity character. Regarding the structure of  $K_2^2(2^n)$ , Lemma 2.11 and [6 Theorem 38.1], the irreducible characters of  $K_2^2(q)$  in the case of  $q$  even are  $\mu_k \times 1_G$ ,  $\mu_k \times \psi$ ,  $\mu_k \times \chi_i$  and  $\mu_k \times \theta_j$  where  $k = 0, 1$  and  $1 \leq i \leq \frac{q-2}{2}, 1 \leq j \leq \frac{q}{2}$ .

## 3. The rational Character table of $K_2^2(2^n)$

LEMMA 3.1. Let  $G = K_2^2(2^n)$ , then the Galois orbit sums in  $\text{Irr}(G)$  are as follows:

a)  $\mu_k \times \chi_e = \sum_{\alpha \in \Gamma} (\mu_k \times \chi_i)^\alpha$

where,  $e = (i, q-1)$  and  $1 \leq i \leq \frac{q-2}{2}$ ,  $k = 0, 1$  and  $\Gamma = \Gamma(Q(\mu_k \times \chi_i) : \mathbb{Q}) = \Gamma(Q(\chi_i) : \mathbb{Q})$ .

b)  $\mu_k \times \theta_f = \sum_{\alpha \in \Gamma} (\mu_k \times \theta_j)^\alpha$

where,  $f = (j, q+1)$  and  $1 \leq j \leq \frac{q}{2}$ ,  $k = 0, 1$  and  $\Gamma = \Gamma(Q(\mu_k \times \theta_j) : \mathbb{Q}) = \Gamma(Q(\theta_j) : \mathbb{Q})$ .

c)  $\mu_k \times \theta_{\frac{q+1}{4}}$  when,  $q \equiv -1 \pmod{3}$ .

d)  $1_G, \psi$ .

Let  $G = K_2^2(2^n)$ , then the Galois orbit sums in  $\text{Irr}(G)$  are as follows:

- a)**  $\mu_k \times \chi_e = \sum_{\alpha \in \Gamma} (\mu_k \times \chi_i)^\alpha$   
 where,  $e = (i, q-1)$  and  $1 \leq i \leq \frac{q-2}{2}$   $k = 0, 1$  and  $\Gamma = \Gamma(Q(\mu_k \times \chi_i) : Q) = \Gamma(Q(\chi_i) : Q)$ .  
**b)**  $\mu_k \times \theta_f = \sum_{\alpha \in \Gamma} (\mu_k \times \theta_j)^\alpha$   
 where,  $f = (j, q+1)$  and  $1 \leq j \leq \frac{q}{2}$ ,  $k = 0, 1$  and  $\Gamma = \Gamma(Q(\mu_k \times \theta_j) : Q) = \Gamma(Q(\theta_j) : Q)$ .  
**c)**  $\mu_k \times \theta_{(\frac{q+1}{4})}$  when,  $q \equiv -1 \pmod{3}$ .  
**d)**  $1_G, \psi$ .

we know :

If  $\chi_i, \theta_j$  are irreducible characters of  $G$ , then  $\mu_k \times \chi_e = \sum_{\alpha \in \Gamma_1} (\mu_k \times \chi_i)^\alpha$  and  $\mu_k \times \theta_f = \sum_{\alpha \in \Gamma_2} (\mu_k \times \theta_j)^\alpha$  are rational valued characters of  $G$ , where  $\Gamma_1 = \Gamma(Q(\chi_i) : Q)$  and  $\Gamma_2 = \Gamma(Q(\theta_j) : Q)$ .

**COROLLARY 3.2.** *Let  $G = K_2^2(2^n)$ , then characters of an irreducible  $Q(G)$ -module are as follows:*

$\mu_k \times \chi_e = \sum_{\alpha \in \Gamma_1} (\mu_k \times \chi_i)^\alpha$ ,  $\mu_k \times \theta_f = \sum_{\alpha \in \Gamma_2} (\mu_k \times \theta_j)^\alpha$ ,  $\mu_k \times \theta_{(\frac{q+1}{4})}$ ,  $1_G, \psi$   
 where,  $e = (i, q-1)$ ,  $1 \leq i \leq \frac{q-2}{2}$  and  $k = 0, 1$ ;  $f = (j, q+1)$ ,  $1 \leq j \leq \frac{q}{2}$  and  $\Gamma_1 = \Gamma(Q(\mu_k \times \chi_i) : Q)$ ,  $\Gamma_2 = \Gamma(Q(\mu_k \times \theta_j) : Q)$ .

**LEMMA 3.3.** *Let  $G = K_2^2(2^n)$ , then for the irreducible characters  $\mu_k \times \chi_e, \mu_k \times \theta_f$  where,  $k = 0, 1$  of rational representations of  $G$  we have*

- a)**  $(\mu_0 \times \chi_e)(1, 1) = (\mu_0 \times \chi_e)(x, 1) = (\mu_1 \times \chi_e)(1, 1) = \frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$   
 $(\mu_0 \times \chi_e)(1, c) = (\mu_0 \times \chi_e)(x, c) = (\mu_1 \times \chi_e)(1, c) = \frac{1}{2}\varphi(\frac{q-1}{e})$   
 $(\mu_0 \times \chi_e)(1, a^l) = (\mu_0 \times \chi_e)(x, a^l) = (\mu_1 \times \chi_e)(1, a^l) = S_1(e, e')$   
 $(\mu_0 \times \chi_e)(1, b^m) = (\mu_0 \times \chi_e)(x, b^m) = (\mu_1 \times \chi_e)(1, b^m) = 0$   
 $(\mu_1 \times \chi_e)(x, 1) = -\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$   
 $(\mu_1 \times \chi_e)(x, c) = -\frac{1}{2}\varphi(\frac{q-1}{e})$   
 $(\mu_1 \times \chi_e)(x, a^l) = -S_1(e, e')$   
 $(\mu_1 \times \chi_e)(x, b^m) = 0$ .  
**b)**  $(\mu_0 \times \theta_f)(1, 1) = (\mu_0 \times \theta_f)(x, 1) = (\mu_1 \times \theta_f)(1, 1) = \frac{1}{2}(q-1)\varphi(\frac{q+1}{e})$   
 $(\mu_0 \times \theta_f)(1, c) = (\mu_0 \times \theta_f)(x, c) = (\mu_1 \times \theta_f)(1, c) = -\frac{1}{2}\varphi(\frac{q+1}{e})$   
 $(\mu_0 \times \theta_f)(1, a^l) = (\mu_0 \times \theta_f)(x, a^l) = (\mu_1 \times \theta_f)(1, a^l) = 0$   
 $(\mu_0 \times \theta_f)(1, b^m) = (\mu_0 \times \theta_f)(x, b^m) = (\mu_1 \times \theta_f)(1, b^m) = -S_2(e, e')$   
 $(\mu_1 \times \theta_f)(x, 1) = -\frac{1}{2}(q-1)\varphi(\frac{q+1}{e})$   
 $(\mu_1 \times \theta_f)(x, c) = \frac{1}{2}\varphi(\frac{q+1}{e})$   
 $(\mu_1 \times \theta_f)(x, a^l) = S_2(e, e')$   
 $(\mu_1 \times \theta_f)(x, b^m) = 0$ .

Now in the above notation we have :

**THEOREM 3.4.** *The character table of the irreducible rational representation of  $K_2^2(2^n)$  is as follow:*

**Table (I)**  
**The irreducible rational characters of  $K_2^2(q)$ ,  $q \not\equiv -1 \pmod{3}$**

	$\mu_0 \times 1_G$	$\mu_0 \times \psi$	$\mu_0 \times \chi_e$ $1 \leq i \leq \frac{q-2}{2}$	$\mu_0 \times \theta_f$ $1 \leq j \leq \frac{q}{2}$	$\mu_1 \times 1_G$	$\mu_1 \times \psi$	$\mu_1 \times \chi_e$ $1 \leq i \leq \frac{q-2}{2}$	$\mu_1 \times \theta_f$ $1 \leq j \leq \frac{q}{2}$
(1, 1)	1	$q$	$\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$	$\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$	1	$q$	$\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$	$\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$
(1, c)	1	0	$\frac{1}{2}\varphi(\frac{q-1}{e})$	$-\frac{1}{2}\varphi(\frac{q+1}{f})$	1	0	$\frac{1}{2}\varphi(\frac{q-1}{e})$	$-\frac{1}{2}\varphi(\frac{q+1}{f})$
$(1, a^l)$ $1 \leq l \leq \frac{q-2}{2}$	1	1	$S_1(e, e')$	0	1	1	$S_1(e, e')$	0
$(1, b^m)$ $1 \leq m \leq \frac{q}{2}$	1	-1	0	$-S_2(f, f')$	1	-1	0	$-S_2(f, f')$
(x, 1)	1	$q$	$\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$	$\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$	-1	$-q$	$-\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$	$-\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$
(x, c)	1	0	$\frac{1}{2}\varphi(\frac{q-1}{e})$	$-\frac{1}{2}\varphi(\frac{q+1}{f})$	-1	0	$-\frac{1}{2}\varphi(\frac{q-1}{e})$	$\frac{1}{2}\varphi(\frac{q+1}{f})$
$(x, a^l)$ $1 \leq l \leq \frac{q-2}{2}$	1	1	$S_1(e, e')$	0	-1	-1	$-S_1(e, e')$	0
$(x, b^m)$ $1 \leq m \leq \frac{q}{2}$	1	-1	0	$-S_2(f, f')$	-1	1	0	$S_2(f, f')$

For the case  $q \equiv -1 \pmod{3}$ , we have the above table with two extra columns, as follow:

**Table (II)**  
 $q \equiv -1 \pmod{3}$

	$\mu_0 \times \theta_{\frac{q+1}{3}}$	$\mu_1 \times \theta_{\frac{q+1}{3}}$
(1, 1)	$q-1$	$q-1$
(1, c)	-1	-1
$(1, a^l)$ $1 \leq l \leq \frac{q-2}{2}$	0	0
$(1, b^m)$ $1 \leq m \leq \frac{q}{2}$	$(-1)^{m+1}$	$(-1)^{m+1}$
(x, 1)	$q-1$	$-(q-1)$
(x, c)	-1	1
$(x, a^l)$ $1 \leq l \leq \frac{q-2}{2}$	0	0
$(x, b^m)$ $1 \leq m \leq \frac{q}{2}$	$(-1)^{m+1}$	$(-1)^m$

### References

- Behraves, H. (1997) *Quasi-permutation representations of p-groups of class 2*, J. London Math. Soc., (2)55, 251-26.
- Behraves, H. (1998) *The rational character table of special linear groups*, J. Sci. I.R.I., Vol. 9. No. 2, 173-180.
- Behraves, H. Daneshkhah, A. Darafsheh, M.R. and Ghorbany, M. (2001) *The rational character table and quasi-permutation representations of the group PGL(2, q)*, Italian Journal of Pure and Applied Mathematics, N.11, 9-18.
- Burns, J.M., Goldsmith, B. Hartley, B and Sandling, R. (1994) *on quasi-permutation representations of finite groups*, Glasgow Math, J., 36, 301-308.
- Darafsheh, M.R. and Nowroozi Larki, F. (1999) *Equivalence Classes of Matrices in GL(2, q) and SL(2, q) and related topics*, Korean J. Comput. Appl. Math. 6, no. 2, 331-344.
- Dornhoff, L. (1987) *Group Representation Theory Part A*, Marcel Dekker, New York.
- Fiet, W. (1987) *Extension of Cuspidal Characters of GL(m, q)*, Publications Mathematicae, 34, 273-297.
- Isaacs, I.M. (1976) *Character theory of finite groups*, Academic Press, Newyork.
- Wong, W.J. (1963) *Linear groups analogous to permutation groups*, J. Austral. Math. Soc, Sec. A3, 180-184.



## Some quotients of the Rees Algebra

Nematollah Shirmohammadi

Faculty of Mathematics, Statistics and Computer Science,  
University of Tabriz, Tabriz, Iran.  
Email: shirmohammadi@tabrizu.ac.ir

**ABSTRACT.** This paper devotes to some quotient rings of the Rees algebra associated to a commutative ring. The family of these quotient rings generalizes both the classical concept of idealization by Nagata and the amalgamated duplication of a ring. Among other things, we provide some ideal-theoretic structure of the family.

**Keywords:** Cohen–Macaulay ring, Duplication, Idealization, Rees algebra

**AMS Mathematics Subject Classification [2020]:** 13A30, 13H10

### 1. Introduction

Assume that  $R$  is a commutative ring with unity, and that  $I \neq 0$  is a proper ideal of  $R$ . Let  $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$  be the Rees algebra associated to the ring  $R$  with respect to  $I$ . Given a monic polynomial  $t^2 + at + b \in R[t]$ , we study the quotient ring  $\mathcal{R}_+ / (I^2(t^2 + at + b))$ , where  $(I^2(t^2 + at + b))$  is the contraction to  $\mathcal{R}_+$  of the ideal generated by  $t^2 + at + b$  in  $R[t]$ . We denote such ring by  $R(I)_{a,b}$ . This construction generalizes the amalgamated duplication of a ring along an ideal, which is introduced and studied in [?], as well as the Nagata’s idealization with respect to an ideal [?, Page 2].

In the first section, we introduce the family of rings  $R(I)_{a,b}$ . Then we show that idealization and duplication are particular cases of them. We finally determine Noetherianity of the family. In the second section, after studying the ideal-theoretic structure and spectrum of  $R(I)_{a,b}$ , in case that  $R$  is local, we prove that the rings  $R(I)_{a,b}$  have the same Krull dimension and that they are Cohen-Macaulay if and only if  $R$  is Cohen-Macaulay and  $I$  is maximal Cohen-Macaulay.

### 2. Basic Properties

Let  $R$  be a commutative ring with unity and  $I$  a proper ideal of  $R$ . Let  $t$  be an indeterminate. The Rees algebra associated to  $R$  and  $I$  is defined as the following graded subring of  $R[t]$ :

$$\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n \subseteq R[t].$$

The first lemma calculates the contraction to  $\mathcal{R}_+$  of the ideal generated by a monic polynomial in  $R[t]$ , and, then shows the form of elements of the obtained quotient ring.

LEMMA 2.1. *Let  $f(t) \in R[t]$  be a monic polynomial of degree  $k$ . Then*

- (1)  $f(t)R[t] \cap \mathcal{R}_+ = \{f(t)g(t); g(t) \in I^k \mathcal{R}_+\}$ . Denote this ideal by  $(I^k f(t))$ .
- (2) Each element of the factor ring  $\mathcal{R}_+/(I^k f(t))$  is represented by a unique polynomial of  $\mathcal{R}_+$  of degree  $< k$ .
- (3) The ring extensions  $R \subseteq \mathcal{R}_+/(I^k f(t)) \subseteq R[t]/(f(t))$  are both integral, and the three rings have the same Krull dimension.

PROOF. For (1) and (2), see [?, Lemmas 1.1 and 1.2]. For (3), note that the ring  $R$  is a subring of  $\mathcal{R}_+/(I^k f(t))$  by (2) and, moreover, the class of  $t$  in  $R[t]/(f(t))$  is integral over  $R$  and over  $\mathcal{R}_+/(I^k f(t))$  as well.  $\square$

We now consider particular choices of the polynomial  $f(t)$  above and get known constructions.

Recall that the Nagata's idealization, or simply idealization, of  $R$  with respect to an ideal  $I$  of  $R$  (that could be defined for any  $R$ -module  $M$ ) is defined as the  $R$ -module  $R \oplus I$  endowed with the multiplication  $(r, i)(s, j) = (rs, rj + si)$  and it is denoted by  $R \rtimes I$ .

The duplication of  $R$  with respect to  $I$  is defined as follows:

$$R \bowtie I = \{(r, r + i) \mid r \in R, i \in I\} \subset R \times R.$$

Note that  $R \bowtie I \cong R \oplus I$  endowed with the multiplication  $(r, i)(s, j) = (rs, rj + si + ij)$ . The following proposition shows that, for particular choices of the polynomial  $f(t)$ , one gets the idealization as well as the duplication.

PROPOSITION 2.2. *We have the following isomorphisms of rings:*

- (1)  $\mathcal{R}_+/(I^2 t^2) \cong R \rtimes I$ ;
- (2)  $\mathcal{R}_+/(I^2(t^2 - t)) \cong R \bowtie I$ .

PROOF. (1) Define the map

$$\alpha : \mathcal{R}_+/(I^2 t^2) \rightarrow R \rtimes I$$

by  $\alpha(r + it + (I^2 t^2)) = (r, i)$ . It is easy to see that  $\alpha$  is an isomorphism of rings.

(2) Similarly to (1), the map

$$\beta : \mathcal{R}_+/(I^2(t^2 - t)) \rightarrow R \bowtie I$$

defined by  $\beta(r + it + (I^2(t^2 - t))) = (r, r + i)$  is an isomorphism of rings.  $\square$

The previous proposition makes natural to consider the family  $R(I)_{a,b} := \mathcal{R}_+/(I^2(t^2 + at + b))$ , where  $a, b \in R$ ; so that  $R(I)_{0,0} \cong R \rtimes I$  and  $R(I)_{-1,0} \cong R \bowtie I$ . Both idealization and duplication can be realized in other cases, this is the content of the next proposition.

- PROPOSITION 2.3. (1) *If  $t^2 + at + b = (t - \alpha)^2$ , with  $\alpha \in R$ , then  $R(I)_{a,b} \cong R \rtimes I$ .*  
 (2) *If  $t^2 + at + b = (t - \alpha)(t - \beta)$ , with  $(t - \alpha)$  and  $(t - \beta)$  comaximal ideals of  $R[t]$ , then  $R(I)_{a,b} \cong R \bowtie I$ .*

PROOF. See [?, Proposition 1.5].  $\square$

Continuing the study of the family of rings of the form  $R(I)_{a,b}$ , we show that many relevant properties are independent by the member of the family. From now on, we denote each element of  $R(I)_{a,b}$  simply by  $r + it$  ( $r \in R, i \in I$ ). The next proposition justifies this for Noetherian property.

PROPOSITION 2.4. *The following conditions are equivalent:*

- (i)  $R$  is a Noetherian ring;
- (ii)  $R(I)_{a,b}$  is a Noetherian ring for all  $a, b \in R$ ;
- (iii)  $R(I)_{a,b}$  is a Noetherian ring for some  $a, b \in R$ .

PROOF. See [?, Proposition 1.11]. □

### 3. Main results

This section devotes to study the ideal-theoretic structure of  $R(I)_{a,b}$  and expresses some local cohomology modules of  $R(I)_{a,b}$  with that of  $R$  and  $I$ . One notices that as  $R$ -module  $R(I)_{a,b} \cong R \oplus I$  and the natural injection  $R \rightarrow R(I)_{a,b}$  is a ring homomorphism. Hereafter we consider the contraction and extension with respect to this natural embedding. In the sequel, for an arbitrary ideal  $J$  of  $R$ , we use  $\mathbb{J}$  to denote  $J \oplus I$ .

LEMMA 3.1. *Let  $J$  be an ideal of  $R$ . Then*

- (1)  $J \oplus JI$  is an ideal of  $R(I)_{a,b}$ .
- (2)  $J^e = J \oplus JI$ , the extension is taken with respect to the natural embedding  $R \rightarrow R(I)_{a,b}$ .
- (3) If  $I \subseteq J$ , then  $\mathbb{J}^2 = J^2 \oplus JI$ .

PROOF. See [?, Lemma 2.1] □

Using the chain of inclusions  $R \subseteq R(I)_{a,b} \subseteq R[t]/(t^2 + at + b)$  and the fact that these extensions are integral (by Lemma ??), we can get information on  $\text{Spec}(R(I)_{a,b})$  with respect to  $\text{Spec}(R)$ .

PROPOSITION 3.2. *For each prime ideal  $P$  of  $R$ , there are at most two prime ideals of  $R(I)_{a,b}$  lying over  $P$ . Moreover, if  $t^2 + at + b$  is irreducible on  $R/\mathfrak{m}$  for any maximal ideal  $\mathfrak{m}$  of  $R$ , then there is exactly one prime ideal of  $R(I)_{a,b}$  lying over  $P$ .*

PROOF. See [?, Proposition 1.9]. □

Assume that  $R$  is local, with maximal ideal  $\mathfrak{m}$ . Then it is known that both  $R \rtimes I$  and  $R \times I$  are local with maximal ideals  $\mathfrak{m} \oplus I$  (in the first case under the isomorphism  $R \rtimes I \cong R \oplus I$ ). More generally, we have the following proposition.

PROPOSITION 3.3.  *$R$  is local if and only if  $R(I)_{a,b}$  is local. In this case, the maximal ideal of  $R(I)_{a,b}$  is  $\mathfrak{m} \oplus I$  (as  $R$ -module).*

PROOF. See [?, Proposition 2.1]. □

Under some conditions, one can say more about prime ideals of  $R(I)_{a,b}$ .

LEMMA 3.4. *Let  $\mathfrak{p} \in \text{Spec}(R)$ . Then*

- (1) If  $bI^2 \subseteq \mathfrak{p}$ , then  $\mathbb{P} = \mathfrak{p} \oplus I \in \text{Spec}(R(I)_{a,b})$ .
- (2) If  $I \subseteq \mathfrak{p}$ , then  $\sqrt{\mathfrak{p}^e} = \mathbb{P}$ . In particular, if  $(R, \mathfrak{m})$  is local, then  $\sqrt{\mathfrak{m}^e} = \mathbb{M}$ .

PROOF. The proof of (1) is similar to Lemma ??(1) (note that, by assumption,  $ijk \in \mathfrak{p}$ , for any  $i, j \in I$ ). For the proof of (2), one only need to see, by the Lemma ??, that  $\mathbb{P}^2 \subseteq \mathfrak{p}^e \subseteq \mathbb{P}$ . □

The following auxiliary lemma plays a substantial role in the study of Cohen-Macaulay property.

**THEOREM 3.5.** *Let  $\mathfrak{p} \in \text{Spec}(R)$  and  $I \subseteq \mathfrak{p}$ . Then  $H_{\mathbb{P}}^i(R(I)_{a,b}) \cong H_{\mathfrak{p}}^i(R) \oplus H_{\mathfrak{p}}^i(I)$  for all  $i \in \mathbb{N}$ . In particular, If  $(R, \mathfrak{m})$  is local, then  $H_{\mathbb{M}}^i(R(I)_{a,b}) \cong H_{\mathfrak{m}}^i(R) \oplus H_{\mathfrak{m}}^i(I)$  for all  $i \in \mathbb{N}$ .*

**PROOF.** (1). By Lemma ??(2),  $\sqrt{\mathfrak{p}^e} = \mathbb{P}$ . Now using [3, Theorem 4.2.1], we have

$$H_{\mathbb{P}}^i(R(I)_{a,b}) = H_{\mathfrak{p}^e}^i(R(I)_{a,b}) \cong H_{\mathfrak{p}}^i(R(I)_{a,b}) \cong H_{\mathfrak{p}}^i(R \oplus I) \cong H_{\mathfrak{p}}^i(R) \oplus H_{\mathfrak{p}}^i(I).$$

□

As a consequence, using the theorem above, one can again reveal that  $\dim R(I)_{a,b} = \dim R$ , and that Cohen-Macaulay property of  $R(I)_{a,b}$  does not depend on the choice of  $a$  and  $b$  [?, Proposition 1.2]. The first one is clear, we insert the second in the following corollary.

**COROLLARY 3.6.** *Let  $(R, \mathfrak{m})$  be a local ring. The following conditions are equivalent:*

- (1)  $R(I)_{a,b}$  is a Cohen-Macaulay ring for all  $a, b \in R$ .
- (2)  $R(I)_{a,b}$  is a Cohen-Macaulay ring for some  $a, b \in R$ .
- (3)  $R$  is a Cohen-Macaulay ring and  $I$  is a maximal Cohen-Macaulay  $R$ -module.

**PROOF.** Let  $a, b \in R$  be arbitrary. By definition, we have  $R(I)_{a,b}$  is Cohen-Macaulay if and only if  $H_{\mathbb{M}}^i(R(I)_{a,b}) = 0$  for all  $i < \dim R(I)_{a,b}$  if and only if  $H_{\mathfrak{m}}^i(R) = 0 = H_{\mathfrak{m}}^i(I)$  for all  $i < \dim R$  if and only if  $R$  is Cohen-Macaulay and  $I$  is a maximal Cohen-Macaulay  $R$ -module. □

#### 4. Conclusion

The family  $R(I)_{a,b}$  provides a versatile framework that unifies several important extensions of rings along ideals. Its ideal-theoretic structure and homological invariants, such as local cohomology, can be explicitly expressed in terms of those of  $R$  and  $I$ . The independence of the Noetherian and Cohen–Macaulay properties from  $a, b$  illustrates the robustness of this construction.

#### References

1. Azimi, Y. and Shirmohammadi, N. (2022) *Families of generalized Cohen–Macaulay and filter rings*, J. Algebra Appl., **21**, 2250214.
2. Barucci, V., D’Anna, M. and Strazzanti, F. (2015) *A family of quotients of the rees algebra*, Comm. Algebra, **43**, 130–142.
3. Barucci, V., D’Anna, M. and Strazzanti, F. (2016) *Families of gorenstein and almost gorenstein rings*, Ark. Mat., **54** (2016) 321–338.
4. D’Anna, M. and Fontana, M. (2007) *An amalgamated duplication of a ring along an ideal: The basic properties*, J. Algebra Appl., **6**, 443–459.
5. Nagata, M. (1962) *Local Rings*, Interscience, New York.



## Vector space generated by the multiplicative commutators of a division ring

Asghar Madadi

Department of Mathematics, Za. C, Islamic Azad University, Zanjan, Iran.

Email: as.madadi@iau.ac.ir

---

**ABSTRACT.** Let  $D$  be a division ring with center  $F$  and  $T(D)$  be the vector space over  $F$  generated by all multiplicative commutators in  $D$ . In this paper, it is shown that if  $D$  is algebraic over  $F$  and  $\text{Char}(D) = 0$ , then  $D = T(D)$ . Among other results it is shown that in characteristic zero if  $T(D)$  is algebraic over  $F$ , then  $D$  is algebraic over  $F$ .

**Keywords:** Multiplicative Commutator, Trace Function, Division Ring.

**AMS Mathematics Subject Classification [2020]:** 17A35, 17C60.

---

### 1. Introduction

Throughout this paper  $D$  is a division ring with center  $F$ . An element of the form  $xyx^{-1}y^{-1} \in D$  is called a *multiplicative commutator*, and  $D'$  and  $[D, D]$  denote the multiplicative and additive commutator subgroup of  $D$ , respectively. Also we denote by  $T(D)$  the vector space generated by the set of all multiplicative commutators of  $D$  over  $F$ . An element  $a \in D$  is said to be *algebraic* over  $F$  if  $a$  satisfies a non-zero polynomial in  $F[x]$ . A set  $S \subseteq D$  is called *algebraic* if each of its elements is algebraic over  $F$ . When  $K$  is a finite dimensional extension of  $F$ , then we denote by  $\text{Tr}_{K/F}$ , the regular trace of  $K$  over  $F$ . If  $a \in D$ , then  $F(a)$  denotes the subfield of  $D$  generated by  $F$  and  $\{a\}$ . An element  $a \in D$  is said to be *radical* over  $F$  if there exists an integer  $n = n(a)$  such that  $a^n \in F$ . A set  $S \subseteq D$  is said to be radical over  $F$ , if each element of  $S$  is radical over  $F$ . Also we denote by  $\text{Char}(D)$  the characteristic of  $D$ .

The division ring generated by additive commutators or multiplicative commutators of  $D$  is the whole  $D$  [3, pp. 205, 211]. In the algebraic and zero characteristic case, it was proved that  $D$  is generated as a vector space over the center by the union of its additive commutators and the unity, see [1]. As a generalization we provide another generating structure for a division ring as a vector space generated by all multiplicative commutators over its center. We prove this in the algebraic case when the characteristic is zero. Besides, consider a special property  $P$  in a ring (for example commutativity, algebraicity or some finiteness conditions), there are a lot of studies in literature to specify a set or a substructure  $S$ , such that the property  $P$  for  $S$  implies the property  $P$  for the whole ring, for instance see [1, 2] and [4–6].

Most of these studies have focused on the set of multiplicative and additive commutators and their generating subgroups  $D'$  and  $[D, D]$  in division rings. We show that the subspace  $T(D)$  reflects some properties to the whole division ring.

## 2. Main results

We start this section with the following lemma.

**Lemma 2.1.** *Let  $D$  be a division ring with center  $F$ . Then for each algebraic element  $a \in D$ , there exists an element  $d \in T(D) \cap F(a)$  such that  $Tr_{F(a)/F}(a) = ad$ .*

The following lemma is an immediate consequence of the previous lemma.

**Lemma 2.2.** *Let  $D$  be a division ring with center  $F$  and  $a \in D$  be an algebraic element over  $F$  such that  $Tr_{F(a)/F}(a) \neq 0$ . Then  $a^{-1} \in T(D)$ .*

Now, we have the following result.

**Theorem 2.3.** *Let  $D$  be a division ring with center  $F$ . Then  $T(D)$  contains all separable elements of  $D$ .*

The following is a simple corollary of the previous result.

**Theorem 2.4.** *Let  $D$  be an algebraic division ring over its center  $F$ . If  $Char(D) = 0$ , then  $T(D) = D$ .*

A theorem due to Kaplansky [3, p. 246] states that if  $D$  is radical over  $F$ , then  $D = F$ . There are various kinds of generalizations of this theorem. The next theorem is another one.

**Theorem 2.5.** *Let  $D$  be a division ring with center  $F$ . If  $T(D)$  is radical over  $F$ , then  $D = F$ .*

Mahdavi-Hezavehi together with his colleagues in [6] conjectured that one could conclude the algebraicity of a division ring over its center from the algebraicity of all its multiplicative commutators. They were able to deduce the conjecture is true in the case  $D'$  is algebraic over the center, but in general the problem is still open. Also it is proved that when  $[D, D]$  is algebraic over  $F$ , then  $D$  is algebraic over  $F$ , see [1]. In what follows we give an affirmative answer to the conjecture when the center of division ring is uncountable.

**Theorem 2.6.** *Let  $D$  be a division ring with uncountable center  $F$  such that all of its multiplicative commutators are algebraic over  $F$ . Then  $D$  is algebraic over  $F$ . In particular, if  $D$  is a division ring with uncountable center  $F$  and  $T(D)$  is algebraic over  $F$ , then  $D$  is algebraic over  $F$ .*

Now, a question is naturally proposed: Whether a division ring is finite dimensional when  $T(D)$  is a finite dimensional vector space over  $F$ ? This is the content of the following theorem. This theorem is along of some analogous results which state that if each element of a specific set has a minimal polynomial of bounded degree, then  $D$  is finite dimensional

over  $F$  [6]. Note that finite dimensionality of  $T(D)$  as a vector space does not imply that all elements of  $T(D)$  or multiplicative commutators are algebraic over  $F$ .

**Theorem 2.7.** *Let  $D$  be a division ring with center  $F$ . If  $\dim_F T(D) = n < \infty$ , then  $\dim_F D < \infty$ .*

Now, we would like to prove that if  $D$  is a division ring with center  $F$  and  $T(D)$  is algebraic over  $F$ , then  $D$  is algebraic over  $F$ . Before stating the proof we need a lemma.

**Lemma 2.8.** *Let  $D$  be a division ring with center  $F$ ,  $T(D)$  is algebraic over  $F$  and  $\text{Char}(D) = 0$ . Then for any two algebraic elements  $a, b \in D$ , the set  $S = \{a + b, aba, a^2b\}$  is algebraic over  $F$ .*

**Theorem 2.9.** *Let  $D$  be a division ring with center  $F$  and  $\text{Char}(D) = 0$ . Then  $T(D)$  is algebraic over  $F$  if and only if  $D$  is algebraic over  $F$ .*

### 3. Conclusion

Let  $T(D)$  be the vector space over  $F$  generated by all multiplicative commutators in division ring  $D$  with center  $F$ . We provide another generating structure for a division ring as a vector space generated by all multiplicative commutators over its center. Also, we show that the subspace  $T(D)$  reflects some properties to the whole division ring.

### References

1. S. Akbari, M. Ariannejad, M. L. Mehrabadi, *On additive commutator groups in division rings*, Results Math. **33** (1998) 9-21.
2. M. A. Chebotar, Y. Fong, P. H. Lee, *On division rings with algebraic commutators of bounded degree*, Manuscripta Math. **113** (2004) 153-164.
3. T. Y. Lam, *A First Course in Noncommutative Rings*, 2nd edn., Graduate Texts in Mathematics, Vol. 131 (Springer, New York, 2001).
4. M. Mahdavi-Hezavehi, *On derived groups of division rings*, Commun. Algebra **23**(3) (1995) 913-926.
5. M. Mahdavi-Hezavehi, S. Akbari-Feyzaabaadi, *On a question of Herstein concerning commutators in division rings*, Rev. Roumaine Math. Pures Appl. **41**(3-4) (1996) 201-204.
6. M. Mahdavi-Hezavehi, S. Akbari-Feyzaabaadi, M. Mehraabaadi, H. Haji-Abolhassan, *On derived groups of division rings II*, Commun. Algebra **23**(8) (1995) 2881-2887.



## Homological properties of some quotients of the Rees Algebra

Nematollah Shirmohammadi

Faculty of Mathematics, Statistics and Computer Science,  
University of Tabriz, Tabriz, Iran.  
Email: shirmohammadi@tabrizu.ac.ir

**ABSTRACT.** We study a family of quotient rings of the Rees algebra that generalizes both Nagata's idealization and the amalgamated duplication of a ring. For this family, we establish characterizations of the generalized Cohen-Macaulay and filter ring properties.

**Keywords:**  $f$ -ring, Generalized Cohen-Macaulay ring, Rees algebra

**AMS Mathematics Subject Classification [2020]:** 13A30, 13H10

### 1. Introduction

Throughout this paper  $R$  is a commutative ring with unity, and  $I \neq 0$  a proper ideal of  $R$ . Let  $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$  be the Rees algebra associated to the ring  $R$  with respect to  $I$ . In [?], the authors introduce and study the family of quotient rings

$$R(I)_{a,b} = \mathcal{R}_+ / (I^2(t^2 + at + b)),$$

where  $(I^2(t^2 + at + b))$  is the contraction to  $\mathcal{R}_+$  of the ideal generated by  $t^2 + at + b$  in  $R[t]$ . This construction generalizes the amalgamated duplication of a ring along an ideal as well as the Nagata's idealization with respect to an ideal [?]. Cohen-Macaulay, Gorenstein and almost Gorenstein properties of the family investigated in [?].

In this paper, we characterize when  $R(I)_{a,b}$  is a generalized Cohen-Macaulay or  $f$ -ring, and show that these properties are independent of the choices of  $a$  and  $b$ .

### 2. Auxiliaries

Let  $R$  be a commutative ring with unity and  $I$  a proper ideal of  $R$ . Let  $t$  be an indeterminate. The Rees algebra associated to  $R$  and  $I$  is defined as the following graded subring of  $R[t]$ :

$$\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n \subseteq R[t].$$

It was shown in [?, Lemma 1.1] that the contraction to  $\mathcal{R}_+$  of the ideal generated by a monic polynomial  $f(t) \in R[t]$  is

$$(I^k f(t)) := f(t)R[t] \cap \mathcal{R}_+ = \{f(t)g(t); g(t) \in I^k \mathcal{R}_+\}.$$

Then each element of the factor ring  $\mathcal{R}_+/(I^k f(t))$  is represented by a unique polynomial of  $\mathcal{R}_+$  of degree  $< k$  [?, Lemma 1.2]. It follows that the ring  $R$  is a subring of  $\mathcal{R}_+/(I^k f(t))$ . Moreover, the class of  $t$  in  $R[t]/(f(t))$  is integral over  $R$  and over  $\mathcal{R}_+/(I^k f(t))$  as well; so that the ring extensions  $R \subseteq \mathcal{R}_+/(I^k f(t)) \subseteq R[t]/(f(t))$  are both integral, and the three rings have the same Krull dimension. In the sequel, We consider particular choices of the polynomial  $f(t)$  above and get known constructions.

Recall that the Nagata's idealization, or simply idealization, of  $R$  with respect to an ideal  $I$  of  $R$  (that could be defined for any  $R$ -module  $M$ ) is defined as the  $R$ -module  $R \oplus I$  endowed with the multiplication  $(r, i)(s, j) = (rs, rj + si)$  and it is denoted by  $R \rtimes I$ .

The duplication of  $R$  with respect to  $I$  is defined as follows:

$$R \rtimes I = \{(r, r + i) \mid r \in R, i \in I\} \subset R \times R.$$

Note that  $R \rtimes I \cong R \oplus I$  endowed with the multiplication  $(r, i)(s, j) = (rs, rj + si + ij)$ . The following proposition shows that, for particular choices of the polynomial  $f(t)$ , one gets the idealization as well as the duplication.

From now on, set  $R(I)_{a,b} := \mathcal{R}_+/(I^2(t^2 + at + b))$ , where  $a, b \in R$ ; so that one has  $R(I)_{0,0} \cong R \rtimes I$  and  $R(I)_{-1,0} \cong R \rtimes I$  [?, Proposition 1.4]. We denote each element of  $R(I)_{a,b}$  simply by  $r + it$  ( $r \in R, i \in I$ ) and show that many relevant properties are independent by the member of the family.

The following Lemma reveals some information about ideal-theoretic structure of  $R(I)_{a,b}$  [?, Lemma 2.1]. One notices that as  $R$ -module  $R(I)_{a,b} \cong R \oplus I$  and the natural injection  $R \rightarrow R(I)_{a,b}$  is a ring homomorphism. Hereafter we consider the contraction and extension with respect to this natural embedding. In the sequel, for an arbitrary ideal  $J$  of  $R$ , we use  $\mathbb{J}$  to denote  $J \oplus I$ .

LEMMA 2.1. *Let  $J$  be an ideal of  $R$ . Then*

- (1)  $J \oplus JI$  is an ideal of  $R(I)_{a,b}$ .
- (2)  $J^e = J \oplus JI$ , the extension is taken with respect to the natural embedding  $R \rightarrow R(I)_{a,b}$ .
- (3) If  $I \subseteq J$ , then  $\mathbb{J}^2 = J^2 \oplus JI$ .

Using the chain of inclusions  $R \subseteq R(I)_{a,b} \subseteq R[t]/(t^2 + at + b)$  and the fact that these extensions are integral, we can get information on  $\text{Spec}(R(I)_{a,b})$  with respect to  $\text{Spec}(R)$ .

PROPOSITION 2.2. *The followings hold:*

- (1) For each prime ideal  $P$  of  $R$ , there are at most two prime ideals of  $R(I)_{a,b}$  lying over  $P$ . Moreover, if  $t^2 + at + b$  is irreducible on  $R/\mathfrak{m}$  for any maximal ideal  $\mathfrak{m}$  of  $R$ , then there is exactly one prime ideal of  $R(I)_{a,b}$  lying over  $P$ .
- (2)  $R$  is local if and only if  $R(I)_{a,b}$  is local. In this case, the maximal ideal of  $R(I)_{a,b}$  is  $\mathbb{M} := \mathfrak{m} \oplus I$  (as  $R$ -module).
- (3) Let  $\mathfrak{p} \in \text{Spec}(R)$ . If  $bI^2 \subseteq \mathfrak{p}$ , then  $\mathbb{P} = \mathfrak{p} \oplus I \in \text{Spec}(R(I)_{a,b})$ .
- (4) If  $I \subseteq \mathfrak{p}$ , then  $\sqrt{\mathfrak{p}^e} = \mathbb{P}$ . In particular, if  $(R, \mathfrak{m})$  is local, then  $\sqrt{\mathfrak{m}^e} = \mathbb{M}$ .

PROOF. See [?, Propositions 1.9 and 2.1] for (1) and (2), and see [?, Lemma 2.2].  $\square$

The following theorem which is proved in [?, Lemma 2.3] plays a substantial role in the study of (generalized) Cohen–Macaulay property.

THEOREM 2.3. *Let  $\mathfrak{p} \in \text{Spec}(R)$  and  $I \subseteq \mathfrak{p}$ . Then  $H_{\mathbb{P}}^i(R(I)_{a,b}) \cong H_{\mathfrak{p}}^i(R) \oplus H_{\mathfrak{p}}^i(I)$  for all  $i \in \mathbb{N}$ . In particular, If  $(R, \mathfrak{m})$  is local, then  $H_{\mathbb{M}}^i(R(I)_{a,b}) \cong H_{\mathfrak{m}}^i(R) \oplus H_{\mathfrak{m}}^i(I)$  for all  $i \in \mathbb{N}$ .*

As a consequence, one can reprove the Cohen-Macaulay property of  $R(I)_{a,b}$  does not depend on the choices of  $a$  and  $b$ .

### 3. Main Results

In the sequel, we assume that  $(R, \mathfrak{m})$  is a Noetherian local ring,  $I$  is an ideal of  $R$ , and that  $M$  is a nonzero finitely generated  $R$ -module.

The notion of  $M$ -filter regular sequence is defined as a sequence  $x_1, \dots, x_n$  of elements in  $\mathfrak{m}$  such that  $x_i \notin \mathfrak{p}$  for all  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{i-1})M) \setminus \{\mathfrak{m}\}$  and for all  $i = 1, \dots, n$ . The filter depth,  $f\text{-depth}(I, M)$ , of  $I$  on  $M$  is defined as the length of any maximal  $M$ -filter regular sequence in  $I$ . To prove the next lemma use the following characterization of  $f\text{-depth}(I, M)$  (see [?, Section 3]):

$$f\text{-depth}(I, M) = \inf\{r \mid H_r^f(M) \text{ is not an Artinian } R\text{-module}\}.$$

LEMMA 3.1. *Let  $\mathfrak{p} \in \text{Spec}(R)$ . Then*

- (1)  $f\text{-depth}(\mathfrak{p}^e, R(I)_{a,b}) = \min\{f\text{-depth}(\mathfrak{p}, R), f\text{-depth}(\mathfrak{p}, I)\}$ .
- (2) *If  $\text{ht}(\mathfrak{p}) \leq f\text{-depth}(\mathfrak{p}, I)$ , (e.g. if  $\text{ht}(\mathfrak{p}) \leq \text{grade}(\mathfrak{p}, I)$ ), then  $f\text{-depth}(\mathfrak{p}^e, R(I)_{a,b}) = f\text{-depth}(\mathfrak{p}, R)$ .*

PROOF. See [?, Lemma 3.1]. □

The authors in [?] present two generalization of the notion of Cohen-Macaulay rings, namely generalized Cohen-Macaulay and filter rings ( $f$ -rings). In the following, we investigate when the family  $R(I)_{a,b}$  has these properties. Recall that the ring  $R$  of dimension  $d > 0$  is said to be *generalized Cohen-Macaulay* if  $H_m^i(R)$  is of finite length for all  $i < d$ . Theorem ?? in conjunction with [?, Lemma 2.12] follows the next theorem [?, Theorem 4.2].

THEOREM 3.2. *The following conditions are equivalent:*

- (1)  $R(I)_{a,b}$  is a generalized Cohen-Macaulay ring for all  $a, b \in R$ .
- (2)  $R(I)_{a,b}$  is a generalized Cohen-Macaulay ring for some  $a, b \in R$ .
- (3)  $R$  and  $I$  are generalized Cohen-Macaulay and  $\dim I \in \{0, \dim R\}$ .

The following result follows from Theorem ??.

COROLLARY 3.3. *The trivial extension  $R \times I$  is generalized Cohen-Macaulay if and only if  $R$  and  $I$  are generalized Cohen-Macaulay and  $\dim I \in \{0, \dim R\}$ .*

$f$ -modules were introduced in [?] as modules for which every system of parameters is a filter regular sequence. The ring  $R$  is called an  $f$ -ring if it is an  $f$ -module over itself.

Let  $\mathfrak{p}$  be a prime ideal of  $R$  and  $M$  be an  $R$ -module. By  $\text{ht}_M(\mathfrak{p})$ , we mean the Krull dimension of the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$ .

THEOREM 3.4. *Let  $R(I)_{a,b}$  be an  $f$ -ring. Then*

- (1)  $R$  is an  $f$ -ring.
- (2)  $f\text{-depth}(\mathfrak{p}, I) = \text{ht}(\mathfrak{p}) = f\text{-depth}(\mathfrak{p}, R)$  for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .
- (3)  $I$  is an  $f$ -module.
- (4)  $I_{\mathfrak{p}}$  is maximal Cohen-Macaulay for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .

PROOF. See [?, Theorem 4.4]. □

In the following theorem we provide characterization of  $R(I)_{a,b}$  to be an  $f$ -ring [?, Theorem 4.5].

THEOREM 3.5. *The following statements are equivalent:*

- (1)  $R(I)_{a,b}$  is an  $f$ -ring.
- (2)  $R$  is an  $f$ -ring and  $f\text{-depth}(\mathfrak{p}, I) = \text{ht}(\mathfrak{p})$  for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .
- (3)  $R$  is an  $f$ -ring and  $I_{\mathfrak{p}}$  is maximal Cohen-Macaulay for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .

COROLLARY 3.6.  $R \bowtie I$  is an  $f$ -ring if and only if  $R$  is an  $f$ -ring and  $f\text{-depth}(\mathfrak{p}, I) = \text{ht}(\mathfrak{p})$  for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$  if and only if  $R$  is an  $f$ -ring and  $I_{\mathfrak{p}}$  is maximal Cohen-Macaulay for any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ .

#### 4. Conclusion

The family  $R(I)_{a,b}$  unifies several important extensions of rings along ideals. The independence of the generalized Cohen–Macaulay and  $f$ -ring properties from  $a, b$  illustrates the robustness of this construction.

#### References

1. Azimi, Y. and Shirmohammadi, N. (2022) *Families of generalized Cohen–Macaulay and filter rings*, J. Algebra Appl., **21**, 2250214.
2. Barucci, V., D’Anna, M. and Strazzanti, F. (2015) *A family of quotients of the rees algebra*, Comm. Algebra, **43**, 130–142.
3. Barucci, V., D’Anna, M. and Strazzanti, F. (2016) *Families of gorenstein and almost gorenstein rings*, Ark. Mat., **54**, 321–338.
4. Cuong, N. T., Schenzel, P. and Trung, N. V. (1978) *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr., **85**, 57–73.
5. D’Anna, M. and Fontana, M. (2007) *An amalgamated duplication of a ring along an ideal: The basic properties*, J. Algebra Appl., **6**, 443–459.
6. Sahandi, P., Shirmohammadi, N. and Sohrabi, S. (2016) *Cohen-Macaulay and Gorenstein properties under the amalgamated construction*, Comm. Algebra, **44**, 1096–1109.



## Derived Length Bounds for Soluble-by-Finite Subgroups in Valued Division Algebras

Mehran Motiee<sup>1,\*</sup> and Behnam Talaei<sup>2</sup>

<sup>1</sup>Faculty of Basic Sciences, Babol Noshirvani University of Technology, Babol, Iran

Email: motiee@nit.ac.ir

<sup>2</sup>Faculty of Basic Sciences, Babol Noshirvani University of Technology, Babol, Iran

Email: behnamtalaei@nit.ac.ir

**ABSTRACT.** We study soluble-by-finite subgroups of the multiplicative group of a valued division algebra. When the residue division algebra is a field and the valuation is strongly tame, every soluble-by-finite subgroup of the multiplicative group is soluble with derived length at most three. In two important cases (totally ramified algebras and prime power-degree algebras without a primitive root of unity in the center) the bound improves to two. Several examples constructed via iterated Laurent-series and symbol algebras illustrate the necessity of the hypotheses.

**Keywords:** Division algebras, soluble-by-finite subgroups, valuation theory

**AMS Mathematics Subject Classification [2020]:** 16K20 (primary); 16W25, 12J10 (secondary)

### 1. Introduction

Let  $D$  be a finite-dimensional division algebra with center  $F$ . The dimension of  $D$  over  $F$ , denoted  $[D : F]$ , is always a perfect square, and the degree of  $D$  is defined as  $\deg(D) = \sqrt{[D : F]}$ . The multiplicative group  $D^*$  of non-zero elements of  $D$  is a rich object of study in algebra, as it encodes significant information about the structure of  $D$  and its subfields. Investigating subgroups of  $D^*$  is a classical problem with deep connections to various areas of mathematics, including Galois theory, group theory, and noncommutative algebra. In particular, soluble-by-finite subgroups—those subgroups that possess a soluble normal subgroup of finite index—are of special interest due to their close ties to the Galois groups of maximal subfields of  $D$  [2]. Notably, Shirvani proved that every soluble-by-finite subgroup of  $D^*$  contains an abelian subgroup of finite index, with the index bounded by a function of the degree of  $D$  [3].

These subgroups often reflect the arithmetic and algebraic properties of  $D$ , and their structure can provide insights into the division algebra's behavior under various constraints.

\*Speaker.

The introduction of a valuation on  $D$  adds a powerful analytic tool to this investigation. A valuation imposes a topological and algebraic structure that interacts with the group-theoretic properties of  $D^*$ . By considering valued division algebras  $(D, v)$ , where  $v : D^* \rightarrow \Gamma$  is a valuation to a totally ordered abelian group  $\Gamma$ , we can leverage valuation-theoretic properties—such as the residue division ring  $\overline{D}$  and the value group  $\Gamma_D$ —to constrain the structure of subgroups of  $D^*$ . This paper focuses on soluble-by-finite subgroups of  $D^*$  in the presence of a valuation, exploring how conditions such as strong tameness, the residue division algebra being a field, or total ramification influence their solubility and derived length. Our main results establish uniform bounds on the derived length of such subgroups under specific valuation-theoretic hypotheses, and we provide examples to demonstrate the necessity of these conditions. These findings contribute to the broader understanding of how valuation structures interplay with the group-theoretic properties of division algebras.

## 2. Preliminaries

In this section, we provide the foundational definitions and concepts used throughout the paper, including valuations, residue division rings, symbol algebras, and related notions for valued division algebras.

**DEFINITION 2.1 (Valuation).** Let  $D$  be a division algebra. A *valuation* on  $D$  is a map  $v : D^* \rightarrow \Gamma$ , where  $\Gamma$  is a totally ordered abelian group, satisfying the following properties for all  $a, b \in D^*$ :

- (1)  $v(ab) = v(a) + v(b)$ ,
- (2)  $v(a + b) \geq \min\{v(a), v(b)\}$ .

We extend  $v$  to  $D$  by defining  $v(0) = \infty$ , where  $\infty > \gamma$  for all  $\gamma \in \Gamma$ . The *value group* is  $\Gamma_D = v(D^*)$ . The *valuation ring*  $O_D = \{a \in D \mid v(a) \geq 0\}$  is a subring of  $D$ , and its unique maximal ideal is  $M_D = \{a \in D \mid v(a) > 0\}$ .

**DEFINITION 2.2 (Residue Division Ring).** The *residue division ring* of a valued division algebra  $(D, v)$  is the quotient  $\overline{D} = O_D/M_D$ . This is a division algebra over the residue field  $\overline{F} = O_F/M_F$ , where  $O_F = O_D \cap F$  and  $M_F = M_D \cap F$  for the center  $F$  of  $D$ . The residue division ring  $\overline{D}$  inherits a structure from  $D$ , and its properties (e.g., whether it is a field or a noncommutative division algebra) play a critical role in the structure of subgroups of  $D^*$ .

We say  $D$  is *strongly tame* if the residue characteristic of  $\overline{F}$  does not divide the degree  $n$ . A division algebra  $D$  is *totally ramified* if  $[\overline{D} : \overline{F}] = 1$ , meaning  $\overline{D} = \overline{F}$ , and the ramification index  $|\Gamma_D : \Gamma_F|$  equals  $n$ . The center  $F$  is *Henselian* if the valuation  $v|_F$  satisfies Hensel's lemma, ensuring unique lifting of factorizations from the residue field to  $O_F$ .

**DEFINITION 2.3 (Symbol Algebra).** Let  $F$  be a field containing a primitive  $n$ -th root of unity  $\omega_n$ , and let  $a, b \in F^*$ . The *symbol algebra*  $(a, b)/(F, \omega_n)$  is an  $F$ -central simple algebra generated by elements  $i$  and  $j$  subject to the relations:

$$i^n = a, \quad j^n = b, \quad ji = \omega_n ij.$$

The dimension of  $(a, b)/(F, \omega_n)$  over  $F$  is  $n^2$ , and it is a central division algebra under certain conditions on  $a$  and  $b$  [1, §11]. Symbol algebras are fundamental in constructing examples of division algebras with specific properties, such as noncommutative residue division rings.

### 3. Main results

The key theorems of this study are outlined below.

**THEOREM 3.1.** *Let  $D$  be a valued  $F$ -central division algebra of degree  $n$ . If  $D$  is strongly tame and the residue division algebra  $\overline{D}$  is a field, then every soluble-by-finite subgroup of  $D^*$  is soluble with derived length at most 3.*

**THEOREM 3.2.** *Let  $D$  be a valued  $F$ -central division algebra of degree  $n$ . If  $D$  is strongly tame and totally ramified, then every soluble-by-finite subgroup of  $D^*$  is soluble with derived length at most 2.*

**THEOREM 3.3.** *Let  $D$  be a valued  $F$ -central division algebra of prime power degree  $p^m$ . If  $D$  is strongly tame and  $F$  does not contain a primitive  $p$ -th root of unity, then every soluble-by-finite subgroup of  $D^*$  is soluble with derived length at most 2.*

**REMARK 3.4.** The hypotheses (residue ring a field, strong tameness, or total ramification) are used critically in the proofs; examples in Section 4 show they cannot be dropped in general.

### 4. Examples

We summarize representative examples showing failure of the above theorems when  $\overline{D}$  is not a field or tameness does not hold.

For a division ring  $D$ , define the iterated Laurent-series division ring  $D((x_1)) \cdots ((x_n))$  by successively forming Laurent-series extensions. An element of  $D((x_1)) \cdots ((x_n))$  is a formal series of the form  $\sum_{k \in \mathbb{Z}^n} a_k x_1^{k_1} \cdots x_n^{k_n}$ , where  $a_k \in D$ ,  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ , and the support  $\{k \mid a_k \neq 0\}$  is well-ordered with respect to the right-to-left lexicographic ordering on  $\mathbb{Z}^n$ . The *right-to-left lexicographic valuation*  $v : D((x_1)) \cdots ((x_n))^* \rightarrow \mathbb{Z}^n$  is defined as follows: for a non-zero element  $f = \sum_{k \in \mathbb{Z}^n} a_k x_1^{k_1} \cdots x_n^{k_n}$ , let  $k = (k_1, \dots, k_n)$  be the smallest multi-index in the right-to-left lexicographic order such that  $a_k \neq 0$ ; then  $v(f) = k$ . The value group is  $\Gamma = \mathbb{Z}^n$ , ordered lexicographically, and the residue division ring is  $\overline{D((x_1)) \cdots ((x_n))} \cong D$ . Thus, starting with a non-field division ring  $D$  yields valued division algebras with a non-field residue, which can lead to non-soluble subgroups in  $D^*$ .

Let  $F$  contain primitive roots of unity as needed and form symbol algebras  $(a, b)/(F, \omega_n)$ . Extending these to iterated Laurent centers and taking tensor products produces valued central division algebras  $T(k; n_1, \dots, n_r)$  whose value groups and residues can be arranged to violate the hypotheses (e.g. make  $\overline{D}$  noncommutative).

**EXAMPLE 4.1.** Let  $H = (-1, -1)/\mathbb{R}$  (real quaternions). Set  $D_1 = H((x_1))((y_1))$ . Tensoring with a suitable totally-ramified symbol algebra  $T(\mathbb{R}; 2)$  yields  $D$  with  $\overline{D} \cong H$  (not a field). Since  $H^*$  contains a subgroup isomorphic to  $SL_2(\mathbb{Z}_5)$  (finite but non-solvable),  $D^*$  contains a soluble-by-finite non-solvable subgroup.

**EXAMPLE 4.2.** For any positive integer  $d$  construct a central division  $k$ -algebra  $D$  of degree  $d$  whose  $D^*$  contains a soluble subgroup of derived length  $dl(d) + 1$  [4]. Form  $M = D \otimes_F T(k; (m, \dots, m))$ . Then  $\overline{M} = D$  while the ramification index  $|\Gamma_M : \Gamma_F| = m^{2r}$  can be made arbitrarily large, showing no control of solvable length by ramification alone.

**EXAMPLE 4.3.** Let  $L/F$  be cyclic of degree  $m = p^n$  with generator  $\sigma$ . Define  $D = L((x))$  with relation  $xa = \sigma(a)x$ . Then  $D$  is central with center  $F((x^m))$ ,  $\overline{D} \cong L$  (a field), and the subgroup  $G = L^*\langle x \rangle$  is solvable of derived length 2.

These examples illustrate the necessity of the hypotheses in the main theorems. Example 1 demonstrates that when the residue division algebra  $\bar{D}$  is not a field, soluble-by-finite subgroups of  $D^*$  may fail to be soluble, directly contrasting Theorem 3.1. Example 2 shows that without the strong tameness condition or control over ramification, the derived length of soluble subgroups can be arbitrarily large, exceeding the bounds of 2 or 3 provided in Theorems 3.1, 3.2, and 3.3. In contrast, Example 3 provides a construction where the derived length reaches 2 under conditions aligning with the theorems, suggesting that the bounds are tight and cannot be improved without additional assumptions.

## 5. Conclusion

We proved uniform derived-length bounds for soluble-by-finite subgroups of  $D^*$  under valuation-theoretic hypotheses (residue field and tameness), and displayed constructions that demonstrate necessity of these hypotheses. The author conjectures the bound 3 in Theorem 3.1 might be improvable to 2 in full generality.

## References

1. P. K. Draxl, *Skew fields* (London Mathematical Society Lecture Note Series, Vol. 81), Cambridge University Press (1983).
2. R. Hazrat, M. Mahdavi-Hezavehi and M. Motiee, *Multiplicative groups of division rings*, Proc. R. Ir. Acad. Vol. 114A, No. 1 (2014), pp. 37–114.
3. M. Shirvani, *On soluble-by-finite subgroups of division algebras*, J. Algebra 294 (2005), 255–277.
4. B. A. F. Wehrfritz, *The derived length of a soluble subgroup of a finite-dimensional division algebra*. *Glasg. Math. J.*, Vol. 4 (2006), 119–124.



## Conditions on Lie ideals in rings

Asghar Madadi

Department of Mathematics, Za. C, Islamic Azad University, Zanjan, Iran.

Email: as.madadi@iau.ac.ir

**ABSTRACT.** We show that if  $I$  is a non-central Lie ideal of a ring  $R$  with  $\text{Char}(R) \neq 2$ , such that all of its non-zero elements are invertible, then  $R$  is a division ring. We prove that if  $R$  is an  $F$ -central algebra and  $I$  is a Lie ideal without zero divisor such that the set of multiplicative cosets  $\{aF \mid a \in I\}$  is of finite cardinality, then either  $R$  is a field or  $I$  is central. We show the only non-central Lie ideal without zero divisor of a non-commutative central  $F$ -algebra  $R$  with  $\text{Char}(R) \neq 2$  and radical over the center is  $[R, R]$ , the additive commutator subgroup of  $R$ .

**Keywords:** Division ring, Lie ideal, Quaternion algebra.

**AMS Mathematics Subject Classification [2020]:** 16K40, 17A35.

### 1. Introduction

Throughout this paper  $R$  is a unitary ring with center  $Z(R)$  and  $F$  is a field. For a pair of elements  $a, b$  of  $R$  we denote by  $[a, b] = ab - ba$  the Lie product of  $a$  and  $b$ . Also elements of  $R$  with such a representation as  $ab - ba$  for two elements  $a, b$  in  $R$  are called additive commutators. For two subsets  $A$  and  $B$  let  $A \setminus B = \{a \in A \mid a \notin B\}$ . An additive subgroup  $I$  of  $R$  is said to be a Lie ideal of  $R$  if  $[r, a] \in I$  for every  $r \in R$  and  $a \in I$ . Also, for subsets  $A, B$  of  $R$  we denote by  $[A, B]$  the additive subgroup of  $R$  generated by all  $[a, b]$  with  $a \in A$  and  $b \in B$ . By this notation  $[R, R]$  is called the additive commutator subgroup of  $R$ . An element  $a \in R$  is said to be radical over the center or  $Z(R)$  if there exists a positive integer  $n = n(a)$  such that  $a^n \in Z(R)$ . A subset  $S \subseteq R$  is said to be radical over  $Z(R)$ , if each element of  $S$  is radical over  $Z(R)$ . We denote the characteristic of  $R$  by  $\text{Char}(R)$ . For a subset  $S \subseteq R$ , the centralizer of  $S$  in  $R$  is defined by  $C_R(S) = \{r \in R \mid rs = sr \text{ for all } s \in S\}$ . An element  $a$  of  $R$  is called a zero divisor if there exists a non-zero  $b \in R$  such that either  $ab = 0$  or  $ba = 0$ . We say the ring  $R$  is an  $F$ -central algebra if  $R$  is an algebra over  $F$  and  $Z(R) = F$ . A derivation on  $R$  is an additive group homomorphism  $d : R \rightarrow R$  satisfying  $d(r_1 r_2) = (d(r_1))r_2 + r_1(d(r_2))$ .

In Theorem 1.1, stated below, Bergen et al. [2] proved that a unitary ring  $R$  with a derivation such that all of its non-zero images are invertible, except for a special case, either is a division ring  $D$  or  $M_2(D)$ , the ring of  $2 \times 2$  matrices over a division ring  $D$ . Bergen and Carini gave a generalization of this result to the case of a Lie ideal. More precisely, a semiprime unitary ring  $R$  with a derivation such that all of its non-zero images

over a non-central Lie ideal are invertible, either is a division ring  $D$  or  $M_2(D)$  for a division ring  $D$ .

In this paper, we study the rings with specific Lie ideals. Our study begins with two theorems having a similar idea as above. We prove that when the non-zero additive commutators or the non-zero elements of a Lie ideal  $I$  of a ring  $R$  are invertible then  $R$  is a division ring. As a consequence we present a commutativity condition over a ring. We show that if  $I$  is a Lie ideal of an  $F$ -central algebra  $R$  without zero divisor such that the set of multiplicative cosets  $\{aF \mid a \in I\}$  is of finite cardinality, then either  $R$  is a field or  $I$  is central. We prove that any division ring  $D$  with  $Char(D) \neq 2$  which contains a non-central Lie ideal  $I$  without zero divisor and radical over the center is isomorphic to a generalized quaternion algebra and  $I = [D, D]$ . At last we prove that when  $R$  is an  $F$ -central algebra and  $Char(R) \neq 2$  and  $I$  is a Lie ideal without zero divisor, if the residual additive group  $(\frac{I+F}{F}, +)$  is of finite cardinality, then  $I$  is central. First, we recall the following theorems.

**Theorem 1.1.** [2] *Let  $R$  be a ring with 1 and  $d \neq 0$  a derivation of  $R$  such that, for each  $x \in R$ ,  $d(x) = 0$  or  $d(x)$  is invertible in  $R$ . Then  $R$  is either*

- (i) *a division ring  $D$ , or*
- (ii)  *$M_2(D)$  for some division ring  $D$  or*
- (iii)  *$D[x]/(x^2)$ , for some division ring  $D$ , where  $char D = 2$ ,  $d(D) = 0$  and  $d(x) = 1 + ax$  for some  $a$  in the center  $Z(D)$  of  $D$ .*

*Furthermore, if  $2R \neq 0$  then  $R = M_2(D)$  is possible if and only if  $D$  does not contain all quadratic extensions of  $Z(D)$ , the center of  $D$ ; equivalently if and only if some element in  $Z(D)$  is not a square in  $D$ .*

**Theorem 1.2.** [5] *Let  $D$  be a division ring with center  $F$ , such that  $(xy - yx)^{n(x,y)} \in F$ ,  $n(x, y) \geq 1$  for all  $x, y \in D$ , then  $\dim_F(D) \leq 4$ .*

**Theorem 1.3.** [4, p. 5] *Let  $D$  be a division algebra with center  $F$  and  $Char(D) \neq 2$  and let  $I$  be a Lie ideal of  $D$ . Then either  $I \subseteq F$  or  $[D, D] \subseteq I$ .*

## 2. Main results

We show that the invertibility condition on some special subsets or substructures of a ring may implies the invertibility of all elements of the ring. In particular, we show that a ring with all its non-zero additive commutators invertible is a division ring. This is the content of the following theorem, which is really an easy consequence of Theorem 1.1 above.

**Theorem 2.1.** *Let  $R$  be a non-commutative ring with center  $Z(R)$ , and with all its non-zero additive commutators invertible. Then  $R$  is a division ring.*

Clearly  $[R, R]$  is a Lie ideal in any ring  $R$ , containing all additive commutators. When  $R$  is a division ring with  $Char(R) \neq 2$ , then by Theorem 1.3 all non-central Lie ideals contain  $[R, R]$ , but in general there is not a clear relation between a Lie ideal and  $[R, R]$ . So one may ask what would be the case when the same condition, as above theorem, be valid on a Lie ideal of a ring. In the following theorem we show that only a division ring may contains a Lie ideal such that all of its non-zero elements are invertible.

**Theorem 2.2.** *Let  $R$  be a ring and  $Char(R) \neq 2$ . If  $I$  is a non-central Lie ideal of  $R$ , all of whose non-zero elements are invertible, then  $R$  is a division ring.*

We use the previous theorem to present a commutativity condition in terms of Lie ideals without zero divisor.

**Theorem 2.3.** *Let  $F$  be a field and  $R$  be an  $F$ -central algebra with a Lie ideal  $I$  without zero divisor. If the set of multiplicative cosets  $\{aF \mid a \in I\}$  is of finite cardinality, then either  $R$  is a field or  $I$  is central.*

To present our next result, we need to recall the following:  
Let  $F$  be a field with  $\text{Char}(F) \neq 2$ . By [1] when  $R$  is a finite dimensional  $F$ -algebra, then  $[R, R]$  is a hyperplane in  $R$ . Consider the generalized quaternion algebra

$$D = \left( \frac{a, b}{F} \right) = \{ \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in F \},$$

where  $i^2 = a$ ,  $j^2 = b$ ,  $ij = k$  and  $a, b \in F$  [3, p. 136]. Then one can easily show that  $[D, D] = \text{span}(\{i, j, k\}) = \{ \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_1, \alpha_2, \alpha_3 \in F \}$ . In what follows we show that only generalized quaternion algebras  $D$  may contain non-central radical Lie ideal  $I$  without zero divisor and in this case  $I = [D, D]$ .

**Theorem 2.4.** *Let  $R$  be a non-commutative central  $F$ -algebra and  $\text{Char}(R) \neq 2$ . If  $I$  is a non-central Lie ideal of  $R$  without zero divisor and radical over  $F$ , then  $I = [R, R]$  and  $R$  is a generalized quaternion algebra.*

We need the following technical Lemma to give our next result.

**Lemma 2.5.** *Let  $R$  be a ring and  $a, y \in R$ , such that  $a$  is not zero divisor. If  $ay \in Z(R)$ , then  $ay = ya$ .*

**Theorem 2.6.** *Let  $R$  be a central  $F$ -algebra with  $\text{Char}(R) \neq 2$  and let  $I$  be a Lie ideal of  $R$  without zero divisor. If the residual additive group  $(\frac{I+F}{F}, +)$  is of finite cardinality, then  $I$  is central.*

### 3. Conclusion

In this paper, we study the rings with specific Lie ideals. We prove that when the non-zero additive commutators or the non-zero elements of a Lie ideal  $I$  of a ring  $R$  are invertible then  $R$  is a division ring. As a consequence we present a commutativity condition over a ring.

### References

1. S. Akbari, M. Ariannejad and M. L. Mehrabadi, *On additive commutator groups in division rings*, Results Math. **33** (1998) 9-21.
2. J. Bergen, I. N. Herstein and C. Lanski, *Derivations with invertible values*, Canad. J. Math. **35**(2) (1983) 300-310.
3. B. Farb and R. K. Dennis, *Non-Commutative Algebra*, GTM, No. 144 (Springer-Verlag, New York, 1993).
4. I. N. Herstein, *Topics in Ring Theory*, Chicago Lectures in Mathematics (The university of Chicago Press, London, 1969).
5. I. N. Herstein, C. Procesi and M. Schacher, *Algebraic valued functions on non-commutative rings*, J. Algebra **36** (1975) 128-150.



## Algorithmic Framework for Symbolic-Numeric Sign Analysis of Mathematical Expressions

Mahdi Dehghani Darmian<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.

Email: m.dehghanidarmian@ipm.ir

**ABSTRACT.** This paper presents a comprehensive algorithmic framework for analyzing the sign behavior of mathematical expressions over the real domain. The **Maple** implementation combines symbolic computation, numerical analysis, and interval arithmetic to determine whether an expression is always positive, always negative, or changes sign, while precisely identifying the intervals of positivity and negativity. The algorithm offers a robust solution that handles a wide range of mathematical expressions, including polynomials, radicals, trigonometric functions, and rational expressions.

**Keywords:** sign analysis, symbolic-numeric computation, root finding, interval arithmetic

**AMS Mathematics Subject Classification [2020]:** 68W30 and 65G40

### 1. Introduction

The determination of sign properties of mathematical expressions constitutes a fundamental problem in computational mathematics with far-reaching applications across scientific computing, formal verification, optimization, and control theory. The ability to algorithmically ascertain whether an expression maintains constant sign or exhibits sign changes over a domain is crucial for proving inequalities, establishing stability conditions, identifying feasible regions in optimization, and guiding numerical methods away from singularities. The problem appears deceptively simple: given a real-valued function  $f(x)$ , determine its sign behavior over  $\mathbb{R}$ . However, the computational challenges are substantial, encompassing symbolic complexity of expressions, numerical instability near roots, completeness guarantees, and efficient handling of diverse function classes, including polynomials, rational functions, radicals, and transcendental elements.

The literature on sign analysis and real root finding spans several decades, with contributions from computer algebra, numerical analysis, and real algebraic geometry.

**Computer Algebra Approaches:** The foundational work in computer algebra established robust methods for polynomial systems. [1] provided comprehensive algorithms for real algebraic geometry, including cylindrical algebraic decomposition (CAD) for quantifier elimination, which can solve sign determination problems for polynomial systems. [3]

\*Speaker.

presented fundamental algorithms for polynomial root isolation, building upon classical techniques like Sturm sequences and Descartes' rule of signs. For symbolic simplification and manipulation, [2] remains the authoritative reference, covering canonical forms and expression normalization essential for reliable sign analysis.

**Numerical Analysis Perspectives:** From the numerical standpoint, [5] established the definitive framework for analyzing numerical stability and developing robust evaluation strategies near roots and singularities. The comprehensive treatment of root-finding algorithms in [4] provides practical hybrid methods combining bisection, Newton-Raphson, and other iterative techniques with careful attention to convergence guarantees and error control.

**Hybrid Symbolic-Numeric Methods:** More recently, the limitations of purely symbolic and purely numerical approaches have motivated hybrid strategies. [6] surveyed advances in real quantifier elimination and decision methods that blend algebraic and numerical techniques and demonstrated the power of combining symbolic manipulation with numerical evaluation through their work on special functions, establishing patterns for robust computational mathematics.

**Interval Arithmetic and Validated Computation:** The development of interval arithmetic by Moore and others provided tools for rigorous sign analysis with error bounds, though practical implementations often face challenges with dependency problems and computational complexity.

**1.1. Contributions and Paper Organization.** Despite these advances, a unified framework that seamlessly integrates symbolic and numeric approaches while providing completeness guarantees for diverse expression classes remains an open challenge. Existing methods typically excel in one domain—either handling polynomials symbolically with CAD or addressing transcendental functions numerically—but struggle with mixed expressions common in practical applications. This paper presents a hybrid symbolic-numeric algorithm designed to determine the sign (positive, negative, or zero) of a mathematical expression  $f(x)$  over the entire real line. The core insight, rooted in classical analysis, is that for continuous functions, sign changes can only occur at points where the function is zero or undefined. Therefore, by identifying all these "critical points," the real number line can be partitioned into intervals within which the function's sign is constant. The sign on each interval is then determined by evaluating the function at a single, well-chosen test point. The master algorithm, `ANALYZEEXPRESSION`, integrates several sub-algorithms for simplification, root-finding, and interval analysis to implement this principle in a robust and computationally tractable manner, handling a wide range of functions from polynomials to transcendental expressions. Consequently, this paper presents a comprehensive algorithmic framework that bridges this gap through several key contributions:

- A hybrid symbolic-numeric architecture that leverages the strengths of both paradigms while mitigating their individual limitations
- A complete sign analysis algorithm with theoretical guarantees for continuous functions based on the Intermediate Value Theorem
- Handling of diverse expression classes through adaptive root-finding strategies
- Practical numerical safeguards for stable evaluation near roots and singularities
- Extensive experimental validation across polynomial, rational, radical, and transcendental expressions

Due to space constraints, we only include the main `Maple` procedure implementing our algorithms. Illustrative examples demonstrate their functionality.

## 2. Core Algorithm: ANALYZEEXPRESSION

The strategy of the ANALYZEEXPRESSION algorithm is a direct application of the Intermediate Value Theorem (IVT) from calculus. The IVT states that if a continuous function  $f$  takes values  $f(a)$  and  $f(b)$  at two points  $a$  and  $b$ , then it must take any value between  $f(a)$  and  $f(b)$  at some point in the interval  $(a, b)$ . A critical corollary for sign analysis is that if  $f$  is continuous on an interval  $I$  and  $f(x) \neq 0$  for all  $x \in I$ , then  $f$  cannot change sign on  $I$ ; it must be either entirely positive or entirely negative [5].

The algorithm's pipeline, therefore, is:

- (1) **Simplify** the expression to a canonical form to facilitate subsequent analysis.
- (2) **Identify Critical Points** where the sign *could* change (real roots, poles).
- (3) **Partition** the real line into intervals using these critical points as boundaries.
- (4) **Classify** the sign of  $f(x)$  on each interval by safe evaluation at an interior point.
- (5) **Synthesize** the results into a complete sign profile.

---

### Algorithm 1 Master Procedure: ANALYZEEXPRESSION

---

**Require:** A mathematical expression  $f$  in variable  $x$

**Ensure:** A description of the sign of  $f(x)$  for all  $x \in \mathbb{R}$

- 1:  $f \leftarrow \text{SIMPLIFY}(f)$
  - 2: **if** ISCONSTANT( $f$ ) **then**
  - 3:     **return** CONSTANTSIGNANALYSIS( $f$ )
  - 4: criticalPoints  $\leftarrow$  FINDREALROOTS( $f, x$ )
  - 5: intervals  $\leftarrow$  PARTITIONREALLINE(criticalPoints)
  - 6: signMap  $\leftarrow$  CLASSIFYINTERVALS( $f, \text{intervals}, x$ )
  - 7: **return** SYNTHESISRESULTS(signMap, criticalPoints)
- 

The algorithm employs several key subroutines to implement its sign analysis pipeline. The SIMPLIFY procedure normalizes expressions to canonical forms using algebraic simplification rules and trigonometric identities [2]. FINDREALROOTS identifies all points where sign changes may occur by combining symbolic methods (such as Sturm sequences for polynomial components [1, 3]) with validated numerical techniques (including interval bisection for transcendental functions [4, 5]). PARTITIONREALLINE divides the real number line into intervals bounded by the detected critical points. CLASSIFYINTERVALS determines the sign on each interval through safe evaluation at carefully chosen interior points, employing numerical safeguards to avoid instability near roots and singularities [5]. Finally, SYNTHESISRESULTS compiles the individual interval classifications into a complete sign profile. The most computationally intensive step is FINDREALROOTS, which uses a hybrid symbolic-numeric approach [6] to ensure all roots are captured while maintaining computational tractability across diverse expression classes.

## 3. Running Examples

This section presents illustrative examples of the algorithm's output, demonstrating its ability to handle diverse mathematical expressions.

**Example 1: Rational Expression with Trigonometric Component.** For the expression  $\frac{\sin^2(x)-1}{x^2-4}$ , the algorithm produces:

```
> AnalyzeExpression((sin(x)^2-1)/(x^2-4));
```

```
Positive on: (-2, 2)
Negative on: (-∞, -2) ∪ (2, ∞)
Undefined at: {-2, 2}
```

**Example 2: Exponential Rational Expression.** For the expression  $\frac{e^x-2}{e^x-4}$ , the algorithm produces:

```
> AnalyzeExpression((exp(x)-2)/(exp(x)-4));
```

```
Positive on: (-∞, ln(2)] ∪ (2*ln(2), ∞)
Negative on: (ln(2), 2*ln(2))
Undefined at: {2*ln(2)}
Zero points: {ln(2)}
```

**Example 3: Polynomial Rational Expression.** For the expression  $\frac{x^2-7x+12}{x^2-4}$ , the algorithm produces:

```
> AnalyzeExpression((x^2-7*x+12)/(x^2-4));
```

```
Positive on: (-∞, -2) ∪ (2, 3] ∪ [4, ∞)
Negative on: [3, 4] ∪ (-2, 2)
Undefined at: {-2, 2}
Zero points: {3, 4}
```

**Example 4: Exponential Expression with Radical Denominator.** For the expression  $\frac{e^x-1}{\sqrt{x^2-4}}$ , the algorithm produces:

```
> AnalyzeExpression((exp(x)-1)/sqrt(x^2-4));
```

```
Positive on: (2, ∞)
Negative on: (-∞, -2)
Undefined at: {-2, 2}
Zero points: { }
```

These examples demonstrate the algorithm's capability to precisely identify intervals of positivity and negativity, locate zeros, and detect points of undefined behavior across different classes of mathematical expressions.

## References

1. Basu, S., Pollack, R., Roy, M. F. *Algorithms in Real Algebraic Geometry*. Springer, 2nd edition, 2006.
2. Geddes, K. O., Czapor, S. R., and Labahn, G. *Algorithms for Computer Algebra*. Kluwer Academic Publishers, 1992.
3. Collins, G. E. *Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition*. Second GI Conf. Automata Theory and Formal Languages, Springer LNCS 33, 1975.
4. Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press, 3rd edition, 2007.
5. Stoer, J. and Bulirsch, R. *Introduction to Numerical Analysis*. Springer, 3rd edition, 2002.
6. Sturm, T. A survey of some methods for real quantifier elimination, decision, and satisfiability and their applications. *Mathematics in Computer Science*, 3(4):465–492, 2009.



## Nakayama Lemma in the Category $\mathbf{Act-S}$

Kamal Ahmadi

Assistant Professor, Department of Mathematics, Za. C., Islamic Azad University, Zanjan, Iran.

Email: kamal.ahmadi.math@iau.ac.ir

**ABSTRACT.** In this talk, we are going to study some forms of Nakayama's lemma on  $\mathbf{Act-S}$ . We attempt to present the similarities and differences of this lemma and some of its applications for acts and modules.

**Keywords:** monoid, Nakayama's lemma,  $S$ -act

**AMS Mathematics Subject Classification [2020]:** 20M30, 20M50

### 1. Introduction

Let  $S$  be a monoid with identity 1. Recall that a (*right*)  $S$ -act is a non-empty set  $A$  equipped with a map  $\mu : A \times S \rightarrow A$  called its action, such that, denoting  $\mu(a, s)$  by  $as$ , we have  $a1 = a$  and  $a(st) = (as)t$ , for all  $a \in A$ , and  $s, t \in S$ . An element  $\theta \in A$  is called a *zero element* of  $A$  if  $\theta s = \theta$  for every  $s \in S$ . Let  $A$  be an  $S$ -act and  $B \subseteq A$  a non-empty subset. Then  $B$  is called a *subact* of  $A$  if  $bs \in B$  for all  $s \in S$  and  $b \in B$ . In particular, if  $I$  is a (proper) ideal of  $S$ , then

$$(1) \quad AI := \{as \mid a \in A, s \in I\}$$

is a subact of  $A$ . An equivalence relation  $\rho$  on an  $S$ -act  $A$  is called a *congruence* on  $A$  if  $apa'$  implies  $(as)\rho(a's)$  for  $a, a' \in A$  and  $s \in S$ . Any subact  $B \subseteq A$  defines the *Rees congruence*  $\rho_B$  on  $A$ , by setting  $a\rho_B a'$  if  $a, a' \in B$  or  $a = a'$ . We denote the resulting factor act by  $A/B$  and call it the *Rees factor act* of  $A$  by the subact  $B$ . Clearly,  $A/B$  has a zero which is the class consisting of  $B$ , all other classes are one-element sets. Moreover, any subact  $B \subseteq A$  gives rise to a kernel congruence  $\ker \pi$  where  $\pi : A \rightarrow A/B$  is the canonical epimorphism. The category of all  $S$ -acts, with action-preserving ( $S$ -act) maps ( $f : A \rightarrow B$  with  $f(as) = f(a)s$ , for  $s \in S$ ,  $a \in A$ ), is denoted by  $\mathbf{Act-S}$ . Clearly  $S$  itself is an  $S$ -act with its operation as the action.

Throughout this paper,  $S$  is a monoid with at least a right non-invertible element, all  $S$ -acts will be right  $S$ -acts and all ideals of  $S$  are right ideals, zero element of an  $S$ -act, if it exists, is unique. If  $S$ -act  $A$  has a unique zero element  $\theta$ , then  $\theta \in B$  for any subact  $B$  of  $A$ . The set of all idempotents of  $S$  is denoted by  $E(S)$ . It is known that the set

$$(2) \quad \{s \mid s \text{ is a right non-invertible element of } S\}$$

is the only maximal right ideal of  $S$ . In this note, we reserve  $\mathfrak{M}$  to denote, always, this unique maximal right ideal of  $S$ . For more information on  $S$ -acts we refer the reader to [9]. Nakayama's Lemma was first discovered in the special case of ideals in a commutative ring by W. Krull and then in general by G. Azumaya [5]. The lemma is named after the Japanese mathematician T. Nakayama and introduced in its present form in [10]. D. D. Anderson and E. W. Johnson have proved some versions of Nakayama's Lemma for lattices and commutative monoids with a unique zero element in [2] and [1], respectively. Some generalizations of Nakayama's Lemma have been given and studied, in the literatures. For example, A. Azizi [4] introduced Nakayama property for modules over a commutative ring with identity. He says that an  $R$ -module  $M$  has Nakayama property if  $IM = M$ , where  $I$  is an ideal of  $R$ , implies that there exists  $a \in R$  such that  $aM = 0$  and  $a - 1 \in I$ . Then Nakayama's Lemma states that every finitely generated  $R$ -module has Nakayama property. He has proved that  $R$  is a perfect ring if and only if every  $R$ -module has Nakayama property. Besides, we remark that there are generalizations in other contexts, we refer the reader to [6] and [11]. It is a significant tool in algebraic geometry, because it allows local data on algebraic varieties, in the form of modules over local rings, to be studied pointwise as vector spaces over the residue field of the ring. Nakayama's Lemma for  $R$ -modules governs the interaction between the Jacobson radical of a ring and its finitely generated modules. There are several equivalent forms of Nakayama's Lemma in algebra. We express one here. Let  $R$  be a ring with identity 1, and  $A$  a finitely generated right  $R$ -module. If  $I$  is a right ideal of  $R$  contained in the Jacobson radical of  $R$ ,  $J(R)$ , and  $AI = A$  then  $A = 0$ . Notice that for a ring  $R$  with identity 1,  $J(R)$  is a two-sided ideal of  $R$ , but for a monoid  $S$ ,

$$(3) \quad \mathfrak{M} = \{s \in S \mid st \neq 1 \text{ for all } t \in S\}$$

is the only maximal right ideal of  $S$  and then for any proper ideal  $I$  of  $S$  we have  $I \subseteq \mathfrak{M}$ . We can not talk about Jacobson radical of a monoid, because  $\mathfrak{M}$  is the only maximal right ideal of it. We therefore consider Nakayama's Lemma in **Act**- $S$  where  $S$  is a monoid with a unique two-sided maximal ideal  $\mathfrak{M}$ . In [9, Example 3.18.10], the authors present a monoid  $S$  in which  $\mathfrak{M}$  is not a two-sided ideal. Moreover, there are other examples of monoids  $S$  that are not commutative, but their maximal ideals are two-sided. For example, given  $S = (M_n(\mathbb{R}), \cdot)$ , the monoid of all  $n \times n$  matrices with real number entries under usual multiplication of matrices. Since  $ab = 1$  implies  $ba = 1$  for  $a, b \in S$ , the unique maximal ideal of  $S$  is two-sided (see Lemma 2.1 below). Besides, there are many examples of finitely generated  $S$ -acts  $A$  with a zero element  $\theta$  in which for a proper ideal  $I$  of  $S$ ,  $AI = A$ , but  $A \neq \{\theta\}$ ; take any monoid  $S$  and an arbitrary finite set  $A$  with  $|A| > 1$ . Then  $A$  becomes a right  $S$ -act by trivial action, i.e.,  $as = a$  for all  $a \in A, s \in S$ . Therefore,  $AI = A$  for every proper ideal  $I$ , although,  $A \neq \{\theta\}$ . As another example of this situation, we will provide in Example 2.7 of next section, a finitely generated  $\mathbb{N}$ -act with a unique zero for which  $A(2\mathbb{N}) = A$ , but  $A \neq \{\theta\}$ .

## 2. Main Results

In this section first we present a lemma which determines monoids in which their unique maximal ideals are two-sided.

LEMMA 2.1. *Let  $S$  be a monoid. Then the following statements are equivalent:*

- 1)  $\mathfrak{M}$  is a two-sided ideal of  $S$ .
- 2)  $st = 1$  implies  $ts = 1$ , for all  $s, t \in S$ .

The next lemma guarantees that every finitely generated  $S$ -act with a unique zero element has a maximal subact.

**LEMMA 2.2.** *Let  $S$  be a monoid and let  $A$  be a finitely generated  $S$ -act with a unique zero element  $\theta$  and  $A \neq \{\theta\}$ . Then every proper subact of  $A$  is contained in a maximal subact. In particular,  $A$  has a maximal subact.*

Now we are ready to state the first version of Nakayama's lemma.

**THEOREM 2.3.** *Let  $S$  be a monoid in which its unique maximal right ideal  $\mathfrak{M}$  is two-sided. Moreover, let  $A$  be an  $S$ -act and  $B$  a maximal subact of  $A$  in which there exists  $a \in A \setminus B$  such that  $\mathfrak{M} = \{s \in S \mid as \in B\}$ . Then for every proper ideal  $I$  of  $S$  we have  $AI \neq A$ .*

The next example illustrates Theorem 2.3.

**EXAMPLE 2.4.** Let  $S = (\mathbb{N}, \cdot)$  be the monoid of natural numbers with the usual multiplication. Then  $A = \{2, 3, \dots\}$  is a subact of  $\mathbb{N}$ . Let  $U$  be the set of all prime numbers. Then  $U$  is a set of generating elements of  $A$ . Note that  $U$  is the least generating set of  $A$ , i.e.  $A$  is not finitely generated. The set  $B = \{3, 4, \dots\}$  is a maximal subact of  $A$  in which  $\mathfrak{M} = \{n \in \mathbb{N} \mid 2n \in B\}$ . All assumptions of Theorem 2.3 hold for  $A$ . Therefore for any proper ideal  $I$  of  $S$ ,  $AI \neq A$ .

In the next lemma we will see that the second condition of Theorem 2.3 is equivalence to the implication that ' $as = a$  implies  $s \notin \mathfrak{M}$ '. More precisely, we have:

**LEMMA 2.5.** *Let  $A$  be an  $S$ -act and  $B$  a maximal subact of  $A$ . Then for every  $a \in A \setminus B$  the following statements are equivalent:*

- 1)  $\mathfrak{M} = \{s \in S \mid as \in B\}$ .
- 2)  $as = a$  implies  $s \notin \mathfrak{M}$ .

If  $A$  is a right  $R$ -module on local ring  $R$  (a unitary ring with a unique maximal right ideal) then  $J(R) = \mathfrak{M}$  is the only maximal right ideal of  $R$ . In this case we have,  $a = am$  if and only if  $a = 0$ , for all  $m \in \mathfrak{M}$  and  $a \in A$ . With this explanations, the next corollary provides a version of Nakayama's lemma for  $S$ -acts that we will see it in the following.

**COROLLARY 2.6.** *Let  $S$  be a monoid in which the maximal ideal  $\mathfrak{M}$  is two-sided and  $A$  is an  $S$ -act with a maximal subact  $B$ . Let  $AI = A$ , for some proper ideal  $I$  of  $S$ . Then for every  $a \in A \setminus B$  there exists  $m \in \mathfrak{M}$  such that  $am = a$ .*

**EXAMPLE 2.7.** Let  $A = \mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$  be the additive group of integers modulo  $n$  where  $n > 1$  and let  $(\mathbb{N}, \cdot)$  be the monoid of natural numbers with the usual multiplication. Then,  $A$  is an  $\mathbb{N}$ -act by the action  $[t]k = [tk]$  for every  $k \in \mathbb{N}$  and  $0 \leq t \leq n-1$ . By definition,  $A$  is a finitely generated  $\mathbb{N}$ -act with a unique zero  $[0]$ . However, for the ideal  $I = 2\mathbb{N}$  of  $\mathbb{N}$  we have  $AI = A$  if  $A = \mathbb{Z}_3$ . Indeed,  $B = \{[0]\}$  is the maximal subact of  $\mathbb{Z}_3$ , according to Corollary 2.6 for any  $a \in \mathbb{Z}_3$  there exists  $m \in \mathfrak{M} = \{2, 3, \dots\}$ , such that  $am = a$ .

**DEFINITION 2.8.** Let  $A$  be an  $S$ -act. A nonzero element  $a \in A$  is called *quasi-strongly faithful*, if for  $s \in S$  the equality  $as = a$  implies that  $s \notin \mathfrak{M}$ . One calls  $A$  a *quasi-strongly faithful  $S$ -act* if all of its nonzero elements are quasi-strongly faithful.

**PROPOSITION 2.9.** *Let  $S$  be a monoid in which its unique maximal right ideal  $\mathfrak{M}$  is two-sided.*

- (i) Let  $A$  be an  $S$ -act and  $B$  a maximal subact of  $A$  such that there exists a quasi-strongly faithful element in  $A \setminus B$ . Then:
- $AI = A$  if and only if  $I = S$ ;
  - For every ideal  $I$  of  $S$  with  $I^2 = I$ ,  $AI \cong A$  if and only if  $I = S$ .
- (ii) Let  $A$  be a quasi-strongly faithful  $S$ -act and  $B$  a maximal subact of  $A$ . Then  $AI = A$  if and only if  $I = S$ .

Now we state a second version of Nakayama's Lemma which is quite similar to module theory. Also, in the next corollary we give another similar consequence.

**THEOREM 2.10.** *Let  $S$  be a monoid in which its unique maximal right ideal  $\mathfrak{M}$  is two-sided. Let  $A$  be a finitely generated quasi-strongly faithful  $S$ -act with a unique zero element  $\theta$ . If  $AI = A$  for some proper ideal  $I$  of  $S$ , then  $A = \{\theta\}$ .*

**COROLLARY 2.11.** *Let  $S$  be a monoid in which its unique maximal right ideal  $\mathfrak{M}$  is two-sided. Let  $A$  be a finitely generated quasi-strongly faithful  $S$ -act. If  $B \cup AI = A$  for some proper ideal  $I$  of  $S$  and some subact  $B$  of  $A$ , then  $A = B$ .*

## References

1. D. D. Anderson, E. W. Johnson, Ideal theory in commutative semigroups, *Semigroup Forum*, 30: 127-158 (1984).
2. D. D. Anderson, E. W. Johnson, Join-principally generated multiplicative lattices, *Algebra Univers.*, 19: 74-82 (1984).
3. F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Second edition, Graduate Texts in Mathematics, Springer-Verlag, New York, (1992).
4. A. Azizi, On Generalization of Nakayama's lemma, *Glasg. Math. J.*, 52: 605-617 (2010).
5. G. Azumaya, On maximally central algebras, *Nagoya Math. J.*, 2: 119-150 (1951).
6. P. Balister, S. Howson. Note on Nakayama's lemma for compact  $\Lambda$ -Modules, *Asian J. Math.*, 1: 224-229 (1997).
7. P. M. Cohen, *Further Algebra and Applications*, Springer-Verlag, (2003).
8. D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, (1994).
9. M. Kilp, U. Knauer, A. V. Mikhalev, *Monoids, Act and Categories*, Walter de Gruyter, Berlin (2000).
10. T. Nakayama, A remark on finitely generated modules, *Nagoya Math. J.*, 3: 139-140 (1951).
11. A. Ogus, G. Bergman, Nakayama's lemma for half-exact functors, *Proc. Amer. Math. Soc.*, 31: 67-74 (1972).



## Dedekind-finite Acts over Monoids

Kamal Ahmadi

Assistant Professor, Department of Mathematics, Za. C., Islamic Azad University, Zanjan, Iran.

Email: kamal.ahmadi.math@iau.ac.ir

---

**ABSTRACT.** Cancellation problem is one of the ancient problems and applied in the theory of modules. There are several variations on the notion of cancellation. One of them is Dedekind-finiteness. Dedekind-finite rings and modules are an interesting research in the theory of modules. In this manuscript, we introduce and study Dedekind-finiteness in the theory of acts over monoids.

**Keywords:** monoid, Dedekind-finite,  $S$ -act

**AMS Mathematics Subject Classification [2020]:** 20M30, 20M50

---

### 1. Introduction

Dedekind [4] defined an infinite set as a set that can be placed in one-to-one correspondence with a proper subset of itself. He then defined a finite set as one that is not infinite. It follows that a set  $A$  is finite if and only if every injective function  $f : A \rightarrow A$  is an isomorphism of sets. There is an analogous characterization of finite-dimensional vector spaces (see, e.g., [3, Section 11.7.51] in this regard). In the theory of modules, a module  $A$  (on a unitary ring  $R$ ) is said to be cohopfian if every injective endomorphism of  $A$  is an automorphism and is said to be Dedekind-finite if  $A$  can never be isomorphic to a proper direct summand of itself. An easy argument shows that cohopfian property implies Dedekind-finiteness. We refer the reader to [5] and [7] for more information on Dedekind-finite and cohopfian modules.

Let  $S$  be a monoid with identity 1. Recall that a (*right*)  $S$ -act is a non-empty set  $A$  equipped with a map  $\mu : A \times S \rightarrow A$  called its action, such that, denoting  $\mu(a, s)$  by  $as$ , we have  $a1 = a$  and  $a(st) = (as)t$ , for all  $a \in A$ , and  $s, t \in S$ . An element  $\theta \in A$  is called a *zero* of  $A$  if  $\theta s = \theta$  for every  $s \in S$ . Let  $A$  be an  $S$ -act and  $B \subseteq A$  a non-empty subset. Then  $B$  is called a *subact* of  $A$  if  $bs \in B$  for all  $s \in S$  and  $b \in B$ . Clearly  $S$  itself is an  $S$ -act with its operation as the action. An equivalence relation  $\rho$  on an  $S$ -act  $A$  is called a *congruence* on  $A$  if  $apa'$  implies  $(as)\rho(a's)$  for  $a, a' \in A$  and  $s \in S$ . Let  $f : A \rightarrow B$  be an  $S$ -homomorphism. Then the kernel equivalence relation of  $f$ ,  $\ker f$ , defined by  $a(\ker f)a'$  if and only if  $f(a) = f(a')$  for  $a, a' \in A$ , is an act congruence which is called the *kernel congruence* of  $f$ .

Throughout this paper,  $S$  is a monoid with 0, all  $S$ -acts will be right  $S$ -acts with unique zero  $\theta$  and any subact of an  $S$ -act contains the zero  $\theta$ . The category of all  $S$ -acts with unique zero  $\theta$  and  $S$ -homomorphisms preserving zero (i.e.,  $f : A \rightarrow B$  with  $f(as) = f(a)s$ , for  $s \in S, a \in A$  and  $f(\theta) = \theta$ ), is denoted by  $\mathbf{Act}_0\text{-}S$ . Note that for each  $A \in \mathbf{Act}_0\text{-}S$  we have  $A0 = \{\theta\}$ .

Recall that the category  $\mathbf{Act}_0\text{-}S$  has coproducts of any non-empty families of  $S$ -acts. More precisely, if  $I$  is a non-empty set,  $X_i \in \mathbf{Act}_0\text{-}S, i \in I$ , and  $\theta_i$  be the zero of  $X_i$ , then by [6, Proposition 2.1.15] the coproduct of  $\{X_i : i \in I\}$  is  $\coprod_{i \in I} X_i = (\bigcup_{i \in I} (X_i \setminus \{\theta_i\})) \dot{\cup} \{\theta\}$  with  $x_i s = \theta$ , if  $x_i s = \theta_i$  in  $X_i, \theta s = \theta$  for  $s \in S$ . Likewise, if  $B$  and  $C$  are two subacts of an  $S$ -act  $A$  such that  $A = B \cup C$  and  $B \cap C = \{\theta\}$ , then we write  $A = B \oplus C$ . In this case  $A = B \oplus C$  is called a *decomposition* of  $A$ . Otherwise,  $A$  is called *indecomposable*. By [6, Theorem 1.5.10], every  $S$ -act  $A$  has a unique decomposition into indecomposable subacts. For more information about  $S$ -acts we encourage the reader to see [6].

In this paper, we are going to study Dedekind-finiteness on  $\mathbf{Act}_0\text{-}S$ . In Theorem 2.7, we will indicate when Dedekind-finiteness and cohopfian property are equivalent. We recall [1] that an  $S$ -act  $A$  is *cancellable* if  $A \coprod B \cong A \coprod C$  implies  $B \cong C$  where  $B, C$  are  $S$ -acts. We say an  $S$ -act  $A$  satisfies *internal cancellation* if, for arbitrary subacts  $C, D, E$  and  $F$  of  $A, A = C \oplus D = E \oplus F$  and  $C \cong E$  implies that  $D \cong F$ .

## 2. Main Results

In [1] the authors have discussed some results of cancellation problem in  $\mathbf{Act}_0\text{-}S$ . In this manuscript we continue our study and investigate the notion of Dedekind-finiteness which is similar to cancellation property.

We begin with a definition.

DEFINITION 2.1. By a *Dedekind-finite*  $S$ -act we mean an  $S$ -act  $A$  which is not isomorphic to any proper direct summand of itself. Equivalently,  $A$  is Dedekind-finite if and only if  $B = \{\theta\}$  is the only  $S$ -act for which  $A \coprod B \cong A$ .

The next example shows how one can obtain a non-Dedekind-finite act from an indecomposable act.

EXAMPLE 2.2. Let  $S$  be a monoid. Take a non-zero indecomposable  $S$ -act  $A$  and an arbitrary infinite set  $I$ . Then  $B = \coprod_{i \in I} A_i$ , in which  $A_i = A$  for any  $i \in I$ , is not Dedekind-finite, because  $B \coprod A \cong B$  but  $A \neq \{\theta\}$ .

In the following lemma we state two results. The proofs are direct and are omitted.

LEMMA 2.3. *Let  $A$  be an  $S$ -act. Then*

- i) *if  $A$  is cancellable, then  $A$  is Dedekind-finite.*
- ii) *if  $C \cong C \coprod C$  where  $C$  is a direct summand of  $A$ , then  $A$  is not Dedekind-finite.*

In the following we define cohopfian act and in the next lemma we prove that every cohopfian act is Dedekind-finite.

DEFINITION 2.4. Let  $A$  be an  $S$ -act. Then  $A$  is called *cohopfian* if every injective endomorphism of  $A$  is an automorphism.

LEMMA 2.5. *If an  $S$ -act  $A$  is cohopfian, then  $A$  is Dedekind-finite.*

DEFINITION 2.6. Let  $A$  be an  $S$ -act. Then  $A$  is called *quasi-injective* if, for any subact  $B$  of  $A$ , any  $f \in \text{Hom}_S(B, A)$  can be extended to an endomorphism of  $A$ , it means that the diagram

$$(1) \quad \begin{array}{ccc} B & \xrightarrow{\iota} & A \\ f \downarrow & \swarrow \tilde{f} & \\ A & & \end{array}$$

commutes, where  $\iota$  is the canonical inclusion.

In the following theorem we will show when a cohopfian act is Dedekind-finite and vice versa. But first let us recall [8] that an  $S$ -act  $A$  is *torsion free* if for any  $x, y \in A$  and for any element  $s \in S$  the equality  $xs = ys$  implies  $x = y$ . Note that in [6] torsion free is considered in much weaker sense.

THEOREM 2.7. *Let  $A$  be a torsion free and quasi-injective  $S$ -act. Then the following statements are equivalent:*

- i)  $A$  is a Dedekind-finite  $S$ -act.
- ii)  $A$  is a cohopfian  $S$ -act.

In what follows we are going to study when Dedekind-finite and cancellation property coincide. Let  $A$  be an  $S$ -act and  $A = \bigoplus_{i \in I} A_i$  be the unique decomposition of  $A$  into indecomposable subacts. Define for  $i, j \in I$ ,  $i \sim j$  if and only if  $A_i \cong A_j$ . Then  $\sim$  is an equivalence relation on  $I$ . The equivalence class  $i \in I$  is given by

$$(2) \quad [i] = \{j \in I \mid A_i \cong A_j\}.$$

We call  $A$  *quasi-finitely decomposable* if  $|P| < \infty$  where  $P = \{\text{Card}[i] \mid i \in I\}$ .

With this introduction we have

LEMMA 2.8. *Let  $A$  be a quasi-finitely decomposable  $S$ -act. If  $A$  is not cancellable, then there exists a direct summand  $C$  of  $A$  such that  $C \cong C \amalg C$ .*

COROLLARY 2.9. *Let  $A$  be a quasi-finitely decomposable  $S$ -act. Then  $A$  is cancellable if and only if  $A$  is Dedekind-finite.*

## References

1. K. Ahmadi, A. Madanshekaf, Cancellation Property for Acts over Monoids, Submitted.
2. F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Second edition, Graduate Texts in Mathematics, Springer-Verlag, New York, (1992).
3. N. Bourbaki, Algebra. Part I (Hermann. Paris and Addison-Wesley, Reading. MA. 1974).
4. R. Dedekind, Was sind und was sollen die Zahlen? Braunschweig. 1887. (Reprinted by Vieweg, Braunschweig, 1969).
5. K. R. Goodearl, Von Neumann Regular Rings. Pitman, London, 1979.
6. M. Kilp, U. Knauer, A. V. Mikhalev, Monoids, Acts and Categories With Applications to Wreath Products and Graphs, Walter de Gruyter, Berlin (2000).
7. T. Y. Lam, A crash course on stable range, cancellation, substitution and exchange, J. Algebra Appl., 3 (2004), 301–343.
8. M. Satyanarayana, Quasi and weakly injective  $S$ -systems, Math. Nachr., 71(1), (1976), 183-190.
9. Y. Suzuki, On automorphisms of injective modules, Proc. Japan Acad. 44 (1968), 120-124.



## Formal Concept for $EQ$ -algebras

Roohallah Daneshpayeh<sup>1,\*</sup>, Sirus Jahanpanah<sup>2</sup> and Mohammad Hamidi<sup>3</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: rdaneshpayeh@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: s.jahanpanah@pnu.ac.ir

<sup>3</sup>Department of Mathematics, Payame Noor University (PNU), Tehran Iran.

Email: m.hamidi@pnu.ac.ir

---

**ABSTRACT.** This paper establishes fundamental connections between  $EQ$ -algebras and Formal Concept Analysis (FCA). We introduce the standard formal context for  $EQ$ -algebras using the equivalence operation and explore alternative contexts based on order and filter structures. Our main results demonstrate that for good  $EQ$ -algebras, the concept lattice is isomorphic to the lattice of  $EQ$ -filters. Furthermore, we prove that the energy of the formal context characterizes linear ordering in  $EQ$ -algebras. These findings bridge algebraic structures in fuzzy logic with knowledge representation techniques, providing new tools for analyzing  $EQ$ -algebras through formal concept analysis. The results offer both theoretical insights and practical applications in fuzzy logic and knowledge representation.

**Keywords:** homogeneous structures, Lie group, conformal vector field (at least 2 and at most 5)

**AMS Mathematics Subject Classification [2020]:** 58E11, 53B30, 53C50 (at least 1 and at most 3)

---

### 1. Introduction

The study of fuzzy logics and their algebraic semantics has been a fertile area of research since Zadeh's introduction of fuzzy set theory [5]. While BL-algebras and their variants have been extensively studied as the algebraic foundation for many-valued logics,  $EQ$ -algebras have emerged as a more recent and fundamental structure. Introduced by Novák et al. [3, 4],  $EQ$ -algebras take the fuzzy equality as a primary operation, making them a natural algebraic counterpart for fuzzy type theory. The theory of these algebras, including the study of good  $EQ$ -algebras and their filters, has been significantly advanced by researchers like El-Zekey [1]. Parallel to these developments in algebraic logic, Formal Concept Analysis (FCA), established by Wille [2], has matured into a powerful framework for data analysis and knowledge representation based on lattice theory. The core data structure of FCA is a formal context, a triple defining a relationship between objects

---

\*Speaker.

and attributes, from which a hierarchical concept lattice can be derived. The potential for applying FCA to analyze algebraic structures has been recognized, with initial forays exploring connections to BL-algebras and other fuzzy algebraic systems. This paper seeks to bridge these two domains by applying the methods of FCA to the theory of EQ-algebras. We introduce the standard formal context for an EQ-algebra, defined by its fuzzy equality, and prove fundamental representation theorems. Our main results demonstrate that the conceptual hierarchy of an EQ-algebra, as captured by its concept lattice, is deeply intertwined with its algebraic structure, particularly its lattice of filters. Furthermore, we introduce spectral methods by defining the energy of an EQ-algebra's context, providing a new tool for classifying these algebras. This work continues the trajectory of enriching both algebraic logic and conceptual data analysis through their interdisciplinary connection.

## 2. Preliminaries

In this section, add the notations that apply in our work.

DEFINITION 2.1. [1] An algebra  $(X, \wedge, \ominus, *, 1)$  equipped with binary operations  $\wedge, \ominus, *$  is a (commutative) EQ-algebra, if for all  $x, y, z, t \in X$ :

- (EQ1)  $(X, \wedge, 1)$  is a  $\wedge$ -semilattice ( $x \leq_X 1, x \leq_X y \Leftrightarrow x \wedge y = x$ ),
- (EQ2)  $(X, \ominus, 1)$  is a (commutative) monoid and  $\ominus$  is isotone,
- (EQ3)  $x * x = 1$ ,
- (EQ4)  $((x \wedge y) * z) \ominus (t * x) \leq_X (z * (t \wedge y))$ ,
- (EQ5)  $(x * y) \ominus (z * t) \leq_X (x * z) * (y * t)$ ,
- (EQ6)  $(x \wedge y \wedge z) * x \leq_X (x \wedge y) * x$ ,
- (EQ7)  $(x \wedge y) * x \leq_X (x \wedge y \wedge z) * (x \wedge z)$ ,
- (EQ8)  $x \ominus y \leq_X x * y$ .

THEOREM 2.2. [1] Let  $(X, \wedge, \ominus, *, 1)$  be an EQ-algebra. Then for all  $a, b, c \in X$ :

- (1)  $a \rhd (b \wedge c) \leq_X a \rhd b$ ,
- (2)  $a \rhd b \leq_X (a \wedge c) \rhd b$ ,
- (3)  $a \rhd d \leq_X (b \rhd a) \rhd (b \rhd d)$ .

DEFINITION 2.3. [2] A Formal context is a structured dataset that defines a relationship between a set of objects and a set of attributes. It is classically represented as a cross-table (a binary matrix). Formally, a formal context is a triple  $K = (G, M, I)$ , where,  $G$  is a set of objects,  $M$  is a set of attributes,  $I$  is a binary relation between  $G$  and  $M$  (i.e.,  $I \subseteq G \times M$ ).

## 3. Standard Formal Context for a EQ-algebra

In this section, we consider the notion of EQ-algebras and introduce the concept of formal context for a EQ-algebra and investigate their properties.

DEFINITION 3.1. Let  $\mathbf{E} = (E, \wedge, \otimes, \sim, 1)$  be an EQ-algebra. The standard formal context for  $\mathbf{E}$  is defined as,  $K(\mathbf{E}) = (E, E, I_{\sim})$  where the incidence relation  $I_{\sim} \subseteq E \times E$  is defined by,  $(a, b) \in I_{\sim}$  if and only if  $a \sim b = 1$ , and objects ( $G$ ) are elements of the EQ-algebra  $E$ , attributes ( $M$ ) are elements of the EQ-algebra  $E$ , and incidence  $(I_{\sim})^{**}$  is object  $a$  has attribute  $b$  if and only if  $a$  is equivalent to  $b$  ( $a \sim b = 1$ ). Also the alternative formal contexts for EQ-algebras, are

1. Order-based context, by  $K_{\leq}(\mathbf{E}) = (E, E, I_{\leq})$  where  $(a, b) \in I_{\leq}$  iff  $a \leq b$  (with  $a \leq b$  defined as  $a \wedge b = a$ ).

2. Tensor-based context, by  $K_{\otimes}(\mathbf{E}) = (E, E, I_{\otimes})$ , where  $(a, b) \in I_{\otimes}$  iff  $a \otimes b = a \wedge b$ .
3. Filter-based context, by  $K_F(\mathbf{E}) = (\mathcal{F}(E), E, I_F)$  where  $\mathcal{F}(E)$  is the set of filters of  $\mathbf{E}$ , and  $(F, a) \in I_F$  iff  $a \in F$ .

EXAMPLE 3.2. Consider a 4-element EQ-algebra  $\mathbf{E} = (E, \wedge, \otimes, \sim, 1)$  where  $E = \{0, a, b, 1\}$  with the order  $0 \leq a \leq 1$ ,  $0 \leq b \leq 1$ , and  $a, b$  are incomparable, as follows: where  $c, d, e < 1$  are specific values in  $E$  satisfying the EQ-algebra axioms. Then for

$\wedge$	0	a	b	1		$\otimes$	0	a	b	1		$\sim$	0	a	b	1
0	0	0	0	0		0	0	0	0	0		0	1	0	0	0
a	0	a	0	a		a	0	a	0	a		a	0	1	c	d
b	0	0	b	b		b	0	0	b	b		b	0	c	1	e
1	0	a	b	1		1	0	a	b	1		1	0	d	e	1

TABLE 1. Operation tables for the 4-element EQ-algebra: Meet ( $\wedge$ ), Tensor ( $\otimes$ ), and Equivalence ( $\sim$ )

$E = \{0, a, b, 1\}$ , where alternative formal context based on order relation is as  $K_{\leq}(\mathbf{E}) =$

Object / Attribute	0	a	b	1
0	✓			
a		✓		
b			✓	
1				✓

TABLE 2. Formal context based on equivalence relation  $x \sim y = 1$

$(E, E, I_{\leq})$  where  $(x, y) \in I_{\leq}$  iff  $x \leq y$ . Also Thus,

Object / Attribute	0	a	b	1
0	✓	✓	✓	✓
a		✓		✓
b			✓	✓
1				✓

TABLE 3. Formal context based on order relation  $x \leq y$

THEOREM 3.3. Let  $\mathbf{E} = (E, \wedge, \otimes, \sim, 1)$  be a *\*\*good EQ-algebra\*\** (where  $a \sim b = 1$  if and only if  $a = b$ ) and let  $K(\mathbf{E}) = (E, E, I_{\sim})$  be its standard formal context with incidence defined by  $(a, b) \in I_{\sim}$  iff  $a \sim b = 1$ . Let  $\mathfrak{B}(K(\mathbf{E}))$  be the concept lattice of this context and let  $\mathcal{F}(\mathbf{E})$  be the lattice of *\*\*EQ-filters\*\** of  $\mathbf{E}$  (filters closed under the equivalence relation). Then,  $\mathfrak{B}(K(\mathbf{E})) \cong \mathcal{F}(\mathbf{E})$ .

PROOF. Sketch: Since  $E$  is good EQ-algebra, the formal context  $K(\mathbf{E})$  has the identity incidence relation. Moreover, for the identity incidence relation, the formal concepts are precisely the pairs,  $(X, X)$  for any  $X \subseteq E$ . Since,  $X' = \{b \in E \mid \forall a \in X, a \sim b = 1\} = X$  and  $X'' = \{a \in E \mid \forall b \in X', a \sim b = 1\} = X$ . In a good EQ-algebra, the EQ-filters

Concept	Extent (Objects)	Intent (Attributes)
$C_1$	$\{0, a, b, 1\}$	$\{0\}$
$C_2$	$\{0, a\}$	$\{a, 1\}$
$C_3$	$\{0, b\}$	$\{b, 1\}$
$C_4$	$\{0\}$	$\{0, a, b, 1\}$
$C_5$	$\{1\}$	$\{1\}$
$C_6$	$\emptyset$	$\{0, a, b, 1\}$

TABLE 4. Formal concepts derived from order-based context

are exactly the principal filters generated by individual elements, by  $\mathcal{F}(\mathbf{E}) = \{\uparrow a \mid a \in E\} \cup \{\emptyset, E\}$  where  $\uparrow a = \{x \in E \mid a \leq x\}$ . Define the isomorphism  $\Phi : \mathfrak{B}(K(\mathbf{E})) \rightarrow \mathcal{F}(\mathbf{E})$  by,  $\Phi(\{\{a\}, \{a\}\}) = \uparrow a$  for atomic concepts,  $\Phi((X, X)) = \bigcap_{a \in X} \uparrow a$  for general concepts,  $\Phi((\emptyset, E)) = E$ , and  $\Phi((E, \emptyset)) = \emptyset$ . The inverse map  $\Psi : \mathcal{F}(\mathbf{E}) \rightarrow \mathfrak{B}(K(\mathbf{E}))$  is given by,  $\Psi(F) = (\min(F), \min(F))$  where  $\min(F)$  is the set of minimal elements of  $F$ .

Hence  $\Phi$  and  $\Psi$  preserve the lattice operations and  $\Psi \circ \Phi = \text{id}$  and  $\Phi \circ \Psi = \text{id}$  are finished the proof.  $\square$

**THEOREM 3.4.** *Let  $\mathbf{E} = (E, \wedge, \otimes, \sim, 1)$  be a finite EQ-algebra and let  $K(\mathbf{E}) = (E, E, I_{\sim})$  be its standard formal context with incidence matrix  $M_I$  where  $M_I(x, y) = 1$  if  $x \sim y = 1$ , and 0 otherwise. Define the **energy** of the context as,  $E(K(\mathbf{E})) = \sum_{i=1}^n |\lambda_i|$  where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $M_I$ . Then the following are equivalent:*

1.  $\mathbf{E}$  is linearly ordered.
2.  $E(K(\mathbf{E})) = n$  (where  $n = |E|$ ).
3.  $M_I$  is an upper triangular matrix with 1's on the diagonal.
4. The concept lattice  $\mathfrak{B}(K(\mathbf{E}))$  is a chain.

#### 4. Conclusion

This paper bridges EQ-algebras with formal concept analysis by defining their standard formal context. We prove the concept lattice of a good EQ-algebra is isomorphic to its EQ-filter lattice and establish an energy-based criterion for linear ordering. These findings connect algebraic fuzzy logic with knowledge representation, enabling new structural analysis methods for logical systems via formal concept analysis.

#### Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

#### References

1. M. El-Zekey, *Representable good EQ-algebras*, Soft Computing, **14** (9) (2010), 1011–1023.
2. B. Ganter, G. Stumme, and R. Wille, *Formal Concept Analysis: Foundations and Applications*, Berlin, Germany: Springer, 1999.
3. M. Kondo, *On the structure of generalized fuzzy concept lattices*, Fuzzy Sets and Systems, **150** (2005), 349–362.
4. V. Novák, *EQ-algebras: Primary concepts and properties*, Fuzzy Sets and Systems, **160** (2009), 3326–3336.
5. L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.



## Formal Concept for $BL$ -algebras

Sirus Jahanpanah<sup>1,\*</sup>, Roohallah Daneshpayeh<sup>2</sup> and Mohammad Hamidi<sup>3</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.  
Email: s.jahanpanah@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.  
Email: rdaneshpayeh@pnu.ac.ir

<sup>3</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.  
Email: m.hamidi@pnu.ac.ir

---

**ABSTRACT.** This paper investigates the deep connections between BL-algebras and Formal Concept Analysis (FCA). We define the standard formal context for any BL-algebra using its natural order and establish fundamental representational theorems. Our main results prove that the concept lattice derived from this context is isomorphic to the lattice of deductive filters of the algebra. Furthermore, we demonstrate that the prime filter spectrum forms a context whose concept lattice is anti-isomorphic to the lattice of radical ideals, providing a basis for a sheaf representation. These results bridge algebraic fuzzy logic with conceptual data analysis, offering new tools for understanding the structure of BL-algebras through FCA.

**Keywords:** BL-algebra, Formal Concept Analysis, Concept Lattice, Fuzzy Logic

**AMS Mathematics Subject Classification [2020]:** 03B05

---

### 1. Introduction

The study of fuzzy logics and their algebraic counterparts has been a significant area of research since Zadeh's pioneering work on fuzzy sets in 1965 [4]. BL-algebras (Basic Logic algebras) were introduced by Hájek in 1998 [2] as the algebraic foundation for fuzzy logic systems, providing a generalized framework that encompasses many-valued logics. These structures have since become fundamental in the study of mathematical fuzzy logic and its applications [3]. Parallel to these developments, formal concept analysis (FCA), established by Wille in 1982 [1], has emerged as a powerful method for data analysis and knowledge representation using lattice theory. The connection between these two fields BL-algebras and formal contexts has recently gained attention as researchers seek to bridge algebraic logic with conceptual data analysis.

The fundamental relationship between BL-algebras and formal concept lattices was first explored by Belohlávek, who demonstrated that certain fuzzy algebraic structures could be represented through formal contexts. Subsequent work has revealed deep connections between the lattice of filters in BL-algebras and concept lattices derived from

---

\*Speaker.

appropriately defined formal contexts. This intersection has opened new avenues for analyzing logical structures using FCA techniques and, conversely, for enriching data analysis with logical foundations from fuzzy algebra.

Recent advancements have extended these connections to more complex settings, including fuzzy formal contexts, triadic approaches, and applications in knowledge representation. The energy-based characterization of BL-algebra contexts and spectral analysis of incidence matrices have further enriched this interdisciplinary field. This paper continues this trajectory by establishing new theorems connecting BL-algebra structures with formal concept analysis, providing both theoretical foundations and practical applications that bridge these important mathematical domains.

## 2. Preliminaries

In this section, add the notations that apply in our work.

DEFINITION 2.1. [2] An algebraic structure  $(X, \wedge, \vee, *, \rightarrow, 0, 1)$  of type  $(2, 2, 2, 2, 0, 0)$  is called a *BL-algebra*, if for all  $x, y, z \in X$ , it satisfies the following conditions:

- (BL1)  $(X, \wedge, \vee, 0, 1)$  is a bounded lattice,
- (BL2)  $(X, *, 1)$  is a commutative monoid,
- (BL3)  $x * y \leq z \Leftrightarrow x \leq y \rightarrow z$ ,
- (BL4)  $x \wedge y = x * (x \rightarrow y)$ ,
- (BL5)  $(x \rightarrow y) \vee (y \rightarrow x) = 1$ .

THEOREM 2.2. [2] Let  $(X, \wedge, \vee, *, \rightarrow, 0, 1)$  be a BL-algebra and  $x, y, z \in X$ . Then

- (i)  $x * y \leq x \wedge y$ .
- (ii)  $x \leq y$  if and only if  $x \rightarrow y = 1$ .
- (iii) If  $x \leq y$ , then  $z \rightarrow x \leq z \rightarrow y$ ,  $y \rightarrow z \leq x \rightarrow z$  and  $x * z \leq y * z$ .
- (iv)  $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$ .
- (v)  $1 \rightarrow x = x$ ,  $x \rightarrow x = 1$  and  $x \rightarrow 1 = 1$ .
- (vi)  $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$ .
- (vii)  $x * (y \vee z) = (x * y) \vee (x * z)$ .

DEFINITION 2.3. [1] A Formal context is a structured dataset that defines a relationship between a set of objects and a set of attributes. It is classically represented as a cross-table (a binary matrix). Formally, a formal context is a triple  $K = (G, M, I)$ , where,  $G$  is a set of objects,  $M$  is a set of attributes,  $I$  is a binary relation between  $G$  and  $M$  (i.e.,  $I \subseteq G \times M$ ).

The notation  $(g, m) \in I$  or  $gIm$  means that object  $g$  has attribute  $m$ .

## 3. Standard Formal Context for a BL-algebra

In this section, we consider the notion of BL-algebras and introduce the concept of formal context for a BL-algebra and investigate their properties.

DEFINITION 3.1. Let  $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$  be a BL-algebra and  $x, y, z \in X$ . Then the formal context of  $A$  is defined as,  $K(\mathbf{A}) = (A, A, I)$ , where the incidence relation  $I \subseteq A \times A$  is defined by the order relation of the algebra,  $(a, b) \in I$  if and only if  $a \leq b$ . In this context, objects ( $G$ ) are the elements of the BL-algebra  $A$ , attributes ( $M$ ), are the elements of the BL-algebra  $A$  and incidence relation ( $I$ ) is an object  $a$  has an attribute  $b$  if and only if  $a$  is less than or equal to  $b$  in the lattice order of the BL-algebra.

EXAMPLE 3.2. Let  $\mathbf{B} = (B, \wedge, \vee, \odot, \rightarrow, 0, 1)$  where  $B = \{0, a, b, 1\}$  with  $0 \leq a \leq 1$ ,  $0 \leq b \leq 1$ , and  $a, b$  are incomparable. The operations are defined as:

$$x \odot y = x \wedge y$$

$$x \rightarrow y = \neg x \vee y \quad \text{where } \neg 0 = 1, \neg a = b, \neg b = a, \neg 1 = 0$$

The complete operation tables are as follows:

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	a	0	a	a	b	1	b	1
b	0	0	b	b	b	a	a	1	1
1	0	a	b	1	1	0	a	b	1

$\wedge$	0	a	b	1	$\vee$	0	a	b	1
0	0	0	0	0	0	0	a	b	1
a	0	a	0	a	a	a	a	1	1
b	0	0	b	b	b	b	1	b	1
1	0	a	b	1	1	1	1	1	1

Formal Context:  $K(\mathbf{B}) = (B, B, \leq)$  is as follows:

Object / Attribute	0	a	b	1
0	✓	✓	✓	✓
a		✓		✓
b			✓	✓
1				✓

THEOREM 3.3. Let  $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$  be a  $BL$ -algebra and let  $K(\mathbf{A}) = (A, A, \leq)$  be its standard formal context, where the incidence relation  $I$  is defined by the lattice order  $((a, b) \in I \text{ iff } a \leq b)$ . Let  $\mathfrak{B}(K(\mathbf{A}))$  be the concept lattice derived from this context. Let  $\mathcal{F}(\mathbf{A})$  be the lattice of filters of  $\mathbf{A}$ , ordered by set inclusion. Then,  $\mathfrak{B}(K(\mathbf{A}))$  is isomorphic to  $\mathcal{F}(\mathbf{A})$ .

THEOREM 3.4. Let  $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$  be a  $BL$ -algebra, and let  $\text{Spec}(\mathbf{A})$  be its set of prime filters. Consider the formal context  $K_{\text{Spec}} = (\text{Spec}(\mathbf{A}), A, I)$  where for a prime filter  $F \in \text{Spec}(\mathbf{A})$  and an element  $a \in A$ , we define  $(F, a) \in I$  if and only if  $a \in F$ .

Then the concept lattice  $\mathfrak{B}(K_{\text{Spec}})$  is anti-isomorphic to the lattice of radical ideals of  $\mathbf{A}$ , and provides the basis for the sheaf representation of  $\mathbf{A}$ .

PROOF. Sketch: □

We have the following steps:

1. Construction of the context, objects: prime filters  $F \in \text{Spec}(\mathbf{A})$ , and attributes: elements  $a \in A$  incidence:  $a \in F$ .

2. Derivation operators: For  $\mathcal{F} \subseteq \text{Spec}(\mathbf{A})$  and  $X \subseteq A$ :  $\mathcal{F}' = \{a \in A : \forall F \in \mathcal{F}, a \in F\} = \bigcap \mathcal{F}$   $X' = \{F \in \text{Spec}(\mathbf{A}) : \forall a \in X, a \in F\} = \{F \in \text{Spec}(\mathbf{A}) : X \subseteq F\}$ .

3. Formal concepts: A formal concept is a pair  $(\mathcal{F}, X)$  where,  $\mathcal{F} \subseteq \text{Spec}(\mathbf{A})$ ,  $X \subseteq A$ , and  $\mathcal{F}' = X$  and  $X' = \mathcal{F}$ .

4. *Connection to radical ideals:* for any subset  $X \subseteq A$ , the radical ideal generated by  $X$  is:

$$\sqrt{\langle X \rangle} = \{a \in A : \exists n \in \mathbb{N}, a^n \in \langle X \rangle\}.$$

In BL-algebras, this corresponds precisely to  $X''$  in our formal context.

5. *Anti-isomorphism:* The mapping:  $(\mathcal{F}, X) \mapsto X \quad I \mapsto (\{F \in \text{Spec}(\mathbf{A}) : I \subseteq F\}, I)$  establishes an anti-isomorphism between the concept lattice and the lattice of radical ideals.

**THEOREM 3.5.** *Let  $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$  be a BL-algebra and let  $K(\mathbf{A}) = (A, A, \leq)$  be its standard formal context. Let  $\mathfrak{B}(K(\mathbf{A}))$  be the concept lattice and  $\mathcal{F}(\mathbf{A})$  the lattice of deductive filters of  $\mathbf{A}$ . Then  $\mathfrak{B}(K(\mathbf{A})) \cong \mathcal{F}(\mathbf{A})$*

**PROOF.** Sketch: Define  $\Phi : \mathfrak{B}(K(\mathbf{A})) \rightarrow \mathcal{F}(\mathbf{A})$  by,

$$\Phi((X, Y)) = \{a \in A \mid \exists x_1, \dots, x_n \in X : x_1 \odot \dots \odot x_n \leq a\}$$

and define  $\Psi : \mathcal{F}(\mathbf{A}) \rightarrow \mathfrak{B}(K(\mathbf{A}))$  by

$$\Psi(F) = (F', F'')$$

where  $F' = \{a \in A \mid \forall f \in F, f \leq a\}$  and  $F'' = \{b \in A \mid \forall a \in F', b \leq a\}$ . and show  $\Psi \circ \Phi = id$  and  $\Phi \circ \Psi = id$   $\square$

#### 4. Conclusion

This paper has established fundamental isomorphisms between BL-algebra structures and formal concept lattices. The results provide a powerful bridge between algebraic fuzzy logic and knowledge representation techniques. These connections open new avenues for analyzing logical structures using Formal Concept Analysis and for enriching data analysis with fuzzy algebraic foundations.

#### Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

#### References

1. B. Ganter, G. Stumme, and R. Wille, *Formal Concept Analysis: Foundations and Applications*, Berlin, Germany: Springer, 1999.
2. P. Hajek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.
3. L. Li, *Spectral analysis of concept lattices*, *Discrete Mathematics*, **344** (2021), 112–135.
4. L. A. Zadeh, *Fuzzy sets*, *Information and Control*, **8** (1965), 338–353.



## Some Results In Fuzzy Groupoid-Poset Structure

Karim Ghadimi<sup>1,\*</sup>, Akbar Rezaei<sup>2</sup> and Hee Sik Kim<sup>3</sup>

<sup>1,2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

<sup>3</sup>Department of Mathematics, Hanyang University, Seoul, 04763, Korea.

Email: <sup>1</sup>kghadimi@pnu.ac.ir

Email: <sup>2</sup>rezaei@pnu.ac.ir

Email: <sup>3</sup>heekim@hanyang.ac.kr

**ABSTRACT.** In this note, we study a groupoid  $(X, *)$  equipped with a poset structure  $(X, \leq)$  and fuzzy partially ordered groupoids induced by fuzzy subset  $\mu$ . We define a set  $P_\mu(*; \leq)$  and explore its properties.

**Keywords:** groupoid, partially order groupoid, fuzzy partially order groupoid,  $d/BCK$ -algebra.

**AMS Mathematics Subject Classification [2020]:** 20N02, 03E72, 06F35.

### 1. Introduction

Groupoid theory has garnered significant attention in the literature, with researchers exploring its connections to algebraic structures (e.g., [2]). It has combined with the theory of general algebraic structures. Several properties of the groupoid theory have been developed with  $BCK$ -algebras and its related topics. Neggers and Kim later introduced the concept of  $d$ -algebras [5], a useful generalization of  $BCK$ -algebras, and explored numerous relationships between  $d$ -algebras and  $BCK$ -algebras, showing that  $d$ -algebras are among the least associative algebras. Ahn and Kim [1] developed implicative and commutative  $d$ -algebras that are not  $BCK$ -algebras, proving that these are indeed generalizations of  $BCK$ -algebra concepts.

In this paper, we study a groupoid  $(X, *)$  equipped with a poset structure  $(X, \leq)$  and fuzzy partially ordered groupoids induced by fuzzy subset  $\mu$ . We define a set  $P_\mu(*; \leq)$  and explore its properties.

### 2. Preliminaries

A groupoid  $(X, *)$  is said to be *right (left, resp.) compatible with a poset*  $(X, \leq)$  if  $x \leq y$  implies  $x * z \leq y * z$  ( $z * x \leq z * y$ , resp.) for all  $z \in X$ . A groupoid  $(X, *)$  is said to be *compatible with a poset*  $(X, \leq)$  if it is both right compatible and left compatible with

\*Speaker.

a poset  $(X, \leq)$ . Note that it is equivalent to that  $x \leq y, u \leq w$  implies  $x * u \leq y * w$ . The set  $X$  is said to be *medial* if  $(x * y) * (r * s) = (x * r) * (y * s)$  for all  $x, y, r, s \in X$ .

A *BCK-algebra* [3] is a non-empty set  $X$  with a constant  $0$  and a binary operation “ $*$ ” satisfying the following axioms:

- (I)  $x * x = 0$ ,
- (II)  $0 * x = 0$ ,
- (III)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$ ,
- (IV)  $(x * (x * y)) * y = 0$ ,
- (V)  $((x * y) * (x * z)) * (z * y) = 0$ ,

for all  $x, y, z \in X$ .

**THEOREM 2.1.** ([3]) *Let  $(X, *, 0)$  be a BCK-algebra. Then for any  $x, y$  and  $z$  in  $X$ , the following hold:*

- (i)  $x * y \leq z$  implies  $x * z \leq y$ ,
- (ii)  $(x * z) * (y * z) \leq x * y$ ,
- (iii)  $x \leq y$  implies  $x * z \leq y * z$ ,
- (iv)  $x * y \leq x$ ,
- (v)  $x * 0 = x$ .

An algebra  $(X, *, 0)$  of type  $(2, 0)$  is called a *d-algebra* ([5]) if it satisfies (I), (II) and (III). A *d/BCK-algebra*  $(X, *, 0)$  is said to be *positive implicative* if  $(x * z) * (y * z) = (x * y) * z$  for all  $x, y, z \in X$ . An algebra  $(X, *, \leq)$  is said to be a *partially ordered groupoid* if

- $(X, *)$  is a groupoid,
- $(X, \leq)$  is a partially ordered set.

Let  $(X, *)$  be a groupoid. A non-empty subset  $I$  of  $X$  is said to be a

- *right (left, resp.) ideal*:  $x \in I, y \in X$  implies  $x * y \in I$ , i.e.,  $I * X \subseteq I$  ( $X * I \subseteq I$ , resp.),
- *right (left, resp.) semi ideal*:  $x * y \in I, z \in X$  implies  $(x * z) * (y * z) \in I$  ( $(z * x) * (z * y) \in I$ , resp.),
- *semi ideal*:  $x * y \in I, z \in X$  implies  $(x * z) * (y * z) \in I$  and  $(z * x) * (z * y) \in I$ ,
- *reverse right (left, resp.) semi ideal*:  $(x * z) * (y * z) \in I$  ( $(z * x) * (z * y) \in I$ , resp.),  $z \in X$  implies  $x * y \in I$ ,
- *reverse semi ideal*:  $(x * z) * (y * z) \in I$  and  $(z * x) * (z * y) \in I$ ,  $z \in X$  imply  $x * y \in I$

for all  $x, y \in X$ .

### 3. Some types of fuzzy ideals on groupoids

Let  $(X, *, \leq)$  be a partially ordered groupoid. A map  $\mu : X \rightarrow [0, 1]$  is said to be a *fuzzy subgroupoid* of  $(X, *)$  if, for all  $x, y \in X$ ,

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

A map  $\mu : X \rightarrow [0, 1]$  is said to be a *order preserving mapping* if

$$x \leq y \Rightarrow \mu(x) \leq \mu(y).$$

A map  $\mu : X \rightarrow [0, 1]$  is said to be a *fuzzy partially ordered subgroupoid* if it is both a fuzzy subgroupoid of  $(X, *)$  and a order preserving mapping. A map  $\mu : X \rightarrow [0, 1]$  is said to be *full* if  $\mu(x) \leq \mu(y)$  implies  $x \leq y$ . A map  $\mu : X \rightarrow [0, 1]$  is said to be a *fuzzy right*

(left, resp.) compatible if  $\mu(x) \leq \mu(y)$  implies  $\mu(x * z) \leq \mu(y * z)$  ( $\mu(z * x) \leq \mu(z * y)$ , resp.) for all  $z \in X$ .

A fuzzy subgroupoid  $\mu : (X, *) \rightarrow [0, 1]$  is said to be a

- *fuzzy right (left, resp.) semi-ideal* if, for all  $x, y, z \in X$ ,  $\mu((x * z) * (y * z)) \leq \mu(x * y)$  ( $\mu((z * x) * (z * y)) \leq \mu(x * y)$ , resp.).
- *fuzzy semi-ideal* if, for all  $x, y, z \in X$ ,  $\mu((x * z) * (y * z)) \leq \mu(x * y)$  and  $\mu((z * x) * (z * y)) \leq \mu(x * y)$ .
- *fuzzy reverse right semi-ideal (fuzzy reverse left semi-ideal, resp.)* if, for all  $x, y, z \in X$ ,  $\mu((x * z) * (y * z)) \geq \mu(x * y)$  ( $\mu((z * x) * (z * y)) \geq \mu(x * y)$ , resp.).
- *fuzzy reverse semi-ideal* if, for all  $x, y, z \in X$ ,  $\mu((x * z) * (y * z)) \geq \mu(x * y)$  and  $\mu((z * x) * (z * y)) \geq \mu(x * y)$ .

**THEOREM 3.1.** *Let  $(X, *, 0)$  be a BCK-algebra and  $\mu : (X, *) \rightarrow [0, 1]$  be a fuzzy set. The following hold:*

- (i) if  $\mu$  is a fuzzy right semi-ideal, then  $\mu(0)$  is a minimal element of  $\mu(X)$ ,
- (ii) if  $\mu$  is a fuzzy left semi-ideal, then  $\mu(0)$  is a maximal element of  $\mu(X)$ .

**THEOREM 3.2.** *Let  $(X, *, 0)$  be a BCK-algebra and  $\mu : (X, *) \rightarrow [0, 1]$  be a fuzzy set. The following hold:*

- (i) if  $\mu$  is a fuzzy reverse right semi-ideal, then  $\mu(0)$  is a maximal element of  $\mu(X)$ ,
- (ii) if  $\mu$  is a fuzzy reverse left semi-ideal, then  $\mu(0)$  is a minimal element of  $\mu(X)$ .

Given a groupoid  $(X, *)$  and a poset  $(X, \leq)$ , and let  $\mu : (X, *) \rightarrow [0, 1]$  be a fuzzy subset of  $(X, *)$ . We define a set  $P_\mu(*; \leq)$  as follows:

$$P_\mu(*; \leq) := \{y * x \mid \mu(x) \leq \mu(y)\}.$$

Note that  $(X, *) \neq P_\mu(*; \leq)$ , i.e.,  $P_\mu(*; \leq)$  is only a set, not containing an algebraic structure. We have  $\mu(x) \leq \mu(x)$  for each  $x \in X$  whence  $P_\mu(*; \leq)$  always contains the non-empty subset  $\{x * x \mid x \in X\}$ .

A groupoid  $(X, *)$  is said to be a pogroupoid [4] if (i)  $x * y \in \{x, y\}$ ; (ii)  $x * (y * x) = y * x$ ;

(iii)  $(x * y) * (y * z) = (x * y) * z$  for all  $x, y, z \in X$ .

Let  $\mu : X \rightarrow [0, 1]$  be a fuzzy subset. Define a binary operation  $\odot$  on  $X$  by

$$x \odot y = \begin{cases} x & \text{if } \mu(x) > \mu(y) \\ y & \text{if } \mu(x) \leq \mu(y). \end{cases}$$

for all  $x, y \in X$ . We call such a groupoid  $(X, \odot)$  a selective groupoid induced by a fuzzy subset  $\mu$ , and we denote it by  $(X, \odot)_\mu$ .

**PROPOSITION 3.3.** *Let  $(X, \odot)_\mu$  be a selective groupoid induced by  $\mu$ . Define  $x \leq_\mu y \iff y \odot x = y = x \odot y$ . Then  $P_\mu(*; \leq_\mu) = X$ .*

**THEOREM 3.4.** *Let  $(X, *, 0)$  be a BCK-algebra and let  $\mu : (X, *) \rightarrow [0, 1]$  be a fuzzy set. The following hold:*

- (i) if  $\mu$  is a right semi-ideal, then  $P_\mu(*; \leq) = X$ ,
- (ii) if  $\mu$  is a fuzzy left semi-ideal, then  $0 \in P_\mu(*; \leq)$ ,
- (iii) if  $\mu$  is a fuzzy reverse left semi-ideal, then  $P_\mu(*; \leq) = X$ ,
- (iv) if  $\mu$  is a fuzzy reverse right semi-ideal, then  $0 \in P_\mu(*; \leq)$ .

PROOF. (i) Assume  $\mu : (X, *) \rightarrow [0, 1]$  is a fuzzy right semi-ideal of a *BCK*-algebra  $(X, *, 0)$ . By Theorem 3.1(i),  $\mu(0) \leq \mu(x)$  for all  $x \in X$ , then  $x * 0 \in P_\mu(*; \leq)$ . Thus  $x = x * 0 \in P_\mu(*; \leq)$ , i.e.,  $P_\mu(*; \leq) = X$ .

(ii) Assume  $\mu : (X, *) \rightarrow [0, 1]$  is a fuzzy left semi-ideal of a *BCK*-algebra  $(X, *, 0)$ . By Theorem 3.1(ii),  $\mu(x) \leq \mu(0)$  for all  $x \in X$ , then  $0 * x \in P_\mu(*, \leq)$ . Thus  $0 = 0 * x \in P_\mu(*, \leq)$ .

(iii) Assume  $\mu : (X, *) \rightarrow [0, 1]$  is a fuzzy reverse left semi-ideal of a *BCK*-algebra  $(X, *, 0)$ . By Theorem 3.2(i),  $\mu(0) \leq \mu(x)$  for all  $x \in X$ , then  $x * 0 \in P_\mu(*; \leq)$ . Thus  $x = x * 0 \in P_\mu(*; \leq)$ , i.e.,  $P_\mu(*; \leq) = X$ .

(iv) Assume  $\mu : (X, *) \rightarrow [0, 1]$  is a fuzzy left semi-ideal of a *BCK*-algebra  $(X, *, 0)$ . By Theorem 3.2(ii),  $\mu(x) \leq \mu(0)$  for all  $x \in X$ , then  $0 * x \in P_\mu(*, \leq)$ . Thus  $0 = 0 * x \in P_\mu(*, \leq)$ .  $\square$

**THEOREM 3.5.** *If  $\mu : (X, *) \rightarrow [0, 1]$  is a fuzzy right (left, resp.) compatible over a poset  $(X, \leq)$ , if and only if  $P_\mu$  is a right (left, resp.) semi-ideal of  $(X, *)$ .*

PROOF. ( $\Rightarrow$ ) If  $x * y \in P_\mu$ , then  $\mu(y) \leq \mu(x)$ . Since  $\mu$  is a fuzzy right (left, resp.) compatible over a poset  $(X, \leq)$ , we obtain  $\mu(y * z) \leq \mu(x * z)$  for all  $z \in X$ . It follows that  $(x * z) * (y * z) \in P_\mu$ .

( $\Leftarrow$ ) If  $\mu(x) \leq \mu(y)$ , then  $y * x \in P_\mu$ . Since  $P_\mu$  is a right (left, resp.) semi-ideal of  $(X, *)$ , we obtain  $(y * z) * (x * z) \in P_\mu$  ( $(z * y) * (z * x) \in P_\mu$ ) for all  $z \in X$ . Thus,  $\mu(x * z) \leq \mu(y * z)$  ( $(z * y) * (z * x) \in P_\mu$ ).  $\square$

**PROPOSITION 3.6.** *Let  $\mu : (X, *) \rightarrow [0, 1]$  be a full fuzzy partially ordered subgroupoid. If  $P_\mu(*; \leq)$  is a subgroupoid of groupoid  $(X, *)$ , then  $P_\mu(*; \leq)$  is a singleton set.*

**THEOREM 3.7.** *Let  $\mu : (X, *) \rightarrow [0, 1]$  be a full fuzzy partially ordered subgroupoid and  $(X, *)$  be medial and compatible over a poset  $(X, \leq)$ . Then  $P_\mu(*; \leq)$  is a singleton set.*

## 4. Conclusion

In this study, we explored the interplay between groupoids  $(X, *)$ , posets  $(X, \leq)$  and fuzzy subset  $\mu$  through the lens of the set  $P_\mu(*; \leq) = \{y * x \mid \mu(x) \leq \mu(y)\}$ . As a direction of research, extend these results to BE-algebras, pseudo-BCK algebras, or quantum B-algebras, where implications and order relations play a central role, and investigate whether  $P_\mu(*; \leq)$  induces a topological closure operator or a Galois connection between the lattice of groupoids and the lattice of posets on  $X$ . This work opens doors to unifying order theory and algebraic structures, with potential applications in logic, combinatorics, and theoretical computer science. The interplay between groupoids, posets and fuzzy subset  $\mu$  via  $P_\mu(*; \leq)$  offers a rich framework for future research, especially in non-classical algebras and ordered algebraic structures.

## References

1. Ahn, S. S. and Kim, Y. H. (2009) *Some constructions of implicative/commutative d-algebras*, Bull. Korean Math. Soc., **46**(1), 147–153.
2. Borůvka, O. (1976), *Foundations of the Theory of Groupoids and Groups*. John Wiley & Sons, New York.
3. Meng, J. and Jun, Y. B. (1994), *BCK-algebras*. Kyungmoon Sa, Seoul.
4. Neggers, J. (1976) *Partially ordered sets and groupoids*, Kyungpook Math. J., **16**, 7–20.
5. Neggers, J. and Kim, H. S. (1999) *On d-algebras*, Math. Slovaca, **49**(1), 19–25.
6. Rezaei, A., Park, C. and Kim, H. S. (2023) *Some orders in groupoids and its applications to fuzzy groupoids*, Transactions on Fuzzy Sets and Systems, **4**(1), 4–13.



## Some upper bounds for the index of the second $n$ -center subgroup of an $n$ -abelian group

Azam Hokmabadi<sup>1,\*</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: ahokmabadi@pnu.ac.ir

**ABSTRACT.** Let  $n$  be a positive integer. A group  $G$  is said to be  $n$ -abelian, if  $(xy)^n = x^n y^n$ , for any  $x, y \in G$ . In 1979, Fay and Waals introduced the  $n$ -potent and the  $n$ -center subgroups of a group  $G$ , as  $\gamma_2^n(G) = \langle [x, y^n] | x, y \in G \rangle$  and  $Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}$ , respectively. Also, the second  $n$ -center subgroup,  $Z_2^n(G)$ , is defined by  $Z_2^n(G)/Z^n(G) = Z^n(G/Z^n(G))$ . In this paper, we give some upper bounds for the index of the second  $n$ -center subgroup of an  $n$ -abelian group  $G$ , in terms of the order of the  $n$ -potent subgroup, under some conditions.

**Keywords:**  $n$ -abelian group, the second  $n$ -center subgroup,  $n$ -potent subgroup.

**AMS Mathematics Subject Classification [2020]:** 20F14, 20D99

### 1. Introduction

Let  $n$  be a positive integer. A group  $G$  is said to be  $n$ -abelian, if  $(xy)^n = x^n y^n$ , for all elements  $x$  and  $y$  in  $G$ , from which it follows that  $[x^n, y] = [x, y]^n = [x^n, y^n]$ , where  $[x, y] = x^{-1}y^{-1}xy$ . This concept has been discussed in 1944 by Levi [3], for the first time. In 1979, Fay and Waals [1] introduced the  $n$ -potent and the  $n$ -center subgroups of a group  $G$ , denoted by  $\gamma_2^n(G)$  and  $Z^n(G)$  respectively, as follows:

$$\gamma_2^n(G) = \langle [x, y^n] | x, y \in G \rangle, \quad Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}.$$

It is easy to see that  $\gamma_2^n(G)$  is a fully invariant subgroup and  $Z^n(G)$  is a characteristic subgroup of group  $G$ . The concepts of  $n$ -potent and  $n$ -center subgroups are extensions of the important subgroups  $\gamma_2(G)$  and  $Z(G)$ , respectively. Also, the second  $n$ -center subgroup,  $Z_2^n(G)$ , is defined by  $Z_2^n(G)/Z^n(G) = Z^n(G/Z^n(G))$ .

A famous theorem of I. Schur [8] asserts that if the center of a group  $G$  has finite index, then the derived subgroup of  $G$  is finite. Infinite extra special groups show that the converse of Schur's theorem does not hold, in general. However, P. Hall [2] observed that if  $\gamma_2(G)$  is finite then  $|G : Z_2(G)|$  is finite and bounded above in terms of the order of  $\gamma_2(G)$ . The first explicit bound was given by I.D. Macdonald [4].

In 2020, Pourmirzaei et al. [5] extended Schur's theorem in the variety of  $n$ -abelian groups and proved that the finiteness of the index of the  $n$ -center subgroup implies the finiteness of the order of the  $n$ -potent subgroup, for an  $n$ -abelian group  $G$ . In this paper,

\*Speaker.

we prove that if  $G$  is an  $n$ -abelian group such that  $\gamma_2^n(G)$  is finite and  $\frac{\gamma_2^n(G)}{\gamma_2^n(G) \cap Z^n(G)}$  is a cyclic group, then  $|G : Z_2^n(G)|$  is finite and  $|G : Z_2^n(G)| \leq |\frac{\gamma_2^n(G)}{\gamma_2^n(G) \cap Z^n(G)}|^2$ . Also we show that if  $G$  is a finite  $n$ -abelian group and  $\gamma_2^n(G)$  is a cyclic group of order  $m$ , then  $|G : Z_2(G)| \leq m\phi(m)$ .

## 2. Main results

In order to prove the main results, we need two following lemmas from [5].

**Lemma 2.1.** *Let  $G$  be a group and  $C$  be a proper subgroup of  $G$ . Then*

$$\gamma_2^n(G) = [G - C, G^n].$$

**Lemma 2.2.** *Let  $G$  be an  $n$ -abelian group,  $Z = \gamma_2^n(G) \cap Z^n(G)$  and  $|\gamma_2^n(G) : Z| = m$ . Suppose that  $T$  is a subgroup of  $G$  such that  $\gamma_2^n(G) \leq T$  and the following properties hold.*

- (i)  $\gamma_2^n(G) = \gamma_2^n(T)Z$ ,
- (ii)  $\gamma_2^n(G) \cap Z^n(T) = Z$ ,
- (iii)  $d(\frac{T}{Z}) = K$ .

*Then there exists a subgroup  $M$  of  $G$  such that  $|G : M| \leq m^k$ , and  $[M, G^n, G^n] = 1$ .*

Now we prove the following lemma which plays an important role to prove the main results.

**Lemma 2.3.** *Let  $G$  be an  $n$ -abelian group and  $Z = \gamma_2^n(G) \cap Z^n(G)$ . Let  $\gamma_2^n(G)$  be a  $p$ -group and  $\frac{\gamma_2^n(G)}{Z}$  be a cyclic group of order  $m$ . Then there exist a subgroup  $M$  of  $G$  such that  $|G : M| \leq m^2$  and  $[M, G^n, G^n] = 1$ .*

PROOF. We can find an element  $x \in \gamma_2^n(G)$  such that  $x \notin Z$  and  $x^p \in Z$ . Put  $C = C_G(x^n)$ . It follows that  $C \neq G$  and  $C_G(y^n) \cap \gamma_2^n(G) = Z^n(G) \cap \gamma_2^n(G)$ , for all  $y \in G - C$ . Since  $C$  is a proper subgroup of  $G$ , we have  $\gamma_2^n(G) = [G - C, G^n]$ , by Lemma 2.1. Hence there exist elements  $a \in G - C$  and  $b \in G$  such that  $\gamma_2^n(G) = \langle [a, b^n], Z \rangle$ .

Now suppose  $T = \langle a, b, Z \rangle$ . One can see that  $T$  satisfies the three conditions of Lemma 2.2 with  $k=2$ . Therefore, by Lemma 2.2, there exists a subgroup  $M$  of  $G$  such that  $|G : M| \leq m^2$  and  $[M, G^n, G^n] = 1$ . □

Now we are ready to state the first main result of the paper.

**Theorem 2.4.** *Let  $G$  be an  $n$ -abelian group and  $Z = \gamma_2^n(G) \cap Z^n(G)$ . If  $\gamma_2^n(G)$  is finite and  $\frac{\gamma_2^n(G)}{Z}$  is a cyclic group, then  $|G : Z_2^n(G)|$  is finite and  $|G : Z_2^n(G)| \leq |\frac{\gamma_2^n(G)}{Z}|^2$ .*

PROOF. If  $\gamma_2^n(G)$  is a  $p$ -group then the result follows, by Lemma 2.3.

Let  $\gamma_2^n(G)$  be of order  $m = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$  and  $P_1, P_2, \dots, P_t$  be Sylow subgroups of  $\gamma_2^n(G)$ . Since  $\frac{\gamma_2^n(G)}{Z}$  is cyclic,  $\gamma_2^n(G)$  is abelian. Let  $N_{p_i}$  be the normal  $p_i$ -complement of  $\gamma_2^n(G)$ . Then  $\gamma_2^n(\frac{G}{N_{p_i}})$  is a  $p_i$ -group such that  $\gamma_2^n(\frac{G}{N_{p_i}}) \cong P_i$ . Since  $\frac{Z^n(G)N_{p_i}}{N_{p_i}} \leq Z^n(\frac{G}{N_{p_i}})$ , so  $\frac{\gamma_2^n(G/N_{p_i})}{\gamma_2^n(G/N_{p_i}) \cap Z^n(G/N_{p_i})}$  is a cyclic group of order dividing  $m_{p_i}$ , the  $p_i$ -part of  $m$ .

Now consider the group  $\frac{G}{N_{p_i}}$  and put  $Z_i = \gamma_2^n(\frac{G}{N_{p_i}}) \cap Z^n(\frac{G}{N_{p_i}})$ . Then  $\gamma_2^n(\frac{G}{N_{p_i}})$  is a  $p_i$ -group and  $\frac{\gamma_2^n(G/N_{p_i})}{Z_i}$  is cyclic of order dividing  $m_{p_i}$ . Hence by Lemma 2.3, there exists a subgroup  $\frac{M_i}{N_{p_i}} \leq \frac{G}{N_{p_i}}$  such that  $|\frac{G}{N_{p_i}} : \frac{M_i}{N_{p_i}}| \leq m_{p_i}^2$  and  $[\frac{M_i}{N_{p_i}}, (\frac{G}{N_{p_i}})^n, (\frac{G}{N_{p_i}})^n] = 1$ .

Put  $M = \bigcap_{i=1}^t M_i$ . Then we have  $[M, G^n, G^n] \leq \bigcap_{i=1}^t [M_i, G^n, G^n] \leq \bigcap_{i=1}^t N_{p_i} = 1$  and so  $M \leq Z_2^n(G)$ . On the other hand

$$|G : M| \leq \prod_{i=1}^t |G : M_i| \leq \prod_{i=1}^t \left| \frac{G}{N_{p_i}} : \frac{M_i}{N_{p_i}} \right| \leq \prod_{i=1}^t m_{p_i}^2 = m^2.$$

Therefore we have

$$|G : Z_2^n(G)| \leq |G : M| \leq m^2 = \left| \frac{\gamma_2^n(G)}{Z} \right|^2.$$

□

Now we try to find another upper bound for the index of the second  $n$ -center subgroup of a group  $G$ , in terms of the order of the  $n$ -potent subgroup of  $G$ , when the  $n$ -potent subgroup is cyclic. First, we prove the following theorem which gives an upper bound for the index of the  $n$ -center subgroup of a finite  $n$ -abelian group  $G$ , under some conditions.

**Theorem 2.5.** *Let  $G$  be a finite  $n$ -abelian group. Suppose  $\gamma_2^n(G)$  is a cyclic  $p$ -group of order  $m$ . If  $\gamma_2^n(G) \cap Z^n(G) = 1$ , then  $|G : Z^n(G)| \leq m\phi(m)$ , where  $\phi$  is Euler function.*

PROOF. Put  $D = C_G(\gamma_2^n(G))$ . Using three subgroup lemma, we have  $[D^n, D, G] = 1$  and so  $\gamma_2^n(D) \leq Z^n(G)$ . Then  $\gamma_2^n(D) \leq \gamma_2^n(G) \cap Z^n(G) = 1$  and hence  $\gamma_2^n(D) = [D^n, D] = 1$ . It is clear that  $\frac{G}{D}$  is isomorphic to a subgroup of  $\text{Aut}(\gamma_2^n(G))$ . If  $\gamma_2^n(G)$  is cyclic of order  $p = 2$ , then the unique element of order 2 in  $\gamma_2^n(G)$  is central in  $G$  and so  $\gamma_2^n(G) \cap Z^n(G) \neq 1$ . Hence we have  $p > 2$ . Therefore  $\frac{G}{D}$  is cyclic and

$$(1) \quad \left| \frac{G}{D} \right| \leq \phi(m).$$

Suppose  $\frac{G}{D} = \langle xD \rangle$ . Put  $C = C_D(x^n)$ . Since  $G$  is  $n$ -abelian, we obtain

$$(2) \quad C \leq Z^n(G).$$

On the other hand,  $|D : C| = |[D, x^n]| \leq |\gamma_2^n(G)| = m$ . Hence by (1) and (2), we have

$$|G : Z^n(G)| \leq |G : C| = |G : D| |D : C| \leq m\phi(m)$$

□

Now the second main result of the paper is obtained.

**Corollary 2.6.** *Let  $G$  be finite  $n$ -abelian group and  $\gamma_2^n(G)$  be a cyclic group of order  $m$ . Then*

$$|G : Z_2(G)| \leq m\phi(m)$$

PROOF. Similar to the proof of Theorem 2.4 and using the multiplicative property of Euler function  $\phi$ , we can reduce the proof to the case  $\gamma_2^n(G)$  is a  $p$ -group. If  $\gamma_2^n(G) \cap Z^n(G) \neq 1$ , then  $|Z| \geq p$  and by Theorem 2.4, we have

$$|G : Z_2(G)| \leq \frac{m^2}{p^2} \leq m\phi(m).$$

If  $\gamma_2^n(G) \cap Z^n(G) = 1$ , then  $|G : Z_2(G)| \leq |G : Z^n(G)| \leq m\phi(m)$ , by Theorem 2.5 and the result follows. □

### 3. Conclusion

Let  $G$  be an  $n$ -abelian group and  $Z = \gamma_2^n(G) \cap Z^n(G)$ . In this paper we prove that if  $\gamma_2^n(G)$  is finite and  $\frac{\gamma_2^n(G)}{Z}$  is a cyclic group, then  $|G : Z_2^n(G)|$  is finite and  $|G : Z_2^n(G)| \leq |\frac{\gamma_2^n(G)}{Z}|^2$ . Also we show that if  $G$  is a finite  $n$ -abelian group and  $\gamma_2^n(G)$  is a cyclic group of order  $m$ , then  $|G : Z_2(G)| \leq m\phi(m)$ .

### References

1. Fay, T.H and Waals, G.L. (1983) *Some remarks on  $n$ -potent and  $n$ -abelian groups*, J. Indian. Math. Soc. **47**, 217–222.
2. Hall, P. (1956) *Finite-by-nilpotent groups*, Proc. Camb. Phil. Soc. **52**, 611-616 .
3. Levi F.W. (1944) *Notes on group theory I*, J. Indian. Math. Soc., **8**, 1–7.
4. Macdonald, I.D. (1961) *Some explicit bounds in groups with finite derived groups*, proc. London math. Soc. **11**(3), 23–56.
5. Pourmirzaei, A. (2020) *On the order of the  $n$ -Center factor subgroup of an  $n$ -abelian group*, Global Analysis and discrete Mathematics, **6**, 303–308.





## Various Types of Injectivity in the Categories of Krasner $(m, n)$ -ary Hypermodules

Najmeh Jafarzadeh<sup>1,\*</sup>, Reza Ameri<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: jafarzadeh@phd.pnu.ac.ir

<sup>2</sup>School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

Email: rameri@ut.ac.ir

**Keywords:** Krasner  $(m, n)$ -hypermodule, Injectivity, Baer's criterion)

**AMS Mathematics Subject Classification [2020]:** 20N20 )

**ABSTRACT.** In this paper, we study the concepts of injectivity in the categories of Krasner  $(m, n)$ - hypermodules over a Krasner  $(m, n)$ - hyperring to generalize the injective modules over a ring. As the main result, we introduce the concepts of normal Baerian injectivity and Baerian injectivity as the versions of Baer's criterion in the categories of Krasner  $(m, n)$ - hypermodules. Finally we find the relation between fundamental functor and normal injective  $(m, n)$ -hypermodules

### 1. Introduction

The theory of hyperstructures was introduced by Marty in 1934 during the 8th Congress of the Scandinavian Mathematicians [8]. He introduced hypergroups as a generalization of groups, and then various researches have been worked on this new field of modern algebra and many authors developed it. The notion of the hyperring was studied by Krasner [7] and then some researchers followed him. Also, hypermodules over a hyperring is a generalization of the classical modules over a ring. Several kinds of hyperrings and hypermodules were introduced and studied by many authors, see e.g., [1, 4, 7].

In 1928, Dörnte introduced the concept of  $n$ -ary groups [5] and since then,  $n$ -ary systemes have been studied in different contexts. The research about  $n$ -ary hyperstructure was initiated by Davvaz and Vougiouklis who introduced these structures. Fixing a Krasner hyperring  $R$ , . Also Jafarzadeh and Ameri introduced the category of  $R_{(m,n)}$ -hypermodules over a Krasner  $(m, n)$ -hyperring  $R$  (see [6]). In this paper, for convenience of the reader, we first recall some notations and basic concepts from [6], [9]. Then we study various types of injectivity for a  $(m, n)$ -Krasner hypermodule. In particular, we focus on Baer's criterion in these categories.

\*Speaker.

**1.1. Preliminaries.** [3] A category denoted by  $\mathcal{C}$  consists of

- (i) A class of objects:  $A, B, C, \dots$
- (ii) A class of morphisms or arrows:  $f, g, h, \dots$

with the following data:

- Given morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , that is, with:  $\text{cod}(f) = \text{dom}(g)$  there is given a morphism:  $g \circ f : A \rightarrow C$  called the composition of morphisms  $f$  and  $g$ .
- Associativity:  $h \circ (g \circ f) = (h \circ g) \circ f$  for all  $f : A \rightarrow B, g : B \rightarrow C$  and  $h : C \rightarrow D$ .
- Identity:  $f \circ \text{id}_A = f = \text{id}_B \circ f$  for all  $f : A \rightarrow B$ .

objects  $A$  and  $B$  of category  $\mathcal{C}$  is denoted by  $\text{Mor}_{\mathcal{C}}(A, B)$ .

A morphism  $f \in \text{Mor}_{\mathcal{C}}(B, C)$  is said to be a *mono* (or *monic*) if for all  $g, h \in \text{Mor}_{\mathcal{C}}(A, B)$ ,  $f \circ g = f \circ h$  implies  $g = h$ .

$E \in \text{Ob}(\mathcal{C})$  is called  $\mathcal{C}$ -*injective* if for every mono  $g \in \text{Mor}_{\mathcal{C}}(A, B)$  and every  $f \in \text{Mor}_{\mathcal{C}}(B, E)$ , there exists  $\bar{f} \in \text{Mor}_{\mathcal{C}}(A, E)$  such that  $\bar{f} \circ g = f$ .

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & \swarrow \bar{f} & \\ E & & \end{array}$$

**DEFINITION 1.1.** [2] Let  $M$  be a nonempty set. Then  $(M, f, g)$  is an  $(m, n)$ -*hypermodule* over an  $(m, n)$ -hyperring  $(R, h, k)$ , if  $(M, f)$  is an  $m$ -ary hypergroup and the map  $g : \underbrace{R \times \dots \times R}_{n-1} \times M \rightarrow P^*(M)$

satisfies the following conditions:

- (i)  $g(r_1^{n-1}, f(x_1^m)) = f(g(r_1^{n-1}, x_1), \dots, g(r_1^{n-1}, x_m))$
- (ii)  $g(r_1^{i-1}, h(s_1^m), r_{i+1}^{n-1}, x) = f(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \dots, g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$
- (iii)  $g(r_1^{i-1}, k(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = g(r_1^{n-1}, g(r_m^{n+m-2}, x))$
- (iv)  $0 \in g(r_1^{i-1}, 0, r_{i+1}^{n-1}, x)$

**DEFINITION 1.2.** [2] Let  $(M, f, g)$  be an  $(m, n)$ -ary hypermodule over an  $(m, n)$ -ary hyperring  $(R, h, k)$ , then  $\hat{\epsilon}$  denoted the transitive closure of the relation  $\epsilon = \bigcup_{\alpha \geq 0} \epsilon_\alpha$ , where

$\epsilon_0$  is the diagonal, i.e.  $\epsilon_0 = \{(x, x) | x \in M\}$  and for every integer  $\alpha \geq 1$ ,  $\epsilon_\alpha$  is the relation defined as follows:

$$x \epsilon_\alpha y \text{ if and only if } \{x, y\} \subseteq f_{(\alpha)},$$

for some  $\alpha \in \mathbb{N}$ . if  $x \epsilon_0 y$  (i.e.  $x = y$ ) then we write  $\{x, y\} \subseteq u_{(0)}$ . we define  $\epsilon^*$  as the smallest equivalence relation such that the quotient  $(M/\epsilon^*, f/\epsilon^*, g/\epsilon^*)$  is an  $(m, n)$ -ary module over an  $(m, n)$ -ary hyperring  $R$ , where  $M/\epsilon^*$  is the set of equivalence classes. The  $\epsilon^*$  is called *fundamental equivalence relation*.

**DEFINITION 1.3.** [2] Let  $(M_1, f_1, g_1)$  and  $(M_2, f_2, g_2)$  be two  $(m, n)$ -hypermodules over an  $(m, n)$ -hyperring  $(R, h, k)$ . we say that  $\phi : M_1 \rightarrow P^*(M_2)$  is a multivalued homomorphism of  $(m, n)$ -hypermodules if for all  $x_1^m, x$  of  $M_1$  and  $r_1^{n-1} \in R$ :

$$\phi\left(f_1(x_1, \dots, x_m)\right) \subseteq f_2\left(\phi(x_1), \dots, \phi(x_m)\right)$$

$\phi(g_1(r_1^{n-1}, x)) = g_2(r_1^{n-1}, \phi(x))$ . is said to be a  $R$ -homomorphism, for short  $R - mv$ -homomorphism.

For two multivalued functions  $f : A \rightarrow P^*(B)$  and  $g : B \rightarrow P^*(C)$ , the composition  $g \circ f$  is defined as the following:

$$(1) \quad (g \circ f)(a) = \bigcup_{b \in f(a)} g(b), \quad \forall a \in A,$$

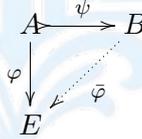
and an identity morphism for an object  $A$  is  $id_A(x) = \{x\}$  for all  $x \in A$ .

## 2. Injectivity

DEFINITION 2.1. [3] *Eventhough we defined these sets similar to the classical case, these definitions are not necessarily categorical concepts.*

- (i) *injective if for all  $a_1, a_2 \in A$ ,  $\varphi(a_1) = \varphi(a_2)$  implies  $a_1 = a_2$ .*
- (ii) *strongly injective if for all  $a_1, a_2 \in A$ ,  $\varphi(a_1) \cap \varphi(a_2) \neq \emptyset$  implies  $a_1 = a_2$ .*

DEFINITION 2.2. *Naturally, it is time to define the concept of injectivity for  $R$ -hypermultiples.* In a category  $\mathcal{C}$ , it is well known an object  $E$  is called  $\mathcal{C}$ -injective, if for every mono  $\psi \in \text{Mor}_{\mathcal{C}}(A, B)$  and every  $\varphi \in \text{Mor}_{\mathcal{C}}(A, E)$ , there exists  $\bar{\varphi} \in \text{Mor}_{\mathcal{C}}(B, E)$  such that  $\bar{\varphi} \circ \psi = \varphi$ . A mono in the category  $R_{(m,n)} - \mathcal{H}\text{mod}$  (resp.,  $R_{(m,n)} - \mathcal{h}\text{mod}$ ) is said to be an  $R_{mv}$ -monomorphism (resp.,  $R$ -monomorphism). So an  $R$ -hypermultiples  $E$  is  $R_{(m,n)} - \mathcal{H}\text{mod}$ -injective (resp.,  $R_{(m,n)} - \mathcal{h}\text{mod}$ -injective), for short  $\mathcal{H}$ -injective (resp.,  $h$ -injective) if for every  $R_{mv}$ -monomorphism  $\psi \in \text{Hom}_R(A, B)$  (resp.,  $R$ -monomorphism  $\psi \in \text{hom}_R(A, B)$ ) and every  $\varphi \in \text{Hom}_R(A, E)$  (resp.,  $\varphi \in \text{hom}_R(A, E)$ ), there exists  $\bar{\varphi} \in \text{Hom}_R(B, E)$  (resp.,  $\bar{\varphi} \in \text{hom}_R(B, E)$ ) such that  $\bar{\varphi} \circ \psi = \varphi$ .



In the following we study some types of injectivity.

DEFINITION 2.3. *An  $(m, n)$ -hypermultiples  $E$  is called*

- (i) *(strongly) injective if for every (strongly) injective  $\psi \in \text{Hom}_R(A, B)$  and every  $\varphi \in \text{Hom}_R(A, E)$ , there exist  $\bar{\varphi} \in \text{Hom}_R(B, E)$ , a set of  $R$ -hypermultiples  $\{X_i\}_{i=1}^m$  ( $m \in \mathbb{N}$ ) and  $\varphi_i \in \text{Hom}_R(X_i, E)$  such that*

$$(\bar{\varphi} \circ \psi)(a) \subseteq f_B(\varphi_1(x_1), \varphi_2(x_2), \dots, \varphi_m(x_m)) \supseteq \varphi(a) \quad \forall a \in A$$

*in which  $x_i \in X_i$ . In this case, we write  $\bar{\varphi} \circ \psi \preceq \varphi$ .*

- (ii) *weakly injective if for every injective single-valued  $\psi \in \text{Hom}_R(A, B)$  (i.e.,  $\psi \in \text{hom}_R(A, B)$ ) and every  $\varphi \in \text{Hom}_R(A, E)$ , there exists  $\bar{\varphi} \in \text{Hom}_R(B, E)$  such that  $\bar{\varphi} \circ \psi = \varphi$ .*

In addition to this concept we introduce the following injectivities.

DEFINITION 2.4. *An  $R$ -hypermultiples  $E$  is called normal injective if for every injective  $\psi \in \text{hom}_R(A, B)$  and every  $\varphi \in \text{hom}_R(A, E)$ , there exists  $\bar{\varphi} \in \text{hom}_R(B, E)$  such that  $\bar{\varphi} \circ \psi = \varphi$ .*

THEOREM 2.1. *Fundamental functor  $\mathcal{F}$  preserves monomorphisms.*

PROPOSITION 2.5. *Fundamental functor  $\mathcal{F}$  preserves normal injective  $(m, n)$ -hypermultiples.*

In the theory of modules, there is a useful result, usually known as Baer's criterion, for checking whether a module is injective: a left  $R$ -module  $E$  is injective if and only if any  $R$ -homomorphism  $\varphi: I \rightarrow E$  defined on a left ideal  $I$  of  $R$  can be extended to all of  $R$ . Inspired by this result, we introduce the following kind of injectivity:

**DEFINITION 2.6.** *An  $(m, n)$ -hypermodule  $E$  is called Baerian injective (resp., normal Baerian injective) if every  $\varphi \in \text{Hom}_R(I, E)$  (resp.,  $\varphi \in \text{hom}_R(I, E)$ ) can be extended to some  $\bar{\varphi} \in \text{Hom}_R(R, E)$  (resp.,  $\bar{\varphi} \in \text{hom}_R(R, E)$ ) for every  $(m, n)$ -hyperideal  $I$  of  $R$ .*

**PROPOSITION 2.7.** *Let  $E$  be an  $(m, n)$ -hypermodule. If  $E$  is a weakly injective  $(m, n)$ -hypermodule, then  $E$  is Baerian injective.*

**PROOF.** It is clear by taking  $g$  as the inclusion  $I \rightarrow R$  in the part (ii) of Definition 2.3. □

Now we are interested to provide some conditions so that the concepts of weakly injectivity and Baerian injectivity are equivalent. Indeed, we state the converse of Proposition 2.7 with some conditions.

**PROPOSITION 2.8.** *Let  $E$  be an  $R$ -hypermodule. If*

- (i)  $f_E(g_E(r, 1_R^{n-2}, x), y, 0^{(m-2)})$  is a singleton for all  $r \in R$  and all  $x, y \in E$ ;
- (ii) for all  $R$ -hypermodule  $B$ ,  $(\psi, \mu) \in \text{Hom}_R(B, E) \times \text{Hom}_R(R, E)$ ,  $r, s \in R$  and  $x, y \in B$ ,

$$\begin{aligned} \psi(f_B(x, -y, 0^{(m-2)})) &= g(h(r, -s, 0^{(m-2)}), 1_R^{n-2}, \mu(1)) \\ &= f_E(g_E(r, 1_R^{n-2}, \mu(1)), -g_E(s, 1_R^{n-2}, \mu(1)), 0^{(m-2)}) \end{aligned}$$

$$\implies f_E(\psi(x), g_E(s, 1_R^{n-2}, \mu(1)), 0^{(m-2)}) = f_E(g_E(r, 1_R^{n-2}, \mu(1)), \psi(y), 0^{(m-2)});$$

- (iii)  $E$  is Baerian injective;

then  $E$  is weakly injective.

### 3. Conclusion

Here we introduced some types of injectivity for  $(m, n)$ -hypermodules. Although  $(m, n)$ -hypermodules over an  $(m, n)$ -hyperring  $R$  are more general, natural and abundant than  $R$ -modules, constructing or searching for an injective  $(m, n)$ -hypermodule  $A$ , which  $R$  is not an  $(m, n)$ -ring and  $A$  is not an abelian  $m$ -group, may seem difficult because we face two complexities named hyperoperation and  $R_{mv}$ -homomorphism. So looking for some examples of  $(m, n)$ -hypermodules having any kind of injectivity is a possible and suggested area of research in future.

### Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the paper.

**References**

1. Amerii, R. (2003) On the categories of hypergroups and hypermodules, J. Discrete Math. Sci. cryptogr., **6**, 121-132.
2. Anvariye SM and Mirvakili,S and Davvazi,B. (2010) *Fundamental relation on  $(m,n)$ -hypermodules over  $(m,n)$ -hyperrings*, Ars combin. **94**,3273-288.
3. Awodey,S. (2010) *Category Theory*,Oxford university press. Oxford and New York.
4. Corsini,P. (1993) *Prolegemena of Hypergroup Theory.*, Aviani, Editor.Tricesimo .
5. Dörnte.W. (1928) *Untersuchungen Über einen verallgemeinerten Gruppenbegriff*, Math. Z., **29**, 1-19.
6. Jafarzadeh, N. and Ameri,R. (2018) *On the relation between categories of  $(m,n)$ -ary hypermodules and  $(m,n)$ -ary modules*, Sigma J. Eng. and Nat. Sci., **9**, 85-99.
7. Krasner.M. (1983) *A class of hyperring and hyperfields*, Int. J. Math. Math. Sci., **6**, 307-311.
8. Marty. F. (1934) *Sur une generalization de group*, In 8<sup>iem</sup> congres des Mathematiciens Scandinaves. Stockholm, , 45-49.
9. Shojaei, H. and Ameri,R. (2016) *Some Results On Categories of Krasner Hypermodules*, J FundamAppl Sci., **3S**, 2298-2306.





## Classification of generalized Heisenberg 3-Lie algebras of dimension 10 and rank 2, 3

Zari Ahmadieh<sup>1,\*</sup>, Hamid Darabi<sup>2</sup> and Farshid Saeedi<sup>3</sup>

<sup>1</sup>Department of Mathematics and Computer Sciences, Ma. C., Islamic Azad University, Mashhad, Iran.

Email: zari.ahmadieh@iau.ac.ir

<sup>2</sup>Department of Mathematics, Esfarayen University of Technology, Esfarayen, Iran.

Email: darabi@esfarayen.ac.ir

<sup>3</sup>Department of Mathematics and Computer Sciences, Ma. C., Islamic Azad University, Mashhad, Iran.

Email: saeedi@mshdiau.ac.ir

---

**ABSTRACT.** In this paper, we classify generalized Heisenberg 3-Lie algebras of dimension 10 and rank 2, 3.

**Keywords:** Lie algebra, generalized Heisenberg 3-Lie algebras, Nilpotent Lie algebra

**AMS Mathematics Subject Classification [2020]:** 17B05, 17B30

---

### 1. Introduction

The classification of low-dimensional Lie algebras is a fundamental problem in Lie algebra theory. It is known that there exists only one Abelian Lie algebra for each dimension. The classification of non-Abelian nilpotent Lie algebras up to dimension five over an arbitrary field is well understood. The classification of six-dimensional nilpotent Lie algebras over an arbitrary field was given by Cicalo et al. [1]. Furthermore, the classification of 7-dimensional nilpotent Lie algebras over a field  $\mathcal{K}$  with  $\text{char}(\mathcal{K}) \neq 2$  and of 8-dimensional 2-step nilpotent complex Lie algebras were presented in [5] and [6], respectively.

Filippov introduced  $n$ -Lie (Filippov) algebras in 1985 as  $n$ -ary, skew-symmetric, multilinear structures satisfying a generalized Jacobi identity, which reduces to the ordinary Lie algebra when  $n = 2$ . The 9-dimensional nilpotent 3-Lie algebras of class two were classified in [2]. In this paper, we classify generalized Heisenberg 3-Lie algebras of dimension 10 and ranks 2 and 3, using a classification method similar to that in [2].

An  $n$ -Lie algebra is called a *generalized Heisenberg  $n$ -Lie algebra of rank  $r$*  when  $A^2 = Z(A)$  and  $\dim A^2 = r$ ; the case  $r = 1$  corresponds to special Heisenberg algebras. For convenience, some known results used in the next sections are summarized in Table 1.

---

\*Speaker.

THEOREM 1.1. [4] Every special Heisenberg  $n$ -Lie algebra has dimension  $mn + 1$  for some natural number  $m$ , and it is isomorphic to

$$H(n, m) = \langle x, x_1, \dots, x_{nm} : [x_{n(i-1)+1}, x_{n(i-1)+2}, \dots, x_{ni}] = x, i = 1, \dots, m \rangle.$$

TABLE 1.  $d$ -dimensional generalized Heisenberg 3-Lie algebras of rank 2.

Name	Non-zero multiplication
$\mathcal{A}_{6,1}$	$[e_1, e_2, e_3] = e_5, [e_2, e_3, e_4] = e_6$
$\mathcal{A}_{7,1}$	$[e_1, e_2, e_3] = e_6, [e_3, e_4, e_5] = e_7$
$\mathcal{A}_{8,1}$	$[e_1, e_2, e_3] = e_7, [e_4, e_5, e_6] = e_8$
$\mathcal{A}_{9,1}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9$
$\mathcal{A}_{9,2}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9$

THEOREM 1.2. [3] Let  $A$  be a  $d$ -dimensional nilpotent  $n$ -Lie algebra, and let  $\dim A^2 = 1$ . Then, for some  $m \geq 1$ ,  $A$  is isomorphic to  $H(n, m) \oplus F(d - mn - 1)$ , where  $F(d - mn - 1)$  is an Abelian  $n$ -Lie algebra of dimension  $d - mn - 1$ .

THEOREM 1.3. [2] The 9-dimensional generalized Heisenberg 3-Lie algebra of rank 2 are  $\mathcal{A}_{6,1} \oplus F(3)$ ,  $\mathcal{A}_{7,1} \oplus F(2)$ ,  $\mathcal{A}_{8,1} \oplus F(1)$ ,  $\mathcal{A}_{9,1}$  and  $\mathcal{A}_{9,2}$ .

## 2. 10-dimensional generalized Heisenberg 3-Lie algebras

Suppose that  $A$  is a 10-dimensional generalized Heisenberg 3-Lie algebra of rank 2 with basis  $\{e_1, \dots, e_{10}\}$ . Also, suppose that  $A^2 = \langle e_9, e_{10} \rangle$ . In this case,  $A/\langle e_{10} \rangle$  is a 9-dimensional nilpotent 3-Lie algebra of class two with derived algebra of dimension 1. By using Theorem 1.2,  $A/\langle e_{10} \rangle$  is isomorphic to  $H(3, 1) \oplus F(5)$  or  $H(3, 2) \oplus F(2)$ .

LEMMA 2.1. Let  $A/\langle e_{10} \rangle \cong H(3, 1) \oplus F(5)$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,1}$ .

PROOF. Let  $A/\langle e_{10} \rangle \cong H(3, 1) \oplus F(5)$ . Then brackets in  $A$  can be written as  $[e_1, e_2, e_3] = e_9 + \alpha e_{10}$ ,  $[e_i, e_j, e_k] = \alpha_{ijk} e_{10}$ , where  $1 \leq i < j < k \leq 8$  and  $\{i, j, k\} \neq \{1, 2, 3\}$ . Regarding a suitable change of basis, one can assume that  $\alpha = 0$ . Since  $\dim(A/\langle e_9 \rangle)^2 = 1$ , by using Theorem 1.2,  $A/\langle e_9 \rangle \cong H(3, 2) \oplus F(2)$ . Theorem 1.1 completes the proof.  $\square$

LEMMA 2.2. Let  $A/\langle e_{10} \rangle \cong H(3, 2) \oplus F(2)$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,i}$ ,  $2 \leq i \leq 7$ .

The following theorem is an immediate consequence of Lemmas 2.1 and 2.2.

THEOREM 2.3. The only 10-dimensional generalized Heisenberg 3-Lie algebras of rank 2 are  $\mathcal{A}_{10,i}$ ,  $1 \leq i \leq 7$  (see Table 2).

TABLE 2. 10-dimensional generalized Heisenberg 3-Lie algebras of rank 2.

Name	Non-zero multiplication
$\mathcal{A}_{10,1}$	$[e_1, e_2, e_3] = e_9, [e_1, e_4, e_5] = [e_6, e_7, e_8] = e_{10}$
$\mathcal{A}_{10,2}$	$[e_1, e_2, e_3] = e_9, [e_4, e_5, e_6] = e_9 + e_{10}, [e_1, e_7, e_8] = e_{10}$
$\mathcal{A}_{10,3}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_9, [e_1, e_2, e_4] = [e_3, e_7, e_8] = e_{10}$
$\mathcal{A}_{10,4}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_9, [e_2, e_3, e_4] = [e_5, e_7, e_8] = e_{10}$
$\mathcal{A}_{10,5}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_9, [e_1, e_2, e_7] = [e_3, e_4, e_8] = e_{10}$
$\mathcal{A}_{10,6}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_9, [e_1, e_6, e_7] = [e_2, e_5, e_8] = e_{10}$

$\mathcal{A}_{10,7}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_9, [e_2, e_3, e_7] = [e_4, e_5, e_8] = e_{10}$
----------------------	---

Suppose that  $A$  is a 10-dimensional generalized Heisenberg 3-Lie algebra of rank 3 with basis  $\{e_1, \dots, e_{10}\}$ . Also, suppose that  $A^2 = \langle e_8, e_9, e_{10} \rangle$ . In this case,  $A/\langle e_{10} \rangle$  is a 9-dimensional nilpotent 3-Lie algebra of class two with derived algebra of dimension 2. By using Theorem 1.3 and Table 1,  $A/\langle e_{10} \rangle$  is isomorphic to  $\mathcal{A}_{6,1} \oplus F(3)$ ,  $\mathcal{A}_{7,1} \oplus F(2)$ ,  $\mathcal{A}_{8,1} \oplus F(1)$ ,  $\mathcal{A}_{9,1}$ , or  $\mathcal{A}_{9,2}$ .

LEMMA 2.4. *Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{6,1} \oplus F(3)$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,i}$ ,  $8 \leq i \leq 13$ .*

PROOF. Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{6,1} \oplus F(3)$ . Then brackets in  $A$  can be written as  $[e_1, e_2, e_3] = e_8 + \alpha e_{10}$ ,  $[e_2, e_3, e_4] = e_9 + \beta e_{10}$ ,  $[e_i, e_j, e_k] = \alpha_{ijk} e_{10}$ , where  $1 \leq i < j < k \leq 7$  and  $\{i, j, k\} \neq \{1, 2, 3\}, \{2, 3, 4\}$ . Regarding a suitable change of basis, one can assume that  $\alpha = \beta = 0$ . Since  $\dim(A/\langle e_8, e_9 \rangle)^2 = 1$ , by using Theorem 1.2,  $A/\langle e_8, e_9 \rangle$  is isomorphic to  $H(3, 1) \oplus F(4)$  or  $H(3, 2) \oplus F(1)$ . Theorem 1.1 implies the followings:

- (a) If  $A/\langle e_8, e_9 \rangle \cong H(3, 1) \oplus F(4)$ , then  $A$  is isomorphic to  $\mathcal{A}_{10,8}$ .
- (b) If  $A/\langle e_8, e_9 \rangle \cong H(3, 2) \oplus F(1)$ , then  $A$  is isomorphic to  $\mathcal{A}_{10,i}$ ,  $9 \leq i \leq 13$ .  $\square$

LEMMA 2.5.

- (a) *Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{7,1} \oplus F(2)$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,i}$ ,  $14 \leq i \leq 22$ .*
- (b) *Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{8,1} \oplus F(1)$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,8}$  or  $\mathcal{A}_{10,i}$ ,  $23 \leq i \leq 25$ .*
- (c) *Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{9,1}$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,8}$ ,  $\mathcal{A}_{10,9}$ ,  $\mathcal{A}_{10,11}$ ,  $\mathcal{A}_{10,12}$ ,  $\mathcal{A}_{10,17}$ ,  $\mathcal{A}_{10,18}$ ,  $\mathcal{A}_{10,19}$ ,  $\mathcal{A}_{10,21}$ ,  $\mathcal{A}_{10,24}$ , or  $\mathcal{A}_{10,i}$ ,  $26 \leq i \leq 36$ .*
- (d) *Let  $A/\langle e_{10} \rangle \cong \mathcal{A}_{9,2}$ . Then  $A$  is isomorphic to  $\mathcal{A}_{10,24}$ ,  $\mathcal{A}_{10,28}$ ,  $\mathcal{A}_{10,35}$ , or  $\mathcal{A}_{10,i}$ ,  $37 \leq i \leq 56$ .*

The following theorem is an immediate consequence of Lemmas 2.4–2.5.

THEOREM 2.6. *The only 10-dimensional generalized Heisenberg 3-Lie algebras of rank 3 are  $\mathcal{A}_{10,i}$ ,  $8 \leq i \leq 56$  (see Table 3).*

TABLE 3. The 10-dimensional generalized Heisenberg 3-Lie algebras of rank 3.

Name	Non-zero multiplication
$\mathcal{A}_{10,8}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_5, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,9}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_1, e_2, e_4] = [e_5, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,10}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_1, e_4, e_5] = [e_2, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,11}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_2, e_3, e_5] = [e_4, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,12}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_1, e_2, e_5] = [e_4, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,13}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_1, e_2, e_5] = [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,14}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,15}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,16}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_2, e_4] = [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,17}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_3, e_4] = [e_5, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,18}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_2, e_4] = [e_5, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,19}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_2, e_6] = [e_4, e_5, e_7] = e_{10}$
$\mathcal{A}_{10,20}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_4, e_6] = [e_2, e_5, e_7] = e_{10}$
$\mathcal{A}_{10,21}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_2, e_6] = [e_3, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,22}$	$[e_1, e_2, e_3] = e_8, [e_3, e_4, e_5] = e_9, [e_1, e_4, e_6] = [e_2, e_3, e_7] = e_{10}$

$\mathcal{A}_{10,23}$	$[e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_9, [e_1, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,24}$	$[e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_9, [e_1, e_2, e_4] = [e_5, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,25}$	$[e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_9, [e_1, e_3, e_4] = [e_2, e_5, e_7] = e_{10}$
$\mathcal{A}_{10,26}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = e_9, [e_5, e_6, e_7] = e_9 + e_{10}, [e_1, e_2, e_4] = e_{10}$
$\mathcal{A}_{10,27}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_3, e_5] = [e_1, e_4, e_6] = e_{10}$
$\mathcal{A}_{10,28}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_3, e_5] = [e_1, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,29}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_3, e_5] = [e_4, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,30}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_3, e_5, e_6] = [e_1, e_2, e_4] = e_{10}$
$\mathcal{A}_{10,31}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,32}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_3, e_4, e_6] = e_{10}$
$\mathcal{A}_{10,33}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_4, e_5] = [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,34}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_4, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,35}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_6, e_7] = [e_2, e_4, e_5] = e_{10}$
$\mathcal{A}_{10,36}$	$[e_1, e_2, e_3] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_5, e_6] = [e_1, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,37}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,38}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = e_{10}$
$\mathcal{A}_{10,39}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_4] = e_{10}$
$\mathcal{A}_{10,40}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_3, e_5] = e_{10}$
$\mathcal{A}_{10,41}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_4, e_5] = e_{10}$
$\mathcal{A}_{10,42}$	$[e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_{8+10}, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_7] = e_{10}$
$\mathcal{A}_{10,43}$	$[e_1, e_2, e_3] = e_8, [e_4, e_5, e_6] = e_{8+10}, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_2, e_3, e_7] = e_{10}$
$\mathcal{A}_{10,44}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = e_9, [e_5, e_6, e_7] = e_9 + e_{10}, [e_1, e_2, e_4] = e_{10}$
$\mathcal{A}_{10,45}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_7] = [e_3, e_5, e_6] = e_{10}$
$\mathcal{A}_{10,46}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_7] = [e_3, e_4, e_5] = e_{10}$
$\mathcal{A}_{10,47}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_4, e_7] = [e_2, e_3, e_5] = e_{10}$
$\mathcal{A}_{10,48}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_4, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,49}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_3, e_6, e_7] = e_{10}$
$\mathcal{A}_{10,50}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_5, e_6] = [e_3, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,51}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_5, e_6] = [e_2, e_3, e_7] = e_{10}$
$\mathcal{A}_{10,52}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_4, e_5] = [e_2, e_3, e_7] = e_{10}$
$\mathcal{A}_{10,53}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_3, e_4, e_7] = e_{10}$
$\mathcal{A}_{10,54}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_4] = [e_3, e_5, e_6] = e_{10}$
$\mathcal{A}_{10,55}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_2, e_5] = [e_3, e_4, e_6] = e_{10}$
$\mathcal{A}_{10,56}$	$[e_1, e_2, e_3] = [e_4, e_5, e_6] = e_8, [e_2, e_3, e_4] = [e_5, e_6, e_7] = e_9, [e_1, e_4, e_5] = [e_2, e_3, e_6] = e_{10}$

## References

1. S. Cicalo, W. A. de Graaf, C. Schneider, Six-dimensional nilpotent Lie algebras, *Linear Algebra Appl.*, 436, no. 1, (2012), 163-189.
2. H. Darabi, M. Imanparast, On classification of 9-dimensional nilpotent 3-ary algebras of class two, *Bull. Iranian Math. Soc.*, **47**, (2021), 929–937.
3. H. Darabi, F. Saeedi, M. Eshtrati, A characterization of finite dimensional nilpotent Filippov algebras, *J. Geom. Phys.* **101**, (2016), 100–107.
4. M. Eshtrati, F. Saeedi, H. Darabi, On the multiplier of nilpotent  $n$ -Lie algebras, *J. Algebra* **450**, (2016), 162–172.
5. M. P. Gong, Classification of nilpotent Lie algebras of dimension 7, Ph.D. thesis, University of Waterloo, Waterloo, Canada, 1998.
6. Z. Yan, S. Deng, The classification of two step nilpotent complex Lie algebras of dimension 8, *Czechoslovak Math. J.*, **63**(3), (2013), 847–863.



## The structures of the prime graphs of almost simple groups related to $PSL_3(p^2)$ -Part(1)

Masoumeh Sajjadi

Department of Mathematics, Payame Noor University, P. O. Box 19395.3697, Tehran, Iran

Email: m.sajjadi@pnu.ac.ir

**ABSTRACT.** In this paper we determine the structures of the prime graphs (Gruenberg–Kegel graph) of some almost simple groups related to  $PSL_3(p^2)$ . The Gruenberg–Kegel graph is a useful tool in the characterization and classification of finite groups, as it reflects relations between element orders and reveals structural properties of the group.

**Keywords:** prime graph, projective special linear group, almost simple group

**AMS Mathematics Subject Classification [2020]:** 20D05, 20D06

### 1. Introduction

Let  $G$  be a finite group. We denote by  $\omega(G)$  the set of orders of elements of  $G$ . This set is closed under divisibility; hence is uniquely determined by a set  $\mu(G)$  of elements in  $\omega(G)$  which are maximal under divisibility relation. All the prime divisors of  $|G|$  is denoted by  $\pi(G)$ . The prime graph (Gruenberg–Kegel graph) of  $G$  is the graph with vertex-set  $\pi(G)$ , two vertices  $p$  and  $q$  being joined by an edge whenever  $G$  has an element of order  $pq$ , and in this case we will write  $p \sim q$ . The number of connected components of  $\Gamma(G)$  is denoted by  $s(G)$  and the set of  $\pi_1(G), \pi_2(G), \dots, \pi_{s(G)}(G)$ , the connected components of  $\Gamma(G)$ , is denoted by  $S(G)$ . We use the notation  $\pi_i$  instead of  $\pi_i(G)$ . If  $2 \in \pi(G)$ , we assume  $2 \in \pi_i$ .

A group  $G$  is an almost simple group, if  $S \trianglelefteq G \lesssim \text{Aut}(S)$ , for some non-abelian group  $S$ . Let  $A$  and  $B$  be two groups then a split extension is denoted by  $A : B$ . If  $L$  is a finite simple group and  $\text{Aut}(L) \cong L : A$ , then if  $B$  is a cyclic subgroup of  $A$  of order  $n$ , we will write  $L : n$  for the split extension  $L : B$ . Moreover if there are more than one subgroup of order  $n$  in  $A$ , then we will denote them by  $L : n_1, L : n_2$ , etc. We know  $\text{Aut}(L_3(p^2)) \cong L_3(p^2) : D_{12}$  where  $D_{12}$  denotes the dihedral group of order 12. We remark that  $D_{12}$  has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to  $D_6 \cong S_3$  and one subgroup of order 4 isomorphic to the Klein's four group denoted by  $2^2$ . The field and the duality automorphisms of  $L_3(p^2)$  are denoted by  $2_1$  and  $2_2$  respectively, and we set  $2_3 = 2_1 * 2_2$  (field\*duality). Therefore up to conjugacy we have the following almost simple groups related to  $L_3(p^2)$ .

LEMMA 1.1. *If  $G$  is an almost simple group related to  $L = L_3(p^2)$ , then  $G$  is isomorphic to one of the following groups:  $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12} \cong \text{Aut}(L)$ .*

In this paper, we determine the structure of the first five extensions of  $L$  mentioned in the above lemma. The Gruenberg–Kegel graph is a useful tool in the characterization and classification of finite groups, as it reflects the relations between element orders and reveals structural properties of the group. It should be noted that among the simple groups for which the characterization of the corresponding almost simple groups has been studied so far, only  $L_3(25)$  possesses an outer automorphism group of order 12, while the others have outer automorphism groups of order at most 6. However, the structures of the prime graphs of almost simple groups related to  $L_3(25)$  falls within the scope of this paper. So far, the characterization of almost simple groups related to simple groups such as  $L_3(25), L_3(37), U_6(2), U_3(5), L_2(49)$  has been investigated, with several examples provided in [2, 4–6]. Importantly, the structures we present here correspond to an infinite family of almost simple group

### 2. The structures of the prime graphs

Let  $G$  be a finite group and  $q = p^2$ , where  $p \geq 5$  is an odd prime. We determine the structures of the prime graphs of almost simple groups  $L = PSL_3(p^2), L : 2_1, L : 2_2, L : 2_3$  and  $L : 3$  as follows. In the figures below, each circle represents a complete subgraph of the graph. The existence of an edge between a vertex and a circle means that the given vertex is adjacent to all the vertices of the graph contained within that circle. The prime numbers, or equivalently the vertices of the graph contained in each circle, are written below the corresponding circle.

Recall that  $\mu(L_3(q)) = \{q-1, \frac{p(q-1)}{(3, q-1)}, \frac{q^2-1}{(3, q-1)}, \frac{q^2+q+1}{(3, q-1)}\}$  (See [1]). Therefore the structure of the prime graph of  $L$  is shown in Figure 1. Note that  $\Gamma(L)$  has two connected components,  $\pi_2 = \pi((q^2+q+1)/3) = \pi((p^2+p+1)(p^2-p+1)/3)$  and  $(q^2+q+1)_3 = 3$  (for a natural  $n$  and a prime  $r$ , we denote by  $n_r$  the  $r$ -part of  $n$ , i.e. the maximal power of  $r$  dividing  $n$ ).

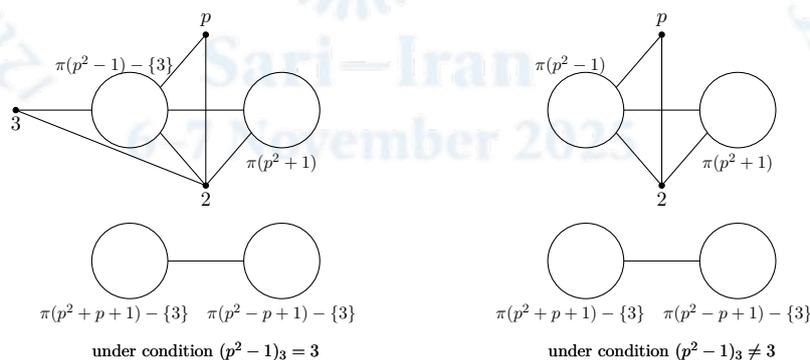


FIGURE 1.  $\Gamma(L)$

Let  $\sigma$  be the field automorphism,  $L : 2_1 \cong P\Sigma L(3, p^2) = \langle L, \text{field auto} \rangle$ . By [3], page 10,  $O(\sigma)$  is connected to every prime dividing  $\frac{p^2+p+1}{(p-1, 3)}$ . Here  $O(\sigma) = 2$ .

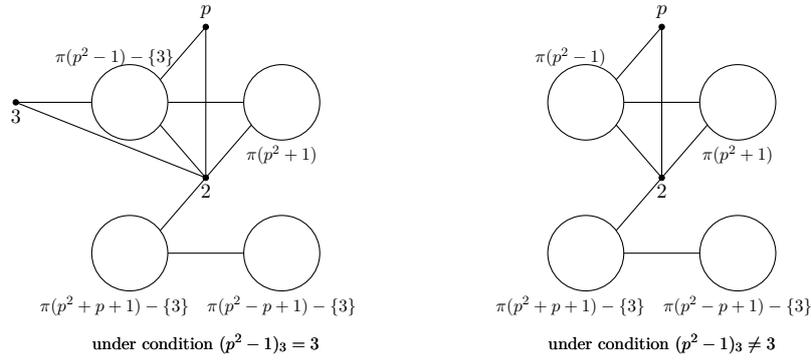


FIGURE 2.  $\Gamma(L : 2_1)$

Let  $\delta$  be the duality automorphism, therefore  $L : 2_2 = \langle L, \text{duality auto} \rangle$ . By [3],  $\Gamma(L : 2_2) = \Gamma(L)$ .

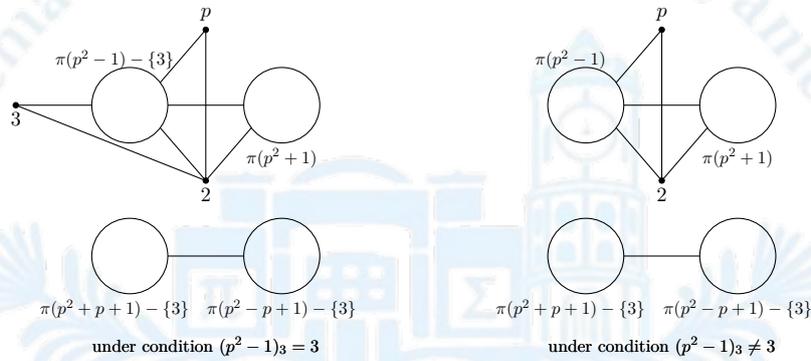


FIGURE 3.  $\Gamma(L : 2_2)$

$L : 2_3 = \langle L, \text{duality} * \text{field} \rangle$ . By [3], page 10,  $O(\delta * \sigma)$  is connected to every prime dividing  $\frac{p^2 - p + 1}{(p + 1, 3)}$ . Here  $O(\delta * \sigma) = 2$ .

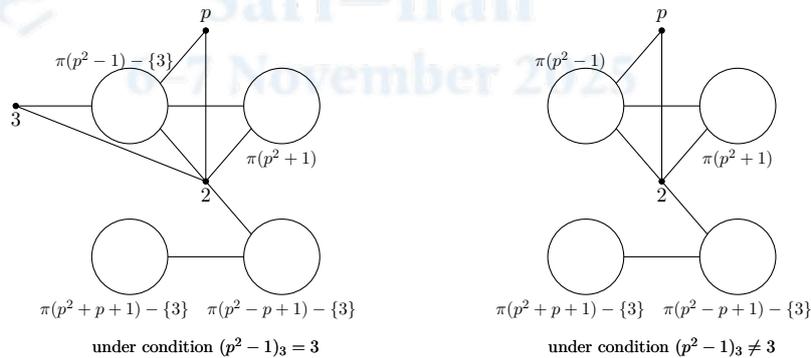


FIGURE 4.  $\Gamma(L : 2_3)$

$L : 3 \cong PGL(3, p^2) = \langle L, \text{diagonal auto} \rangle$ . By considering the maximal tori of  $PGL(3, p^2)$ , one finds that 3 is adjacent to every prime divisor of  $|L|$ .

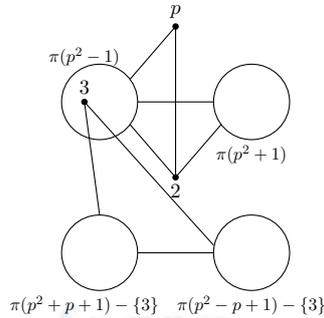


FIGURE 5.  $\Gamma(L : 3)$

### Acknowledgement

The author would like to thank the referee.

### References

1. Darafsheh, M.R., Moghaddamfar, A.R. and Zokayi, A.R. (2004) *A recognition of simple groups  $PSL(3, q)$  by their element orders*, Acta Math. Sci., **24**(1), 45–51.
2. Darafsheh, M.R., Rezaeezadeh, G.R., Bibak, M. and Sajjadi, M. (2013) *OD-characterization of almost simple groups related to  ${}^2E_6(2)$* , Adv. Algebra, **6**, 45–54.
3. Lucido, M.S. (1999) *Prime graph components of finite almost simple groups*, Rend. Semin. Mat. Univ. Padova., **102**, 1–22.
4. Rezaeezadeh, G.R., Darafsheh, M.R., Sajjadi, M. and Bibak, M. (2014) *OD-characterization of almost simple groups related to  $L_3(25)$* , Bull. Iranian Math. Soc., 765–790.
5. Zhang, L.C and Shi, W.J (2008) *OD-characterization of almost simple groups related to  $L_2(49)$* , Arch. Math., **44**(3), 191–199.
6. Zhang, L.C and Shi, W.J (2010) *OD-characterization of almost simple groups related to  $L_2(49)$* , Acta Mech. Sin. Engl. Ser., **26**(1), 161–168.



## The structures of the prime graphs of almost simple groups related to $PSL_3(p^2)$ -Part(2)

Masoumeh Sajjadi

Department of Mathematics, Payame Noor University, P. O. Box 19395.3697, Tehran, Iran

Email: m.sajjadi@pnu.ac.ir

---

**ABSTRACT.** In this paper, we determine the structures of the prime graphs of five almost simple groups related to  $L = PSL_3(p^2)$ .

**Keywords:** prime graph, projective special linear group, almost simple group

**AMS Mathematics Subject Classification [2020]:** 20D05, 20D06

---

### 1. Introduction

Let  $G$  be a finite group. We denote by  $\omega(G)$  the set of orders of elements of  $G$ . This set is closed under divisibility; hence is uniquely determined by a set  $\mu(G)$  of elements in  $\omega(G)$  which are maximal under divisibility relation. All the prime divisors of  $|G|$  is denoted by  $\pi(G)$ . The prime graph (Gruenberg–Kegel graph) of  $G$  is the graph with vertex-set  $\pi(G)$ , two vertices  $p$  and  $q$  being joined by an edge whenever  $G$  has an element of order  $pq$ , and in this case we will write  $p \sim q$ . The number of connected components of  $\Gamma(G)$  is denoted by  $s(G)$  and the set of  $\pi_1(G), \pi_2(G), \dots, \pi_{s(G)}(G)$ , the connected components of  $\Gamma(G)$ , is denoted by  $S(G)$ . We use the notation  $\pi_i$  instead of  $\pi_i(G)$ . If  $2 \in \pi(G)$ , we assume  $2 \in \pi_i$ .

**DEFINITION 1.1.** A group  $G$  is an almost simple group, if  $S \trianglelefteq G \lesssim \text{Aut}(S)$ , for some non-abelian group  $S$

Let  $A$  and  $B$  be two groups then a split extension is denoted by  $A : B$ . If  $L$  is a finite simple group and  $\text{Aut}(L) \cong L : A$ , then if  $B$  is a cyclic subgroup of  $A$  of order  $n$ , we will write  $L : n$  for the split extension  $L : B$ . Moreover if there are more than one subgroup of order  $n$  in  $A$ , then we will denote them by  $L : n_1, L : n_2$ , etc. We know  $\text{Aut}(L_3(p^2)) \cong L_3(p^2) : D_{12}$  where  $D_{12}$  denotes the dihedral group of order 12. We remark that  $D_{12}$  has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to  $D_6 \cong S_3$  and one subgroup of order 4 isomorphic to the Klein's four group denoted by  $2^2$ . The field and the duality automorphisms of  $L_3(p^2)$  are denoted by  $2_1$  and  $2_2$  respectively, and we set  $2_3 = 2_1 * 2_2(\text{field*duality})$ . Therefore up to conjugacy we have the following almost simple groups related to  $L_3(p^2)$ .

LEMMA 1.2. *If  $G$  is an almost simple group related to  $L = L_3(p^2)$ , then  $G$  is isomorphic to one of the following groups:  $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12} \cong \text{Aut}(L)$ .*

In this paper, we determine the structures of the prime graphs of the second five extensions of  $L = L_3(p^2)$ . Therefore, by [3], the determination of the prime graph structures of all almost simple groups related to  $L$  will be completed in this paper.

## 2. The structures of the prime graphs

Let  $G$  be a finite group and  $q = p^2$ , where  $p \geq 5$  is an odd prime. Put  $L = PSL_3(p^2)$ , we determine the structures of the prime graphs of almost simple groups  $L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$  as follows. In the figures below, each circle represents a complete subgraph of the graph.

The existence of an edge between a vertex and a circle means that the given vertex is adjacent to all the vertices of the graph contained within that circle. The prime numbers, or equivalently the vertices of the graph contained in each circle, are written below the corresponding circle. Note that  $\pi((q^2 + q + 1)/3) = \pi((p^2 + p + 1)(p^2 - p + 1)/3)$  and  $(q^2 + q + 1)_3 = 3$  (for a natural  $n$  and a prime  $r$ , we denote by  $n_r$  the  $r$ -part of  $n$ , i.e. the maximal power of  $r$  dividing  $n$ ).

$L : 2^2 = \langle L, \text{duality} \times \text{field} \rangle$ . Therefore, the prime graph of the group  $L : 2^2$  has exactly the same connections as those shown in Figures 2–4 of [3].

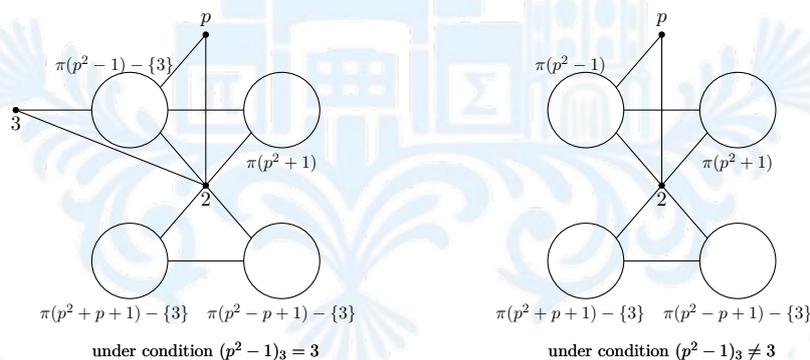


FIGURE 1.  $\Gamma(L : 2^2)$

$L : (D_6)_1 = \langle PGL, field \rangle$ . Therefore, the prime graph of the group  $L : (D_6)_1$  has exactly the same connections as those shown in Figure 2 and Figure 5 of [3].

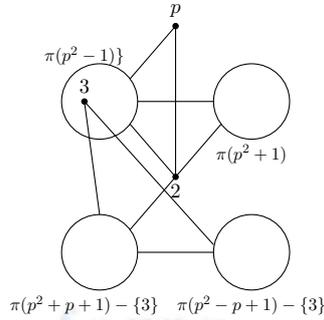


FIGURE 2.  $\Gamma(L : (D_6)_1)$

$L : (D_6)_2 = \langle PGL, duality \rangle$ . Therefore, the prime graph of the group  $L : (D_6)_2$  has exactly the same connections as those shown in Figure 3 and Figure 5 of [3].

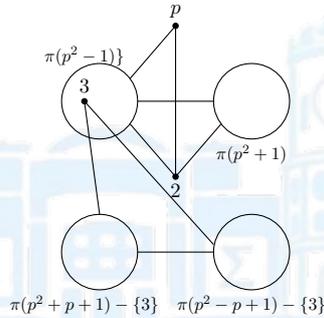


FIGURE 3.  $\Gamma(L : (D_6)_2)$

$L : 6 = \langle PGL, field * duality \rangle$ . Therefore, the prime graph of the group  $L : 6$  has exactly the same connections as those shown in Figure 4 and Figure 5 of [3].

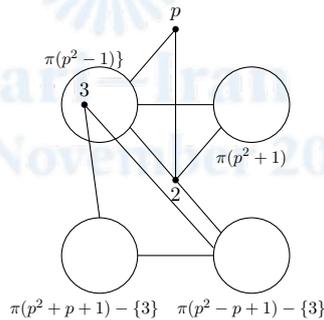


FIGURE 4.  $\Gamma(L : 6)$

$L : (D_{12}) = \text{Aut}(L)$ . In view of what has been stated about the structures of the prime graphs of almost simple groups related to  $L$ , it is clear that the prime graph of the group  $L : D_{12}$  is as follows :

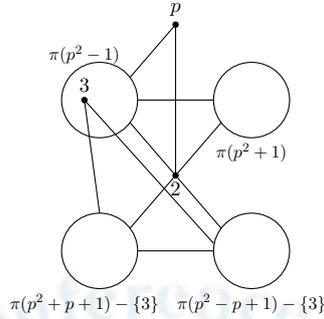


FIGURE 5.  $\Gamma(L : D_{12})$

### Acknowledgement

The author would like to thank the referee.

### References

1. Darafsheh, M.R., Moghaddamfar, A.R. and Zokayi, A.R. (2004) *A recognition of simple groups  $PSL(3, q)$  by their element orders*, Acta Math. Sci., **24**(1), 45–51.
2. Lucido, M.S. (1999) *Prime graph components of finite almost simple groups*, Rend. Semin. Mat. Univ. Padova., **102**, 1–22.
3. Sajjadi, M., (2025) *The structure of the prime graph of almost simple groups related to  $PSL_3(p^2)$ -Part(1)*, 12th National Mathematics Conference of Payame Noor University, Sari-Iran.



## Constructing Graphs from $EQ$ -algebras Using Operation Inverters

Marzieh Rahmati<sup>1\*</sup> and Mohammad Hamidi<sup>2</sup>

<sup>1,2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: m.rahmati@pnu.ac.ir, m.hamidi@pnu.ac.ir

---

**ABSTRACT.** In this paper, we introduce the concept of inverter graph based on the inverters subsets of  $EQ$ -algebras and check their attributes. Indeed, the inverter subsets of  $EQ$ -algebras are the vertices of related graph and the edges extracted by the any two vertices that their inverter subsets can not be joint with respect to the union operation and implication operation.

**Keywords:**  $EQ$ -algebra, inverters set, inverter graph

**AMS Mathematics Subject Classification [2020]:** 06D72, 03F45

---

### 1. Introduction

Algebraic logic, founded by George Boole in 1847, studies logical systems through their algebraic counterparts.  $EQ$ -algebras, introduced by Novák and De Baets in 2009, provide an important generalization of residuated lattices [5]. The combination of graph theory and logic algebra is a novel research in the Mathematics and have some application in the modelify of real problems. For more researh can see the published papers in this regard such as on commuting graphs associated to  $BCI$ -algebras [4]. This paper introduces a new concept explored is the inverter subset assigned to each element within the  $EQ$ -algebra. Utilizing these inverter subsets, we construct the inverter graph  $\mathcal{G}(X)$ . This work bridges the fields of algebraic logic and graph theory, offering new structural insights and interconnections.

#### 1.1. Preliminaries.

**DEFINITION 1.1.** [1] An ordered pair  $D = (V, E)$  is called a digraph, if  $V \neq \emptyset$  and  $E \subseteq V \times V$ , which  $V$  is the set of all vertices and  $V$  is the set of all arcs(directed edge) of  $D$ , respectively. A digraph without of loops is called a simple digraph and a simple digraph without of multi arcs(two arcs with the same ends have the same orientation) is said to be strict and a dimultigraph is a digraph that has multiple directed edges and loops.

---

\*Speaker.

DEFINITION 1.2. [2] A system  $(X, \wedge, \odot, \smile, 1)$  where  $\wedge, \odot, \smile$  are binary operations, is a (commutative) *EQ-algebra*, if  $\forall x, y, z, t \in X$ :

- (EQ1)  $(X, \wedge, 1)$  is a  $\wedge$ -semilattice which contains the element above 1. We place  $x \leq y \iff \wedge(x, y) = x$ ,
- (EQ2)  $(X, \odot, 1)$  is a (commutative) monoid and  $\odot$  is isotone,
- (EQ3)  $x \smile x = 1$ ,
- (EQ4)  $((x \wedge y) \smile z) \odot (t \smile x) \leq_X (z \smile (t \wedge y))$ ,
- (EQ5)  $(x \smile y) \odot (z \smile t) \leq_X (x \smile z) \smile (y \smile t)$ ,
- (EQ6)  $(x \wedge y \wedge z) \smile x \leq_X (x \wedge y) \smile x$ ,
- (EQ7)  $(x \wedge y) \smile x \leq_X (x \wedge y \wedge z) \smile (x \wedge z)$ ,
- (EQ8)  $x \odot y \leq_X x \smile y$ .

The action “ $\odot$ ” is multiplication, and “ $\smile$ ” is fuzzy equality.

In [3], El-Zekey et. al proved that the axiom (EQ7), follows from another axioms, so we can remove it from above axioms. Based [3], put  $x \rhd y = (x \wedge y) \smile x$ ,  $\neg x = x \smile 0$  and  $\tilde{x} = x \smile 1$ , for all  $x, y \in X$ . The *EQ-algebra*  $X$  is called separated, if for all  $x, y \in X$ ,  $x \smile y = 1$  implies  $x = y$ .

## 2. Main results

DEFINITION 2.1. Let  $(X, \wedge, \odot, \smile, 1)$  be an *EQ-algebra*. Then for  $z \in X$ , define  $In(\rhd, z) = \{x \in X \mid x \rhd z = 1\}$  as  $(\rhd)$ -inverter of  $z$  in  $X$ .

DEFINITION 2.2. Let  $(X, \wedge, \odot, \smile, 1)$  be an *EQ-algebra* and  $x, y \in X$ . Then  $\mathcal{G}(X) = (V = V(\mathcal{G}(X)) = X \setminus \{1\}, E = E(\mathcal{G}(X)))$  is called an inverter graph of  $X$ , which for any two distinct vertices  $x$  and  $y$ ,  $(x, y) \in E$  if and only if  $In(\rhd, x) \cup In(\rhd, y) \neq In(\rhd, x \rhd y)$  and  $(y, x) \in E$  if and only if  $In(\rhd, y) \cup In(\rhd, x) \neq In(\rhd, y \rhd x)$ . We emphasize that  $In(\rhd, x) \cup In(\rhd, y) = In(\rhd, x \rhd y)$ , if  $(y, x) \in E$  or there is not any arc between vertices  $x, y$ .

It obvious that  $\mathcal{G}(X) = (V = V(\mathcal{G}(X)) = X \setminus \{1\}, E = E(\mathcal{G}(X)))$  has not loop and is not a simple graph, since for any  $x, y \in V$ , necessarily,  $In(\rhd, x \rhd y) \neq In(\rhd, y \rhd x)$ .

PROPOSITION 2.3. Let  $(X, \wedge, \odot, \smile, 1)$  be an *EQ-algebra* and  $x, y \in X$ .

- (i) If  $(x, y) \notin E(\mathcal{G}(X))$ , then  $In(x) \subseteq In(x \rhd y)$ .
- (ii) If  $(x, y), (y, x) \notin E(\mathcal{G}(X))$ , then  $In(y \rhd x) = In(x \rhd y)$ .

PROOF. (i) Assume  $x, y \in X$ . As  $(x, y) \notin E(\mathcal{G}(X))$ , we get that  $In(\rhd, x) \cup In(\rhd, y) = In(\rhd, x \rhd y)$  and so  $In(\rhd, x) \subseteq In(\rhd, x \rhd y)$ .

(ii) By item (i),  $In(\rhd, x) \cup In(\rhd, y) = In(\rhd, x \rhd y)$  and  $In(\rhd, x) \cup In(\rhd, y) = In(\rhd, y \rhd x)$ . Thus  $In(y \rhd x) = In(x \rhd y)$ .  $\square$

COROLLARY 2.4. Suppose  $(X, \wedge, \odot, \smile, 1)$  be an *EQ-algebra*.

- (i) If  $\forall x, y \in X$ ,  $In(x) \not\subseteq In(x \rhd y)$ , then  $\mathcal{G}(X)$  is a weakly connected graph.
- (ii) If  $\forall x, y \in X$ ,  $In(x) \not\subseteq In(x \rhd y)$  and  $In(y) \not\subseteq In(y \rhd x)$ , then  $\mathcal{G}(X)$  is a strongly connected graph.

EXAMPLE 2.5. Let  $X = \{x, y, z, t, 1\}$  and  $x \leq_X y \leq_X z \leq_X t \leq_X 1$ . Then  $(X, \wedge, \odot, \smile, 1)$  is a non-separated *EQ-algebra* as Tables 1 and 2.  $\mathcal{G}(X)$  is a weakly connected graph, given by Figure 1.

$\odot$	$x$	$y$	$z$	$t$	$1$
$x$	$x$	$x$	$x$	$x$	$x$
$y$	$x$	$x$	$x$	$x$	$y$
$z$	$x$	$x$	$x$	$x$	$z$
$t$	$x$	$x$	$x$	$t$	$t$
$1$	$x$	$y$	$z$	$t$	$1$

TABLE 1.  $(X, \odot)$

$\smile$	$x$	$y$	$z$	$t$	$1$
$x$	$1$	$y$	$x$	$x$	$x$
$y$	$y$	$1$	$y$	$y$	$y$
$z$	$x$	$y$	$1$	$z$	$1$
$t$	$x$	$y$	$z$	$1$	$1$
$1$	$x$	$y$	$z$	$t$	$1$

TABLE 2.  $(X, \smile)$

$\rhd$	$x$	$y$	$z$	$t$	$1$
$x$	$1$	$1$	$1$	$1$	$1$
$y$	$y$	$1$	$1$	$1$	$1$
$z$	$x$	$y$	$1$	$1$	$1$
$t$	$x$	$y$	$z$	$1$	$1$
$1$	$x$	$y$	$1$	$1$	$1$

TABLE 3.  $(X, \rhd)$

Computations show that

$$In(\rhd, x) = \{x\}, In(\rhd, y) = \{x, y\}, In(\rhd, z) = \{x, y, z, 1\},$$

$$\text{and } In(\rhd, t) = In(\rhd, 1) = X.$$

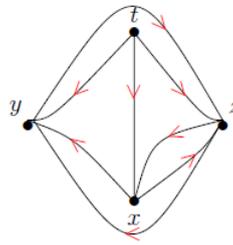


FIGURE 1. Weakly connected graph  $\mathcal{G}(X)$

**THEOREM 2.6.** *Suppose  $(X, \wedge, \odot, \smile, 1)$  be an  $EQ$ -algebra.*

- (i) *If  $\forall x, y, z \in X, In(x \rhd y) \cup In(y \rhd z) \neq In(z \rhd x) \cup In(y)$ , then  $\mathcal{G}(X)$  is an acyclic digraph.*
- (ii) *If  $\forall x, y, z \in X, In(x \rhd y) \cup In(y \rhd z) \neq In(x \rhd z) \cup In(y)$ , then  $\mathcal{G}(X)$  is an acyclic digraph.*

**PROOF.** (i), (ii) Suppose  $x, y, z \in X$ . By Corollary 2.4,  $In(x) \neq In(x \rhd y)$  implies that  $\mathcal{G}(X)$  is a connected digraph. Clearly,  $\mathcal{G}(X)$  is an acyclic digraph, if  $\forall x, y, z \in X$ , we possess the subgraphs as Figures 2 and 3.

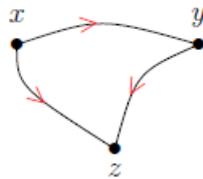


FIGURE 2. Digraph  $G_1$

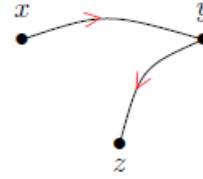


FIGURE 3. Digraph  $G_2$

In digraph  $G_1 = (V_1, E_1)$  Figure 2, as  $(x, y) \in E_1, (y, z) \in E_1$  and  $(x, z) \in E_1$ , we gain that  $In(x) \cup In(y) \neq In(x \rhd y), In(y) \cup In(z) \neq In(y \rhd z)$  and  $In(x) \cup In(z) = In(z \rhd x)$ . Hence  $In(y) \cup In(x) \cup In(z) \neq In(x \rhd y) \cup In(y \rhd z)$  and so  $In(y) \cup In(z \rhd x) \neq In(x \rhd y) \cup In(y \rhd z)$ . In a similar way, in digraph  $G_2 = (V_2, E_2)$ , we get  $In(y) \cup In(z \rhd x) \neq In(x \rhd y) \cup In(y \rhd z)$ . Now apply Proposition 2.3, so we possess  $In(y) \cup In(x \rhd z) \neq In(x \rhd y) \cup In(y \rhd z)$ .  $\square$

EXAMPLE 2.7. Let  $X = \{x, y, z, t, 1\}$  and  $x \leq_X y \leq_X z \leq_X t \leq_X 1$ . Then  $(X, \wedge, \odot, \smile, 1)$  is an EQ-algebra as Tables 4 and 5.

$\odot$	$x$	$y$	$z$	$t$	$1$
$x$	$x$	$x$	$x$	$x$	$x$
$y$	$x$	$x$	$x$	$x$	$y$
$z$	$x$	$x$	$x$	$x$	$z$
$t$	$x$	$x$	$x$	$t$	$t$
$1$	$x$	$y$	$z$	$t$	$1$

TABLE 4.  $(X, \odot)$

$\smile$	$x$	$y$	$z$	$t$	$1$
$x$	$1$	$y$	$x$	$x$	$x$
$y$	$y$	$1$	$y$	$y$	$y$
$z$	$x$	$y$	$1$	$1$	$1$
$t$	$x$	$y$	$z$	$1$	$1$
$1$	$x$	$y$	$z$	$t$	$1$

TABLE 5.  $(X, \smile)$

$\rhd$	$x$	$y$	$z$	$t$	$1$
$x$	$1$	$1$	$1$	$1$	$1$
$y$	$y$	$1$	$1$	$1$	$1$
$z$	$x$	$y$	$1$	$1$	$1$
$t$	$x$	$y$	$1$	$1$	$1$
$1$	$x$	$y$	$1$	$1$	$1$

TABLE 6.  $(X, \rhd)$

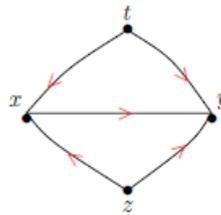


FIGURE 4. Acyclic digraph  $\mathcal{G}(X)$

Computations show that

$$\begin{aligned} In(\rhd, x) &= \{x\}, In(\rhd, y) = \{x, y\}, \\ \text{and } In(\rhd, z) &= In(\rhd, t) = In(\rhd, 1) = X. \end{aligned}$$

THEOREM 2.8. Assume  $(X, \wedge, \odot, \smile, 1)$  be an EQ-algebra. If  $\forall x, y \in X, In(x \rhd y) \neq In(y \rhd x)$ , then  $\mathcal{G}(X)$  is a strongly connected digraph.

PROOF. Assume  $x, y \in X$ . Then  $\mathcal{G}(X)$  is a strongly connected digraph, if for any  $x, y \in X, (x, y) \in E$  and  $(y, x) \in E$  iff  $In(x) \cup In(y) \neq In(x \rhd y)$  and  $In(x) \cup In(y) \neq In(y \rhd x)$ . Hence  $\mathcal{G}(X)$  is a strongly connected digraph, if  $\forall x, y \in X, In(x \rhd y) \neq In(y \rhd x)$ .  $\square$

### 3. Conclusion

This research opens new avenues for applying algebraic logic to problems in computer science, network theory, and discrete mathematics through graph-theoretic representations. The findings suggest potential applications in modeling complex systems where both algebraic and graph-theoretic properties play crucial roles, particularly in knowledge representation and logical modeling.

### References

1. Bondy, J. A, Murty, U. S. R. (2008) *Graphs Theory*, Springer.
2. El-Zekey, M. Novák, V. and Mesiar, R. (2011) *On good EQ-algebras*, Fuzzy Sets and Systems, **178**, 1–23.
3. El-Zekey, M. (2010) *Representable good EQ-algebras*, Soft Computing, **14 (9)**, 1011–1023.
4. Harizavi, H. (2021) *On Commuting Graphs Associated to BCI-Algebras*, Jordan Journal of Mathematics and Statistics (JJMS), **14(3)**, 505-516.
5. Novák, V. De Baets, B. (2009) *EQ-algebras*, Fuzzy Sets and Systems **160** 2956–2978.
6. Novák, V, Dyba, M. (2009) *Non-commutative EQ-logics and their extensions*, in: Proc. World Congress IFSA-EUSFLAT 2009, University of Lisbon, Portugal.



## On Dihypergraphs and Applications

Marzieh Rahmati<sup>1\*</sup> and Mohammad Hamidi<sup>2</sup>

<sup>1,2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: m.rahmati@pnu.ac.ir, m.hamidi@pnu.ac.ir

---

**ABSTRACT.** This paper introduces directed hypergraphs (dihypergraphs), also referring to them with terms like " $m$ -regular." A dihypergraph is characterized by its directed hyperedges. To design efficient algorithms for them, a dihypergraph can often be reduced to a standard directed graph.

**Keywords:** adjacency matrix,  $m$ -regular dihypergraph

**AMS Mathematics Subject Classification [2020]:** 05C65, 05C40, 68R10

---

### 1. Introduction

Directed hypergraphs represent a powerful generalization of classical graphs, are powerful for modeling complex dependencies in various fields. In databases, they represent data dependencies and association rules. In AI, they model complex "if-then" logic for expert systems. They are also crucial in bioinformatics for metabolic pathways and in operations research for sophisticated project scheduling, where multiple prerequisites lead to multiple outcomes. While the theoretical foundations of hypergraphs date to the 1960s by Claude Berge [1], directed hypergraphs emerged prominently in the 1990s to address practical needs in fields like relational databases, expert systems, and combinatorial optimization. A common and effective strategy for working with these structures is to reduce them to standard directed graphs. This reduction allows researchers to leverage the vast arsenal of well-established graph algorithms. Through this practical utility and subsequent theoretical development, the directed hypergraph has evolved from a specialized modeling tool into a profound concept with robust mathematical foundations. Among the works done in the field of directed hypergraphs can be seen in [3–6].

#### 1.1. Preliminaries.

**DEFINITION 1.1.** [2] An ordered pair  $D = (V, E)$  is called a digraph, if  $V \neq \emptyset$  and  $E \subseteq V \times V$ , which  $V$  is the set of all vertices and  $E$  is the set of all arcs (directed edge) of  $D$ , respectively. A digraph without of loops and multi arcs (two arcs with the same ends have the same orientation) is said to be strict and a dimultigraph is a digraph that has multiple directed edges and loops.

---

\*Speaker.

DEFINITION 1.2. [1] A structure  $\mathcal{H}' = (H, E = \{E_i\}_{i=1}^m)$  is called a hypergraph, if for any  $1 \leq i \leq m, \emptyset \neq E_i \subseteq H$  and  $\bigcup_{i=1}^m E_i = H$  and will call  $H$  as the set of all hypervrtices and  $E$  the set of all hyperedges.

DEFINITION 1.3. [3] Let  $G$  be a set and  $E \subseteq P^*(G) \times P(G)$ , where  $P^*(G) = P(G) \setminus \emptyset$  and  $P(G)$  is the power set of  $G$ . Then  $E = (T(E), H(E))$  is called a hyperarc(directed hyperedge), if  $T(E) \cap H(E) = \emptyset$ , where  $T(E)$  is called the tail and  $H(E)$  is called the head of  $E$ , respectively. An ordered pair  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n = \{(T(E_i), H(E_i))\}_{i=1}^n)$  is called a directed hypergraph (dihypergraph), if for any  $1 \leq i \leq n, E_i$  is a directed hyperedge and  $\bigcup_{i=1}^n (T(E_i) \cup H(E_i)) = G$ .

## 2. Main results

From [2], for digraph  $D = (V, E)$ , recall, a directed  $(x, y)$ -path(a dipath from  $x$  to  $y$ ) is a sequence  $P : (x = v_0, e_1, v_1, \dots, v_{l-1}, e_l, v_l = y)$ , which  $v_i \in V, e_i \in E$  are distinct ( $v_{i-1}$  and  $v_i$  are the tail and head of  $e_i$ , respectively). A digraph is (weakly) connected if its underlying graph is connected and it is strongly connected if for any  $x, y$ , there exist two directed  $(x, y)$ -path and  $(y, x)$ -path.

Let  $D = (V, E)$  be a strict digraph and  $|V| = n$  and  $|E| = m$ . For any  $e = (a, b) \in E$ , one can identified  $e \sim (\{a\}, \{b\})$  as  $\varphi : V \rightarrow P^*(V)$  by  $\varphi(x) = \{x\}$ . Thus  $E = (\{a\}, \{b\})$  is a hyperarc and so  $\mathcal{G}' = (V, \{(\{a_i\}, \{b_i\} \mid (a_i, b_i) \in E)\}_{i=1}^m)$  is a dihypergraph. Hence every strict connected digraph is a dihypergraph. We say that a dimultigraph is simple, if is without of loops. It follows that a dimultigraph is a dihypergraph.

EXAMPLE 2.1. Let  $V = \{a, b, c, d, e\}$ . Consider a digraph  $D = (V, \{e_i\}_{i=1}^7)$  in Figure 1. Now, we consider  $E_1 = (\{a\}, \{c, d\}), E_2 = (\{b\}, \{c, d, e\})$  and  $E_3 = (\{c\}, \{a\})$ . Thus we obtain the dihypergraph  $\mathcal{G}' = (V, \{E_i\}_{i=1}^3)$  in Figure 2.

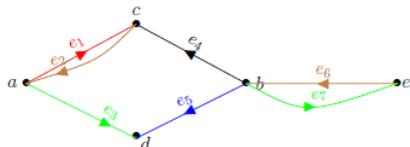


FIGURE 1. Digraph  $D$

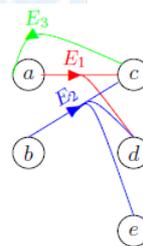


FIGURE 2. Dihypergraph  $\mathcal{G}'$

COROLLARY 2.2. *The following hold:*

- (i) *Every strict connected digraph is a dihypergraph.*
- (ii) *Every simple connected multigraph is a dihypergraph.*

But in what follow, the converse of Corollary 2.2, is not necessarily true.

EXAMPLE 2.3. Consider the dihypergraph  $\mathcal{G}' = (G, \{E_i\}_{i=1}^6)$  in Figure 3, where  $E_1 = (\{a\}, \{d\}), E_2 = (\{a, b\}, \{d, e\}), E_3 = (\{f\}, \{e\}), E_4 = (\{e\}, \{f\}), E_5 = (\{f\}, \{b, c, e\})$  and  $E_6 = (\{b, c\}, \{a\})$  are hyperarcs. The Figure 3, is a dihypergraph, but it is not a digraph, because  $\{E_i\}_{i=1}^6 \subseteq P^*(V) \times P^*(V)$ .

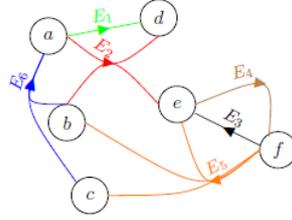


FIGURE 3. Dihypergraph  $\mathcal{G}'$

If  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  is a dihypergraph and for every  $1 \leq i \leq n$ ,  $|T(E_i)| = |H(E_i)| = 1$ , then  $\mathcal{G}'$  is a digraph.

As for directed graphs, the adjacency matrix of a dihypergraph is a  $k \times k$  matrix  $(a_{ij})$  defined as follows.

DEFINITION 2.4. Let  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  be a dihypergraph and  $|G| = k$ . We define an adjacency matrix  $A_k(G) = (a_{ij})_{k \times k}$  of  $G$  by square matrix which rows and columns are indexed by the vertices of  $G$ , where

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ |\{E_i \mid a_i \in T(E_i), a_j \in H(E_i)\}| & \text{if } i \neq j. \end{cases}$$

EXAMPLE 2.5. Consider the dihypergraph  $\mathcal{G}'$  in Figure 3. Then  $A_6(G)$  is the adjacency matrices of dihypergraph  $\mathcal{G}'$  as Figure 4.

$$A = \begin{matrix} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 \end{pmatrix} \end{matrix}$$

FIGURE 4. Adjacency matrix  $A_6(G)$ .

DEFINITION 2.6. Let  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  be a dihypergraph and  $|G| = k$ . Then we say  $\mathcal{G}'$  is an  $m$ -regular dihypergraph, if in the adjacency matrix  $A_k(G)$ , for any  $i \neq j$ ,  $a_{ij} = m$ .

THEOREM 2.7. Let  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  be an  $m$ -regular dihypergraph and  $|G| = k$ . Then  $1 \leq m \leq 2^{k-1} - 1$ .

PROOF. Let  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  be a dihypergraph, where  $G = \{a_1, a_2, \dots, a_k\}$ . Since  $\mathcal{G}'$  is an  $m$ -regular dihypergraph, get  $\bigcup_{i=1}^m (T(E_i) \cup H(E_i)) = G$  and so  $1 \leq m$ . It is clear that  $m$  get the maximom value, if  $|T(E_i)| = 1$ . Assume that  $T(E_i) = \{a_i\}$ . Then  $H(E_i) \subseteq \{a_1, a_2, \dots, a_{i-1}, a_{i+1}, a_{i+2}, \dots, a_k\}$  and so has the following cardinals:

$$\binom{k-1}{1}, \binom{k-1}{2}, \dots, \binom{k-1}{k-2}, \binom{k-1}{k-1}.$$

It follows that  $|H(E_i)| = \binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{k-2} + \binom{k-1}{k-1} = 2^{k-1} - 1 = m$ .  $\square$

Let  $\chi_n(\lambda)$  be the characteristic polynomial of matrix  $A_n(G)$  and for any matrix  $B$ ;  $Ev(B) = \{\lambda \mid \lambda \text{ is an eigenvalue of } B\}$ . In what follows, for any  $m$ -regular dihypergraph  $\mathcal{G}' = (G, \{E_i\}_{i=1}^k)$ , we consider  $\chi_{m,k}(\lambda)$  as characteristic polynomial of matrix of adjacency matrix  $A_k(G) = (a_{ij})_{k \times k}$  of  $G$ .

Let  $m \in \mathbb{R}$  and  $Q_k = (q_{ij})_{k \times k}$  that for all  $1 \leq i, j \leq k, q_{ij} = m$ . Then have the following result.

**THEOREM 2.8.** *Let  $m \in \mathbb{R}$  and  $k \in \mathbb{N}$ . Then*

- (i)  $Q_k = (q_{ij})_{k \times k}$  is a diagonalizable matrix.
- (ii)  $Ev(Q_k + I_k) = Ev(Q_k) + Ev(I_k)$ .

Let  $A_k$  be a matrix and  $m \in \mathbb{N}$ . If  $\lambda \in En(A_k)$ , then  $m\lambda \in En(mA_k)$  and so using Theorem 2.7, have the following result.

**THEOREM 2.9.** *Let  $\mathcal{G}' = (G, \{E_i\}_{i=1}^n)$  be an  $m$ -regular dihypergraph and  $|G| = k$ . Then for any  $1 \leq m \leq 2^{k-1} - 1$ ,*

- (i)  $\chi_{m,k}(\lambda) = (\lambda - m(k - 1))(\lambda + m)^{k-1}$ ;
- (ii)  $Spec(\mathcal{G}') = \begin{pmatrix} -m & m(k - 1) \\ k - 1 & 1 \end{pmatrix}$ .

**EXAMPLE 2.10.** Let  $G = \{a, b, c\}$ . Consider the one-regular dihypergraph  $\mathcal{G}' = (G, \{E_i\}_{i=1}^6)$  in Figure 5. Then  $\chi_{1,3}(\lambda) = (\lambda + 1)^2(\lambda - 2)$ .

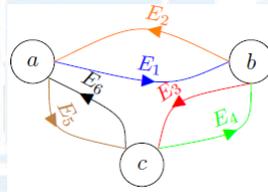


FIGURE 5. One-regular dihypergraph  $\mathcal{G}'$

### 3. Conclusion

The study in question introduces the concept of directed hypergraphs (dihypergraphs), which are described by properties such as being  $m$ -regular. These structures are defined using directed hyperedges, which represent more complex connections than the standard edges of classical graphs.

### References

- [1] Berge, C. (1979) *Graphs and Hypergraphs*, North Holland.
- [2] Bondy, J. A. Murty, U. S. R. (2008) *Graphs Theory*, Springer.
- [3] Gallo, G. Longo, G. and Pallottino, S. (1993) *Directed hypergraphs and applications*, Discrete Appl. Math, **42**, 177–201.
- [4] Hamidi, M. and Borumand Saeid, A. (2019), *Creating and Computing Graphs from Hypergraphs*, Kragujevac Journal of Mathematics, **43**(1), 139–164.
- [5] Thakur, M. and Tripathi, R. (2009) *Linear connectivity problems in directed hypergraphs*, Theoretical Computer Science, **410**, 2592–2618.
- [6] Vietri, A. (2004) *The complexity of arc-colorings for directed hypergraphs*, Discrete Appl. Math., **143**, 266–271.



## A generalization of a converse of Schur's theorem in the variety of $n$ -abelian groups

Azam Hokmabadi<sup>1,\*</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: ahokmabadi@pnu.ac.ir

**ABSTRACT.** For a positive integer  $n$ , a group  $G$  is said to be  $n$ -abelian, if  $(xy)^n = x^n y^n$ , for any  $x, y \in G$ . Fay and Waals introduced the  $n$ -potent and the  $n$ -center subgroups of a group  $G$ , as  $G_n = \langle [x, y^n] | x, y \in G \rangle$  and  $Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}$ , respectively. Pourmirzaei et al. extended a converse of Schur's theorem in the variety of  $n$ -abelian groups and proved that if  $G$  is  $n$ -abelian such that  $\frac{G}{Z^n(G)}$  is finitely generated then the finiteness of  $G_n$  implies the finiteness of  $\frac{G}{Z^n(G)}$  and  $|\frac{G}{Z^n(G)}| \leq |G_n|^{d(\frac{G}{Z^n(G)})}$ . In this paper, we generalize this result and improve the mentioned upper bound.

**Keywords:**  $n$ -abelian group, converse of Schur's theorem

**AMS Mathematics Subject Classification [2020]:** 20F14, 20D99

### 1. Introduction

Let  $n$  be a positive integer. A group  $G$  is said to be  $n$ -abelian, if  $(xy)^n = x^n y^n$ , for all elements  $x$  and  $y$  in  $G$ , from which it follows that  $[x^n, y] = [x, y]^n = [x^n, y^n]$ , where  $[x, y] = x^{-1}y^{-1}xy$ . This concept has been discussed in 1944 by Levi [2], for the first time. In 1979, Fay and Waals [1] introduced the  $n$ -potent and the  $n$ -center subgroups of a group  $G$ , denoted by  $G_n$  and  $Z^n(G)$  respectively, as follows:

$$G_n = \langle [x, y^n] | x, y \in G \rangle, \quad Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}.$$

It is easy to see that  $G_n$  is a fully invariant subgroup and  $Z^n(G)$  is a characteristic subgroup of group  $G$ . The concepts of  $n$ -potent and  $n$ -center subgroups are extensions of the important subgroups  $G'$  and  $Z(G)$ , respectively.

A famous theorem of I. Schur asserts that for a group  $G$  the finiteness of  $G/Z(G)$  implies the finiteness of  $G'$ . The proof of this theorem in fact has been stated by B.H. Neumann [3, Theorem 5.3]. He mentioned that this result can be obtained from an implicit idea of I. Schur, and his proof used Schur's basic idea. Neumann [3] also provided a partial converse of the Schur's theorem as follows:

If  $G$  is finitely generated by  $k$  elements and  $G'$  is finite, then  $G/Z(G)$  is finite and bounded by  $|G/Z(G)| \leq |G'|^k$ .

P. Niroomand [4] generalized this result by proving that if  $G'$  is finite and  $G/Z(G)$  is

\*Speaker.

finitely generated, then  $G/Z(G)$  is finite and  $|G/Z(G)| \leq |G'|^{d(G/Z(G))}$ . B. Sury [6] gave a completely elementary short proof of a further generalization of the Niroomand's result.

Pourmirzaei et al. [5] extended Schur's theorem in the variety of  $n$ -abelian groups and obtained the following theorem.

**Theorem 1.1.** *Let  $G$  be an  $n$ -abelian group and let  $G/Z^n(G)$  be finite. Then  $G_n$  is finite..*

They [5] also proved that if  $G$  is an  $n$ -abelian group such that  $G/Z^n(G)$  is finitely generated then the finiteness of  $G_n$  implies the finiteness of  $G/Z^n(G)$  and  $|G/Z^n(G)| \leq |G_n|^{d(G/Z^n(G))}$ . In this paper, we improve this result and prove that if  $G$  is an  $n$ -abelian group such that  $G/Z^n(G)$  is finitely generated by  $r$  elements, then the finiteness of the set  $S$  of all commutators  $[x, y^n]$  for all  $x, y$  in  $G$ , implies the finiteness of  $G/Z^n(G)$  and  $|G/Z^n(G)| \leq |S|^r$ .

## 2. Main result

**Theorem 2.1.** *Let  $G$  be an  $n$ -abelian group such that the set  $S$  of all commutators  $[x, y^n]$  for all  $x, y$  in  $G$ , is finite. Then  $G_n$  is finite.*

PROOF. Suppose  $S = \{[x_i, y_i^n] | 1 \leq i \leq d\}$  and put  $H = \langle x_1, y_1, \dots, x_d, y_d \rangle$ . Let  $H/Z^n(H) = \langle h_1 Z^n(H), \dots, h_r Z^n(H) \rangle$ , in which  $h_1, \dots, h_r \in H$ . Consider the conjugacy class  $cl(h_i^n)$  in  $H$ . For each  $h \in H$ , there exists an element  $s$  in  $S$  such that  $hh_i^n h^{-1} = sh_i^n$ . So  $|cl(h_i^n)| \leq |S|$ . Therefore,  $|H : C_H(h_i^n)| \leq |S|$ . On the other hand, since  $G$  is  $n$ -abelian, we have  $x \in Z^n(H)$  if and only if  $[x, h_i^n] = 1$ , for all  $1 \leq i \leq r$ , that is  $Z^n(H) = \bigcap_{i=1}^r C_H(h_i^n)$ . Therefore we have

$$|H : Z^n(H)| \leq \prod_{i=1}^r |H : C_H(h_i^n)| \leq |S|^r.$$

Hence  $|H/Z^n(H)|$  is finite and so  $H_n$  is finite, by Theorem 1.1. On the other hand  $G_n = \langle S \rangle \leq H_n$ . Hence  $G_n$  is finite. □

**Theorem 2.2.** *Let  $G$  be an  $n$ -abelian group such that  $G/Z^n(G)$  is finitely generated by  $r$  elements. If the set  $S$  of all commutators  $[x, y^n]$  for all  $x, y$  in  $G$ , is finite, then  $G/Z^n(G)$  is finite and  $|G/Z^n(G)| \leq |S|^r$ .*

PROOF. Suppose  $G/Z^n(G) = \langle g_1 Z^n(G), \dots, g_r Z^n(G) \rangle$ . Similar to the proof of the previous theorem we have  $|G : C_G(g_i^n)| \leq |S|$  and  $Z^n(G) = \bigcap_{i=1}^r C_G(g_i^n)$ . Therefore  $|G : Z^n(G)| \leq \prod_{i=1}^r |G : C_G(g_i^n)| \leq |S|^r$ , and the result follows. □

## 3. Conclusion

Let  $G$  be an  $n$ -abelian group and  $S$  be the set of all commutators  $[x, y^n]$ , for all  $x, y$  in  $G$ . We show that if  $S$  is finite then  $G_n$  is finite. Moreover, we prove that if  $G/Z^n(G)$  is finitely generated by  $r$  elements, then the finiteness of the set  $S$  implies the finiteness of  $G/Z^n(G)$  and  $|G/Z^n(G)| \leq |S|^r$ .

## References

1. Fay, T.H. and Waals, G.L. (1983) *Some remarks on  $n$ -potent and  $n$ -abelian groups*, J. Indian. Math. Soc. **47**, 217–222.
2. Levi F.W. (1944) *Notes on group theory I*, J. Indian. Math. Soc., **8**, 1–7.
3. Neumann, B.H. (1951) *Groups with finite classes of conjugate elements*, Proc. London. Math. Soc. **1**(3), 178–187.
4. Niroomand, P. (2010) *The converse of Schur's theorem*, Arch. Math., **94**, 401–403.
5. Pourmirzaei, A., Hokmabadi, A. and Shakouri, Y. (2020) *Baer's theorem and its converse in the variety of  $n$ -Abelian groups*, U.P.B. Sci. Bull., Series A, **82**, 75–82.
6. Sury B., (2010) *A generalization of a converse of Schur's theorem*, Arch. Math., **95**, 317–318.





## Some aspects of certain automorphism groups of finite $p$ -groups

Rasoul Soleimani

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: r\_soleimani@pnu.ac.ir

**ABSTRACT.** Let  $G$  be a group and  $\text{Aut}^\Phi(G)$  denote the normal subgroup of  $\text{Aut}(G)$ , consisting of all automorphisms which act trivially on the factor group  $G/\Phi(G)$ . In the present paper, we introduce a new subgroup of  $\text{Aut}^\Phi(G)$  for a finite  $p$ -group  $G$ , and then we study some equalities of this group by some subgroups of the automorphism group of a finite  $p$ -group  $G$ .

**Keywords:** Automorphism group, inner automorphism, finite  $p$ -group

**AMS Mathematics Subject Classification [2020]:** 20D45, 20D25, 20D15

### 1. Introduction

Let  $G$  be a finite group. We denote by  $Z(G)$ ,  $\Phi(G)$  and  $\text{Aut}(G)$ , the center, the Frattini subgroup and the group of all automorphisms of  $G$ , respectively. As in Hegarty [3], if  $\alpha \in \text{Aut}(G)$  and  $g \in G$ , then the autocommutator of  $g$  and  $\alpha$  is defined to be  $[g, \alpha] = g^{-1}\alpha(g)$ . Similarly to [5], let  $N$  be a characteristic subgroup of  $G$  and let  $\text{Aut}^N(G)$  be the subgroup of  $\text{Aut}(G)$  consisting of all automorphisms which act trivially on the factor group  $G/N$  or equivalently,  $[g, \alpha] \in N$  for all  $x \in G$  and  $\alpha \in \text{Aut}^N(G)$ . Now let  $M$  be a normal subgroup of  $G$  and  $C_{\text{Aut}^N(G)}(M)$  denote the group of all automorphisms of  $\text{Aut}^N(G)$  fixes  $M$  element-wise. If we choose  $N = Z(G)$ , then  $\text{Aut}^N(G)$  is precisely the group of all central automorphisms of  $G$ . Moreover, if  $N = \Phi(G)$ , then  $\text{Aut}^\Phi(G)$ , where  $\Phi = \Phi(G)$  is a normal subgroup of  $\text{Aut}(G)$  containing  $\text{Inn}(G)$ , the group of all inner automorphisms of  $G$ . By using the above notations, we define

$$L_f(G) = \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in \text{Aut}^\Phi(G)\},$$

which we call the Frattini absolute centre of a group  $G$ . It is easy to check that the Frattini absolute centre of a group  $G$  is a characteristic subgroup of  $G$ . Now let  $G$  be a finite  $p$ -group of class  $c$  such that  $\exp(\Gamma_c(G)) = p^m$  and  $N = \Gamma_c^{p^{m-1}}$ , the group generated by all  $p^{m-1}$ -th powers of elements of  $\Gamma_c$ . Since by [4, Theorem 2],  $C_{\text{Aut}^\Phi(G)}(N) = \text{Aut}^\Phi(G)$ , it follows that  $N \leq L_f(G) \cap \Phi(G)$  and so  $L_f(G)$  is non-trivial. Furthermore if  $\alpha$  runs in the set of all inner automorphisms of  $G$ , then the set of all elements  $g \in G$  such that  $[g, \alpha] = 1$

is the centre  $Z(G)$  of  $G$ . Next we define

$$\text{Aut}_f(G) = \{\alpha \in \text{Aut}^\Phi(G) \mid [g, \alpha] \in L_f(G), \forall g \in G\},$$

and

$$K_f(G) = \langle [g, \alpha] \mid g \in G, \alpha \in C_{\text{Aut}^\Phi(G)}(\text{Aut}_f(G)) \rangle,$$

where  $C_{\text{Aut}^\Phi(G)}(\text{Aut}_f(G))$  is the centralizer of  $\text{Aut}_f(G)$  in  $\text{Aut}^\Phi(G)$ . We can see that  $\text{Aut}_f(G)$  is a normal subgroup of  $\text{Aut}(G)$  contained in  $\text{Aut}^Z(G)$ , where  $Z = Z(G)$  and  $K_f(G)$  is a characteristic subgroup in  $G$ , which is containing the derived subgroup  $G' = \langle [g, \alpha] \mid g \in G, \alpha \in \text{Inn}(G) \rangle$ . It is interesting to find necessary and sufficient conditions for a group  $G$  such that certain subgroups of  $\text{Aut}(G)$  be equal. There are some well-known results in this regard for a finite group  $G$ . Müller in [5, Proposition 3.1 and main theorem] characterized finite  $p$ -groups  $G$  for which  $C_{\text{Aut}^\Phi(G)}(Z(G)) = \text{Inn}(G)$ . He proved that if  $G$  is a finite non-abelian  $p$ -group, then  $C_{\text{Aut}^\Phi(G)}(Z(G)) = \text{Inn}(G)$  if and only if  $\Phi(G) \leq Z(G)$  and  $\Phi(G)$  is cyclic. Moreover, if  $G$  is a finite  $p$ -group which is neither elementary abelian nor extraspecial, then  $\text{Aut}^\Phi(G)/\text{Inn}(G)$  is a non-trivial normal  $p$ -subgroup of the group of outer automorphisms of  $G$ . Curran and McCaughan [1] characterized finite  $p$ -groups  $G$  for which  $\text{Aut}^Z(G) = \text{Inn}(G)$ . They proved that if  $G$  is a finite  $p$ -group, then  $\text{Aut}^Z(G) = \text{Inn}(G)$ , if and only if  $G' = Z(G)$  and  $Z(G)$  is cyclic. Curran [2] considered the case when  $\text{Aut}^Z(G)$  is minimum possible, that is, when  $\text{Aut}^Z(G) = Z(\text{Inn}(G))$ . He proved that if  $\text{Aut}^Z(G) = Z(\text{Inn}(G))$ , then  $Z(G) \leq G'$  and  $Z(\text{Inn}(G))$  is not cyclic.

In this paper, we investigate equalities of the group  $\text{Aut}_f(G)$  with certain subgroups of automorphism group of  $G$ .

## 2. Main results

In this section, we list main properties concerning to the group  $\text{Aut}_f(G)$  of a group  $G$ .

LEMMA 2.1. *Let  $G$  be a finite non-abelian  $p$ -group. Then  $C_{\text{Aut}_f(G)}(Z(G)) = \text{Inn}(G)$  if and only if  $G/L_f(G)$  is abelian,  $K_f(G) \leq Z(G)$  and  $L_f(G)$  is cyclic.*

The following conclusion from [6, Corollary 2.2] plays an important role in our investigate.

LEMMA 2.2. *Let  $A, B$  and  $C$  be finite abelian  $p$ -groups,  $\exp(C) = p^t$  and  $A \leq B$ . Then  $\text{Hom}(A, C) \cong \text{Hom}(B, C)$  if and only if  $A \cong H \times A_1, B \cong H \times B_1$  where all invariants of  $A_1, B_1$  are at least  $t$ ,  $d(A_1) = d(B_1)$  and  $\exp(H) < p^t$ .*

THEOREM 2.3. *Let  $G$  be a finite non-abelian  $p$ -group. Then the following statements are equivalent:*

- (i)  $\text{Aut}_f(G) = C_{\text{Aut}_f(G)}(Z(G));$
- (ii)  $Z(G) \leq K_f(G)G^{p^n}$ , where  $\exp(L_f(G)) = p^n$ ;
- (iii) One of the following statements holds:
  - (a)  $Z(G) \leq K_f(G)$  or
  - (b)  $G/K_f(G)Z(G) \cong A \times X_1, G/G' \cong A \times Y_1$ , where all invariants of  $X_1, Y_1$  are at least  $n$ , with  $\exp(L_f(G)) = p^n, d(X_1) = d(Y_1)$  and  $\exp(A) < p^n$ .

THEOREM 2.4. *Let  $G$  be a finite  $p$ -group such that  $G/L_f(G)$  is abelian. Then the following statements are equivalent:*

- (i)  $\text{Aut}_f(G) = C_{\text{Aut}_f(G)}(Z(G));$

- (ii)  $K_f(G) \leq Z(G)$ ,  $G/Z(G) \cong A \times X_1$ ,  $G/G' \cong A \times Y_1$ , where all invariants of  $X_1$  are exactly equal to  $n$  with  $\exp(L_f(G)) = p^n$ ,  $d(X_1) = d(Y_1)$  and  $\exp(A) < p^n$ ;
- (iii)  $Z(G) = K_f(G)G^{p^n}$ , where  $\exp(L_f(G)) = p^n$ .

In the following two theorems, we give some useful informations about the equalities of the group  $\text{Aut}_f(G)$  and the group of all inner automorphisms of  $G$ .

**THEOREM 2.5.** *Let  $G$  be a finite group such that  $Z(G/L_f(G)) = R/L_f(G)$ . Then*

- (i)  $Z(\text{Inn}(G)) \leq \text{Aut}_f(G)$  if and only if  $R = Z_2(G)$ .
- (ii)  $\text{Aut}_f(G) = Z(\text{Inn}(G))$  if and only if  $\text{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$  and  $R = Z_2(G)$ .
- (iii)  $\text{Aut}_f(G) \leq Z(\text{Inn}(G))$  if and only if  $\text{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong R/Z(G)$ .
- (iv) If  $\text{Aut}_f(G) \leq Z(\text{Inn}(G))$  then  $Z(G) \leq K_f(G)G^{p^n}$ , where  $\exp(L_f(G)) = p^n$ .

**THEOREM 2.6.** *Let  $G$  be a finite non-abelian  $p$ -group. Then the following statements are equivalent:*

- (i)  $\text{Aut}_f(G) = \text{Inn}(G)$ ;
- (ii)  $K_f(G) \leq Z(G)$ ,  $G/L_f(G)$  is abelian,  $L_f(G)$  is cyclic and  $Z(G) = K_f(G)G^{p^n}$ , where  $\exp(L_f(G)) = p^n$ ;
- (iii)  $G/L_f(G)$  is abelian and  $\text{Hom}(G/K_f(G)L_f(G), L_f(G)) \cong G/Z(G)$ .

We end this paper by giving an example of a group which supports Theorem 2.3. Its GAP id is 455.

**EXAMPLE 2.7.** Let  $G = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 \rangle$  be a 2-group of order  $2^8$  with relations:

$$f_5^2, f_8^2, f_7^{-1}f_6^2, f_6^{-1}f_4^2, f_7^{-2}f_8, f_1^2f_4^{-1}, f_2^2f_5, f_3^2f_8, f_1f_3f_2f_1^{-1}f_2^{-1}, \\ [f_3, f_1] = f_8, [f_2, f_3], [f_2, f_4].$$

The nilpotency class of  $G$  is 3, because  $Z(G) = \langle f_4, f_6, f_7, f_3f_5, f_8 \rangle \cong C_2 \times C_{16}$ ,  $Z_2(G) = \langle f_4, f_6, f_7, f_3f_5, f_8, f_5 \rangle$  and  $Z_3(G) = G$ . In this group  $L_f(G) = \langle f_8 \rangle \cong C_2$ ,  $G' = \langle f_3, f_8 \rangle = \langle f_3 \rangle \cong C_4$ , and  $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7, f_8 \rangle \cong C_2 \times C_2 \times C_{16}$ , which shows that  $Z(G) \leq \Phi(G)$ . Also  $\text{Aut}_f(G) \cong C_2 \times C_2$ .

By using GAP, we see that  $C_{\text{Aut}\Phi(G)}(\text{Aut}_f(G)) = \text{Aut}^\Phi(G)$  and by the following automorphisms of  $\text{Aut}^\Phi(G)$ :

$$\beta_1(f_1) = f_1f_3, \beta_1(f_2) = f_2f_5f_7f_8, \beta_1(f_3) = f_3f_8, \beta_1(f_4) = f_4, \beta_1(f_5) = f_5f_8, \beta_1(f_i) = f_i \text{ for } 6 \leq i \leq 8,$$

$$\beta_2(f_1) = f_1f_4, \beta_2(f_2) = f_2f_3f_5f_7f_8, \beta_2(f_3) = f_3, \beta_2(f_4) = f_4f_6, \beta_2(f_5) = f_5, \beta_2(f_6) = f_6f_7, \beta_2(f_7) = f_7f_8, \beta_2(f_8) = f_8,$$

$$\beta_3(f_1) = f_1f_5, \beta_3(f_2) = f_2f_3f_7f_8, \beta_3(f_3) = f_3f_8, \beta_3(f_4) = f_4f_8, \beta_3(f_i) = f_i \text{ for } 5 \leq i \leq 8,$$

$$\beta_4(f_1) = f_1f_6, \beta_4(f_2) = f_2f_3f_5f_7f_8, \beta_4(f_3) = f_3, \beta_4(f_4) = f_4f_7, \beta_4(f_5) = f_5, \beta_4(f_6) = f_6f_8, \beta_4(f_i) = f_i \text{ for } 7 \leq i \leq 8,$$

$$\beta_5(f_1) = f_1f_7, \beta_5(f_2) = f_2f_3f_5f_8, \beta_5(f_3) = f_3, \beta_5(f_4) = f_4f_8, \beta_5(f_5) = f_5f_8, \beta_5(f_i) = f_i \text{ for } 6 \leq i \leq 8,$$

$$\beta_6(f_1) = f_1f_8, \beta_6(f_2) = f_2f_3f_5f_7, \beta_6(f_i) = f_i \text{ for } 3 \leq i \leq 8,$$

we observe that  $\Phi(G) = \langle f_3, f_4, f_5, f_6, f_7, f_8 \rangle \leq K_f(G)$  and so  $K_f(G) = \Phi(G)$ , which shows that  $C_{\text{Aut}_f(G)}(\Phi(G)) = \text{Aut}_f(G)$  because  $C_{\text{Aut}_f(G)}(K_f(G)) = \text{Aut}_f(G)$ . Finally as  $Z(G) \leq \Phi(G) = K_f(G)$ , by Theorem 2.3,  $\text{Aut}_f(G) = C_{\text{Aut}_f(G)}(Z(G))$ .

It is easy to see that the following four automorphisms belong to  $\text{Aut}_f(G)$ :

$$\begin{aligned}\alpha_1(f_1) &= f_1 f_8, \alpha_1(f_2) = f_2 f_8 \text{ and } \alpha_1(f_i) = f_i \text{ for } 3 \leq i \leq 8, \\ \alpha_2(f_1) &= f_1, \alpha_2(f_2) = f_2 f_8 \text{ and } \alpha_2(f_i) = f_i \text{ for } 3 \leq i \leq 8, \\ \alpha_3(f_1) &= f_1 f_8 \text{ and } \alpha_3(f_i) = f_i \text{ for } 2 \leq i \leq 8, \\ \alpha_4(f_i) &= f_i \text{ for } 1 \leq i \leq 8.\end{aligned}$$

### 3. Conclusion

In this paper, first we introduced the central subgroup  $L_f(G)$  of a finite group  $G$ . Next, we continued the study of subgroups of automorphism group by defining the normal subgroup  $\text{Aut}_f(G)$  of  $\text{Aut}^\Phi(G)$ , and then we explore equalities of this new group with certain subgroups of automorphism group of a finite  $p$ -group  $G$ .

### References

1. Curran, M. J. and McCaughan, D. J. (2001) *Central automorphisms that are almost inner*, *Comm. Algebra.*, **29**(5), 2081–2087.
2. Curran, M. J. (2004) *Finite groups with central automorphism group of minimal order*, *Math. Proc. Royal Irish Acad.*, **104A**(2), 223–229.
3. Hegarty, P. V. (1994) *The absolute centre of a group*, *J. Algebra.*, **169**, 929–935.
4. Liebeck, H. (1966) *The automorphism group of finite  $p$ -groups*, *J. Algebra.*, **4**, 426–432.
5. Müller, O. (1979) *On  $p$ -automorphisms of finite  $p$ -groups*, *Arch. Math.*, **32**, 533–538.
6. Soleimani, R. (2023) *Groups with some central automorphisms fixing the central kernel quotient*, *J. Mahani Math. Res.*, **12**(2), 165–177.





## Grey Congruences on S-Acts: A Structural and Categorical Approach

Zohreh Habibi

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Z\_habibi@pnu.ac.ir

Zohreh Habibi<sup>1,\*</sup>, Davood Darvishi<sup>2</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Z\_habibi@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: d\_darvishi@pnu.ac.ir

---

**ABSTRACT.** In this paper, we investigate the notion of grey congruences in the category of  $S$ -acts. We begin by extending the classical concept of congruences by incorporating grey set theory to allow degrees of equivalence under partial information. We prove analogs of the First, Second, and Third Isomorphism Theorems in the grey context. Furthermore, we develop several new structural results on grey congruences, including their behavior under direct products, restrictions to subacts and lattice-theoretic completeness. We show that the collection of grey congruences on an  $S$ -act forms a complete lattice. We analyze the behavior of grey congruences under pullbacks in the category of  $S$ -acts.

**Keywords:** grey congruences,  $S$ -act

**AMS Mathematics Subject Classification [2020]:** 06B10, 18C05, 18E20, 03E72

---

### 1. Introduction

A congruence relation refers to an equivalence relation within an algebraic structure that remains invariant under its operations. Several congruence relations on structures such as rings, semigroups,  $S$ -acts, and  $S$ -posets have been studied. For more details, see [1]. Grey sets extend the concept of fuzzy sets by enabling the representation of uncertainty.

In this study, we intend to define a grey relation and then define the grey congruence relations on  $S$ -act. For this, we provide some preliminaries.

We recall from [4], a *grey number* is characterized by having a well-defined range, with distinct upper and lower limits, but its exact value within that range remains indeterminate, reflecting uncertainty or imprecision. A grey number can be represented as  $g^\pm \in [g^-, g^+] = \{g^- \leq t \leq g^+\}$ , where  $g^\pm$  is a grey number,  $t$  is information,  $g^-$  and  $g^+$  are the lower and upper bounds of the uncertain value.

---

\*Speaker.

In this definition, grey numbers are shown in the form of continuous and have two boundaries.

In [6], it is shown a new representation of grey numbers to consider continuous and discrete grey numbers.

Consider the real set  $\mathbb{R}$ , where a grey number  $g^\pm$  can be represented as a union of several intervals  $g^\pm = \bigcup_{i=1}^n [a_i^-, a_i^+]$ , where  $i$  ranges from 1 to  $n$ , with  $n$  being a finite integer.  $g$  is called a grey number represented by  $g^\pm$ .  $g^- = \inf_{a_i^- \in g^\pm} a_i^-$  and  $g^+ = \sup_{a_i^+ \in g^\pm} a_i^+$  are called the lower and upper of  $g^\pm$ .

For a set  $A \subseteq U$ , if the characteristic function value of  $x$  with respect to  $A$  can be expressed with a grey number  $g_A^\pm(x) \in \bigcup_{i=1}^n [a_i^-, a_i^+] \in D[0, 1]^\pm$

$$\chi_A : U \longrightarrow D[0, 1]^\pm$$

then  $A$  is a *grey set*, where  $D[0, 1]^\pm$  refers to the set of all grey numbers within the interval  $[0, 1]$ . Similar to fuzzy set, a grey set  $A$  is shown

$$A = \frac{g^\pm(x_1)}{x_1} + \frac{g^\pm(x_2)}{x_2} + \dots + \frac{g^\pm(x_n)}{x_n}, \text{ for } x_i \in U. \text{ The grey set denoted as } A = (U, \chi_A).$$

We recall grey lattice operation from [5], which for grey numbers  $x^\pm = [x^-, x^+]$  (lower  $x^-$  and upper  $x^+$ ) and  $y^\pm = [y^-, y^+]$  (lower  $y^-$  and upper  $y^+$ ), the operations of Join and Meet for two grey numbers are defined as follows:

$$\text{Join: } x^\pm \vee y^\pm = [ \min (x^-, y^-), \max (x^+, y^+) ]$$

$$\text{Meet: } x^\pm \wedge y^\pm = [ \max (x^-, y^-), \min (x^+, y^+) ].$$

Now according to the definition Join and Meet, the partial order  $\preceq$  on grey set  $(X, \chi_A)$  is shown as below:

$$x^\pm \preceq y^\pm \iff x^+ \leq y^+ \text{ and } y^- \leq x^-$$

We recall from [2] and [3], the category  $GSet$  of grey sets where morphism between two grey sets  $A = (U, \chi_A)$  and  $B = (U', \chi_B)$  is a function  $f : U \longrightarrow U'$  such that upper  $\chi_A(x) \leq \text{upper } \chi_B f(x)$  and lower  $\chi_B f(x) \leq \text{lower } \chi_A(x)$  for any  $x \in U$ . In this paper, any grey number  $t$  with upper  $g^+$  and lower  $g^-$  is shown as  $t_{g^-}^{g^+}$ .

Now, we recall a (*right*)  $S$ -act on monoid  $S$  is a non-empty set  $A$  together with a map  $A \times S \rightarrow A, (a, s) \mapsto as$ , such that for all  $a \in A, s, t \in S, (as)t = a(st)$  and  $a.1 = a$ . A non-empty subset  $B \subseteq A$  is called a *subact* of  $A$  if  $bs \in B$  for all  $b \in B$  and  $s \in S$ . Let  $A$  and  $B$  be two  $S$ -acts. A mapping  $f : A \rightarrow B$  is called an  $S$ -homomorphism if  $f(as) = f(a)s$  for all  $a \in A, s \in S$ . The category of all  $S$ -acts and  $S$ -homomorphisms between them denoted as **Act- $S$** . In the category of (right)  $S$ -acts (epimorphisms) monomorphisms are (onto) one-to-one  $S$ -homomorphisms.

## 2. Main results

In this section, we define grey relation and grey congruences on  $S$ -acts, and we give some properties of grey congruences on  $S$ -acts. Also, we study three isomorphism theorems for  $S$ -acts using grey congruences. These isomorphism theorems allow us to identify isomorphic  $S$ -acts based on the existence of grey congruences.

**DEFINITION 2.1.** Let  $A_1, A_2, \dots, A_n$  be non-empty  $S$ -acts. A grey  $n$ -ary relation  $\mu$  on  $A_1, A_2, A_3, \dots, A_n$  is defined as a grey set of the product set  $A_1, A_2, \dots, A_n$ , i.e., there

exists a characteristic function  $\mu : A_1 \times A_2 \times \cdots \times A_n \longrightarrow D[0, 1]^\pm$  such that  $\mu(y)$  is a grey number for any  $y \in A_1 \times A_2 \times \cdots \times A_n$ .

**DEFINITION 2.2.** Consider binary relation  $\mu : A \times A \longrightarrow D[0, 1]^\pm$ . It is called  
 (i) Reflexive if  $\mu^+(x, x) = 1$  and  $\mu^-(x, x) = 0$  for any  $x \in A$ ;  
 (ii) Symmetric if  $\mu^+(x, y) = \mu^+(y, x)$  and  $\mu^-(x, y) = \mu^-(y, x)$  for any  $x, y \in A$ ;  
 (iii) If  $(\mu o \mu)^+(x, y) \leq \mu^+(x, y)$  and  $\mu^-(x, y) \leq (\mu o \mu)^-(x, y)$ , for any  $x, y \in A$ , then  $\mu$  is called transitive if  $\mu o \mu(x, y) = \bigvee_{z \in X} (\mu(x, z) \wedge \mu(z, y))$

A grey relation  $\mu$  on an  $S$ -act  $A$  is considered a grey equivalence relation when it satisfies the properties of reflexivity, symmetry, and transitivity.

For two relations  $\theta, \phi$  on  $S$ -act  $A$  define  $\theta \preceq \phi$  if and only if  $\theta(x, y) \preceq \phi(x, y)$ . We define two relations on  $S$ -act  $A$ ,  $\Delta : A \times A \longrightarrow D[0, 1]^\pm$  and  $\nabla : A \times A \longrightarrow D[0, 1]^\pm$  such that  $\Delta(x, y) = t_0^1$  if  $x = y$ ,  $\Delta(x, y) = 0$  if  $x \neq y$  and  $\nabla(x, y) = t_0^1$  for any  $x, y \in A$ .

**DEFINITION 2.3.** A grey equivalence relation  $\theta$  on an  $S$ -act  $A$  is termed a grey congruence if it aligns with the algebraic operation of the  $S$ -act. Specifically, for any  $x, y \in X, s \in S$ ,  $\theta(x, y) \preceq \theta(xs, ys)$  must hold. We denote the set of grey congruences on  $S$ -act  $A$  by  $GC(A)$ .

**PROPOSITION 2.4.** Let  $\alpha$  be a grey congruence on an  $S$ -act  $A$ . Then  $\alpha_{[a]} = \alpha_{[b]}$  if and only if  $\alpha^+(a, b) = 1$ , and  $\alpha^-(a, b) = 0$  for any  $a, b \in A$ .

**PROOF.** Suppose that for any  $c \in A$ ,  $\alpha_{[a]}(c) = \alpha_{[b]}(c)$ . Then  $\alpha(a, c) = \alpha(b, c)$ . In particular,  $\alpha^+(a, b) = \alpha^+(b, b) = 1$  and  $\alpha^-(a, b) = \alpha^-(b, b) = 0$ . Conversely, for any  $c \in A$ , since  $\alpha(a, c) = (\alpha o \alpha)(a, c) \succeq \alpha(a, b) \wedge \alpha(b, c)$  and by assumption, it follow  $\alpha(a, c) \succeq \alpha(b, c)$ . Similarly, we can show that  $\alpha(b, c) \succeq \alpha(a, c)$ . So  $\alpha(a, c) = \alpha(b, c)$  for any  $c \in A$ . Hence  $\alpha_{[a]} = \alpha_{[b]}$ .  $\square$

**DEFINITION 2.5.** Let  $\alpha, \beta$  be two grey congruences on  $S$ -act  $A$  such that  $\alpha \preceq \beta$ , then define the grey relation  $\frac{\beta}{\alpha} : A \longrightarrow D[0, 1]^\pm$  such that  $\frac{\beta}{\alpha}([a]_\alpha, [a']_\alpha) = \beta(a, a')$  for any  $a, a' \in A$ .

**LEMMA 2.6.** The grey relation  $\frac{\beta}{\alpha}$  is well-defined.

**PROOF.** Suppose that  $([a]_\alpha, [a']_\alpha) = ([b]_\alpha, [b']_\alpha)$  for any  $a, a', b, b' \in A$ . We show that  $\frac{\beta}{\alpha}([b]_\alpha, [b']_\alpha) \succeq \frac{\beta}{\alpha}([a]_\alpha, [a']_\alpha)$  using the fact that  $\alpha \subseteq \beta$ . We have  $([a]_\alpha = ([b]_\alpha)$  and  $([a']_\alpha = ([b']_\alpha)$ . So, by Proposition 2.4,  $\alpha^+(a, b) = \alpha^+(a', b') = 1$  and  $\alpha^-(a, b) = \alpha^-(a', b') = 0$ . Now,  $\frac{\beta}{\alpha}([a]_\alpha, [a']_\alpha) = \beta(a, a') = (\beta o \beta)(a, a') \succeq \bigwedge \{ \beta(a, b), \beta(b, b'), \beta(b', a') \} \succeq \bigwedge \{ \alpha(a, b), \beta(b, b'), \alpha(b', a') \} = \beta(b, b') = \frac{\beta}{\alpha}([b]_\alpha, [b']_\alpha)$ . Similarly, we can show that  $\frac{\beta}{\alpha}([b]_\alpha, [b']_\alpha) \succeq \frac{\beta}{\alpha}([a]_\alpha, [a']_\alpha)$ . It follows the result.  $\square$

**PROPOSITION 2.7.** Let  $A$  and  $B$  be  $S$ -acts and let  $\theta_A \in GC(A)$ ,  $\theta_B \in GC(B)$ . Then the relation  $\theta_A \times \theta_B$  defined on  $A \times B$  by

$$(\theta_A \times \theta_B)((a_1, b_1), (a_2, b_2)) = \theta_A(a_1, a_2) \wedge \theta_B(b_1, b_2)$$

is a grey congruence on the direct product  $A \times B$ .

PROOF. Reflexivity, symmetry, and transitivity follow from those of  $\theta_A$  and  $\theta_B$ . Compatibility with the  $S$ -action follows by the component-wise action:  $((a, b)s = (as, bs))$ .  $\square$

We recall that for two  $S$ -homomorphisms  $f : A \rightarrow C$  and  $g : B \rightarrow C$ , their pullback is the subset

$$A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\},$$

with the induced componentwise  $S$ -action  $(a, b)s = (as, bs)$ , making  $A \times_C B$  an  $S$ -act.

PROPOSITION 2.8 (Stability of Grey Congruences under Pullback). *Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be  $S$ -homomorphisms, and let  $\theta$  be a grey congruence on  $C$ . Then the relation  $\rho$  on  $A \times_C B$  defined by*

$$\rho((a_1, b_1), (a_2, b_2)) := \theta(f(a_1), f(a_2))$$

is a grey congruence on the pullback  $A \times_C B$ .

PROOF. Let  $x = (a_1, b_1), y = (a_2, b_2) \in A \times_C B$  with  $f(a_i) = g(b_i)$  for  $i = 1, 2$ . Define  $\rho(x, y) := \theta(f(a_1), f(a_2))$ . Since  $\theta$  is reflexive, symmetric, and transitive, the same properties hold for  $\rho$ :

- Reflexivity:  $\rho(x, x) = \theta(f(a_1), f(a_1)) = t_0^1$ .
- Symmetry:  $\rho(x, y) = \theta(f(a_1), f(a_2)) = \theta(f(a_2), f(a_1)) = \rho(y, x)$ .
- Transitivity:  $\rho(x, z) \leq \rho(x, y) \wedge \rho(y, z)$  via transitivity of  $\theta$ .

For compatibility, let  $s \in S$ .

$$\rho((a_1, b_1)s, (a_2, b_2)s) = \rho((a_1s, b_1s), (a_2s, b_2s)) = \theta(f(a_1s), f(a_2s)) = \theta(f(a_1)s, f(a_2)s)$$

which by compatibility of  $\theta$  satisfies:

$$\theta(f(a_1), f(a_2)) \leq \theta(f(a_1)s, f(a_2)s) \Rightarrow \rho((a_1, b_1), (a_2, b_2)) \leq \rho((a_1s, b_1s), (a_2s, b_2s))$$

Thus,  $\rho$  is an  $S$ -compatible grey congruence.  $\square$

## References

1. Bergman. C., *Universal Algebra: Fundamentals and Selected Topics*, CRC Press, United States, 2011.
2. Hezarjaribi. M. , Darvishi. D. and Habibi. Z., *Category of grey sets*, J. Prime Res. Math, **20**(2024), 89-96.
3. Hezarjaribi. M. and Habibi. Z., *Some properties of grey  $S$ -acts over monoid*, New Math. Nat. Comput, **18**(2) (2022), 313-323.
4. Liu. S. , Gao. T., and Dang. Y., *Grey system theory and its applications*, The Science Press of china, Beijing, 2000.
5. Yamaguchi. D. , Li. G. D. and Nagai. M., *On the combination of rough set theory and grey theory based on grey lattice operations*, in: *S. Greco et al. (Eds.), Proceedings of the Fifth International Conference on Rough Sets and Current Trends in Computing, RSCTC 2006. Lecture Notes in Computer Science*, vol 4259. Springer, Berlin, Heidelberg.
6. Yang. Y and John. R. , *Grey Sets and Greyness*, Information Sciences., **1**(185)(2012), 249-264.



## A note on central reversible group ring

Maryam Masoudi Arani\*

Department of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.  
Email: masoudiar@gmail.com

**ABSTRACT.** In this article, we study conditions under which a group ring  $R[G]$  of a group  $G$  over a ring  $R$  becomes a central reversible ring. The notion of central reversible ring is a generalization of reversible ring. We see that the group ring  $K[G]$  of a group  $G$  over a field  $K$  of characteristic zero is reversible if and only if  $K[G]$  is central reversible if and only if  $K[G]$  is reduced. Also, if  $K$  is a field of characteristic  $p > 0$  and  $G$  is a group which has no finite normal subgroup of order divisible by  $p$ , then  $K[G]$  is reversible if and only if  $K[G]$  is central reversible. Finally, we show that the group ring  $K[Q_8]$  of quaternions  $Q_8$  over a field  $K$  of characteristic zero is reversible if and only if  $K[Q_8]$  is central reversible if and only if the equation  $1 + x^2 + y^2 = 0$  is not solvable in  $K$ .

**Keywords:** reversible ring, central reversible ring, group ring

**AMS Mathematics Subject Classification [2020]:** 16P99; 16U80; 16S34.

### 1. Introduction

In this work, all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent element. If for any ideal  $L$  of  $R$ ,  $L^2 = 0$  yields  $L = 0$ , then  $R$  is called *semiprime*. A ring in which  $rst = 0$  yields  $rts = 0$ , for all  $r, s, t \in R$ , is called *symmetric ring*. A ring  $R$  is said to be *reversible* if for any  $a, b$  of  $R$ ,  $ab = 0$  implies  $ba = 0$ . In this paper, we introduce central reversible rings as a generalization of reversible rings. A ring  $R$  is called *central reversible* if  $ab = 0$  implies  $ba$  belongs to the center of  $R$ , for any  $a, b \in R$ . Clearly, reversible rings are central reversible. We see by an example every central reversible ring is not reversible, in general. In this paper, we study conditions under which a group ring  $R[G]$  of a group  $G$  over a ring  $R$  becomes a central reversible ring. We see that the group ring  $K[G]$  of a group  $G$  over a field  $K$  of characteristic zero is reversible if and only if  $K[G]$  is central reversible. Also, if  $K$  is a field with  $\text{char}(K) = p > 0$  and  $G$  is a group which has no finite normal subgroup of order divisible by  $p$ , then  $K[G]$  is reversible if and only if  $K[G]$  is central reversible if and only if  $K[G]$  is reduced. Finally, we show that the group ring  $K[Q_8]$  of quaternions  $Q_8$  over a field  $K$  of characteristic zero is reversible if and only if  $K[Q_8]$  is central reversible if and only if the equation  $1 + x^2 + y^2 = 0$  is not solvable in  $K$ .

\*Speaker.

**1.1. Introduction to nil reversible ring.** In here, we study some properties of central reversible rings. First, remember the definition of central reversible ring.

DEFINITION 1.1. A ring  $R$  is called central reversible if for each  $a$  and  $b$  of  $R$ ,  $ab = 0$  implies  $ba$  is a central element in  $R$ .

- EXAMPLE 1.2. (1) Every commutative ring is central reversible.  
 (2) reduced rings, central reduced rings, symmetric rings and reversible rings are central reversible.  
 (3) In this part, we see a central reversible ring but is not reversible. Let  $R$  be a commutative reduced ring and let

$$T = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}.$$

- [2, Example 2.2] shows that  $T$  is a central reversible but is not a reversible ring.  
 (4) Let  $K$  be a division ring,  $R = K[x, y]$  and  $I = (xy)$ , where  $xy \neq yx$ . Since  $R$  is a domain, then  $R$  is central reversible. On the other hand,  $(x + I)(y + I) = 0$  but  $(y + I)(x + I) = yx + I \neq 0$  is not central in  $\frac{R}{I}$ . Hence  $\frac{R}{I}$  is not central reversible.

In the following, we show that a set of central reversible rings is closed under finite direct sums.

PROPOSITION 1.3. *Every direct sums of central reversible rings is a central reversible ring.*

PROPOSITION 1.4. *Let  $R$  be a ring and  $I$  is a reduced ideal of  $R$ . If  $\frac{R}{I}$  is a central reversible ring, then  $R$  is a central reversible.*

## 2. Main results

In this section, we study conditions that the group ring  $R[G]$  of a group  $G$  over a ring  $R$  is a central reversible ring. It is a classical result, the group ring over a field of characteristic zero is semiprime. The next Lemma points to this fact.

LEMMA 2.1. *If  $K$  is a field with  $\text{char}(K) = 0$  and  $G$  is a group, then the group ring  $K[G]$  is semiprime.*

[3, Theorem 11.2] which we study in Proposition 2.2, is a result for the group rings of finite groups.

PROPOSITION 2.2. *Let  $K$  be a field with  $\text{char}(K) = p > 0$  and  $G$  be a group, then the group ring  $K[G]$  is semiprime if and only if  $G$  has no finite normal subgroup of order divisible by  $p$ .*

LEMMA 2.3. *Let  $R$  be a semiprime ring. Then,  $R$  is central reversible ring if and only if  $R$  is reversible.*

PROOF. We know that every reversible ring is central reversible. Now, let  $R$  is central reversible and  $x, y \in R$  so that  $xy = 0$ . Since  $R$  is central reversible,  $yx$  is central. So  $yxRyx = yxyxR = 0$  which yields  $yx = 0$ , because  $R$  is a semiprime ring. Therefore  $R$  is a reversible ring.  $\square$

Recall that a ring  $R$  is called *semicommutative* if for any  $x, y \in R$ ,  $xy = 0$  implies  $xRy = 0$ .

THEOREM 2.4. *Let  $R$  be a semiprime ring. Then the following sets are equivalent:*

- (1)  $R$  is reversible.
- (2)  $R$  is central reversible.
- (3)  $R$  is reduced.
- (4)  $R$  is semicommutative.

THEOREM 2.5. *Let  $K$  be a field and  $\text{char}(K) = p > 0$ . If  $G$  is a group which has no finite normal subgroup of order divisible by  $p$ , then the following sets coincide:*

- (1) The group ring  $K[G]$  is reduced.
- (2) The group ring  $K[G]$  is semicommutative.
- (3) The group ring  $K[G]$  is reversible.
- (4) The group ring  $K[G]$  is central reversible.

PROOF. Since  $\text{char}(K) = p > 0$  and  $G$  has no finite normal subgroup of order divisible by  $p$ , the group ring  $K[G]$  is semiprime, by Proposition 2.2. From this, we have this Theorem, by Theorem 2.4.  $\square$

THEOREM 2.6. *Let  $K$  be a field with  $\text{char}(K) = 0$  and  $G$  be a group. Then the following sets coincide:*

- (1) The group ring  $K[G]$  is reduced.
- (2) The group ring  $K[G]$  is semicommutative.
- (3) The group ring  $K[G]$  is reversible.
- (4) The group ring  $K[G]$  is central reversible.

COROLLARY 2.7. *If  $K$  is a field such that  $\text{char}(K) = 0$ , then the following conditions are equivalent:*

- (1) The group ring  $K[Q_8]$  is a reversible ring.
- (2) The group ring  $K[Q_8]$  is a central reversible ring.
- (3) The equation  $1 + x^2 + y^2 = 0$  is not solvable in  $K$ .

PROOF. We obtain this result by [1, Theorem 2.1] and Theorem 2.6.  $\square$

THEOREM 2.8. *Let  $R$  be a ring and  $G$  a group. If the right annihilator of each element of  $R[G]$  is generated by an idempotent of  $R[G]$ , then  $R[G]$  is central reversible ring if and only if  $R$  is reversible.*

THEOREM 2.9. *Let  $R$  be a ring and  $G$  a group. If the left annihilator of each element of  $R[G]$  is generated by an idempotent of  $R[G]$ , then  $R[G]$  is central reversible ring if and only if  $R$  is reversible.*

## References

- [1] Bell, h. e. and Li, y. (2007), *Duo group rings*, J. Pure Appl. Algebra, **3**, 833–838.
- [2] H. Kose, B. Ungor, S. Halicioglu and A. Harmanci. (2014) *A generalization of reversible rings*, IJST, **38**, 43-48
- [3] Sehgal, D. S. (2007), *Hand book of algebra, Group rings*, **3**, Elsevier/North-Holland, Amsterdam.



## Some Objects Of Category Of Grey Sets

Zohreh Habibi<sup>1,\*</sup>, Davood Darvishi<sup>2</sup>, Mostafa Nouri Jouybari<sup>3</sup>

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Z\_habibi@pnu.ac.ir

<sup>2</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: d\_darvishi@pnu.ac.ir

<sup>3</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: m\_njoybari@pnu.ac.ir

---

**ABSTRACT.** In this paper, we introduce the concept of morphisms between two grey sets and defined a new category, namely, GSet, of grey sets and grey morphisms. We investigate some categorical notions such product, coproduct, pullback, and pushout. Additionally, we show that any grey set has an injective hull.

**Keywords:** Coincidence point, category, grey set.

**AMS Mathematics Subject Classification [2020]:** 18A05, 18A20, 03B52.

---

### 1. Introduction

Grey system is one of the most important scientific achievements in the field of how to use uncertain information, which was presented by Julang Deng [?]. Darvishi et al. [?] studied the existing approaches of ordering interval grey numbers in the context of decision-making by surveying existing definitions. The category  $Set(V)$  becomes the category  $Set$  if  $V$  is a singleton and the *fuzzy subsets* category if  $V = [0, 1]$ . Harding et al. [?] studied the 2-category of fuzzy sets and relations. In [?], a new category of fuzzy sets and its properties were defined and studeid. We recall from [?], a *grey numbers* is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. A grey number is expressed mathematically as  $g^\pm \in [g^-, g^+] = \{g^- \leq t \leq g^+\}$  where  $g^\pm$  is a grey number,  $t$  is information, and  $g^-$  and  $g^+$  are the lower and upper limits of the information.

**DEFINITION 1.1.** Let  $X = (U, \chi_A)$  and  $Y = (U', \chi_{A'})$  be two grey sets which  $\chi_A : U \rightarrow D[0, 1]^\pm$  and  $\chi_{A'} : U' \rightarrow D[0, 1]^\pm$ . A grey morphism between grey sets  $X$  and  $Y$  is ordinary function  $f : U \rightarrow U'$  such that (upper) lower  $\chi_A(x) \leq$  (upper) lower  $\chi_{A'}f(x)$ , for shortly,  $\chi_A^\pm(x) \leq \chi_{A'}^\pm f(x)$ .

The composition of two grey morphisms, which is composition of two functions is assotative. The category of grey sets and grey morphismss between is denoted **GSet**.

---

\*Speaker.

Recall from [?], a concrete category over  $X$  is a pair  $(A, U)$ , where  $A$  is a category and  $U : A \rightarrow X$  is a faithful functor. Some times  $U$  is called the forgetfull functor of the concrete category and  $X$  is called the base category for  $(A, U)$ .

## 2. Main results

In this section, we study the existence of product and coproduct in the category  $GSet$ . Additionally, we investigate some categorical concepts such as pushout, pullback and injective objects in the category  $GSet$ , and we demonstrate that any grey set has an injective hull.

**PROPOSITION 2.1.** *The category  $Gset$  has products.*

**PROOF.** Consider a family  $(X_i, \chi_i)_{i \in J}$  of grey sets. We claim  $(X, \chi)$  is the product of this family, which  $X$  is the product  $X_i$ 's and  $\chi$  is the meet of  $\chi_i$ , i.e  $\wedge \chi_i^\pm = \{\wedge \chi_i^-, \wedge \chi_i^+\}$ , for  $i \in I$  and  $\pi_j : (X, \chi) \rightarrow (X_j, \chi_j)$  is projection map. We have  $\chi^\pm(x) = (\wedge_{i \in J} \chi_i^\pm)(x) = \wedge_{i \in J} (\chi_i^\pm(x)) \leq \chi_{j \in J}^\pm(x_j) = \chi_j^\pm(\pi_j(x))$ . So,  $\pi_j \in GSet$ . Now, suppose that there exists a map  $m_j : (Y, \tau) \rightarrow (X_j, \chi_j)$ , by universal properties of product of sets, there exists unique function  $\varphi : (Y, \tau) \rightarrow (\prod_{i \in J} X_i, \chi)$  such that  $\pi_j \varphi = m_j$ . Also, we have  $\tau^\pm \leq \chi_j^\pm m_j = \chi_j^\pm \pi_j \varphi = \chi_j^\pm \varphi$ , hence  $\varphi \in GSet$ . □

**PROPOSITION 2.2.** *The category  $GSet$  has coproducts.*

**PROOF.** For a family  $(X_i, \chi_i)_{i \in J}$  of grey sets, we claim  $(X, \chi)$  is the coproduct, which  $X^\pm$  is disjoint union of  $X_i$ 's and  $\chi^\pm$  is the disjoint union of  $\chi_i^\pm$ . For this, consider  $q_j : (X_j, \chi_j) \rightarrow (\prod_{i \in J} X_i, \prod_{i \in J} \chi_i)$ , which  $q_j : X_j \rightarrow \prod_{i \in J} X_i$  is inclusion maps, such that  $\chi_j^\pm(x_j) = (\prod_{i \in J} \chi_i^\pm q_j(x_j))$ . It is clear that  $q_j$  is a map in  $GSet$ . If there exists any grey map such  $\psi_j : (X_j, \chi_j) \rightarrow (Y, \varsigma)$  in  $GSet$ , by the prpperty of coproduct, there exists a unique function  $\tau : \prod_{i \in J} \chi_i \rightarrow Y$  such that  $\tau q_j = \psi_j$ . Also, since  $\chi_j^\pm \leq \varsigma^\pm \psi_j$ , we have  $\prod_{i \in J} \chi_i \leq \varsigma^\pm \tau$ . Hence  $\tau \in GSet$ . □

We denote the constant characteristic function at a point such as  $m$  by the  $\chi_m$ .

**PROPOSITION 2.3.** *In the category  $Gset$ , the empty set with the constant characteristic function  $\chi_0$  is the initial object and singelton set with the constant characteristic function  $\chi_1$  is the terminal object.*

**PROPOSITION 2.4.** *In the category  $GSet$ , the equalizer of two maps  $f, g : (X, \chi_1) \rightarrow (Y, \chi_2)$  is the pair  $(E, \chi_E)$ , which  $E = \{x \in X | f(x) = g(x)\}$ ,  $\chi_E : E \rightarrow D[0, 1]^\pm$  is the restriction of  $\chi_1$  on  $E$  and the inclusion map  $i : (E, \chi_E) \rightarrow (X, \chi_1)$ .*

**PROPOSITION 2.5.** *In the category  $GSet$ , the coequalizer of two maps  $f, g : (X, \chi_1) \rightarrow (Y, \chi_2)$  is the pair  $(K, \chi_K)$ , which  $K = \frac{Y}{\theta}$ , which  $\theta$  is the equivalence relation generated by of  $\{(f(x), g(x)) | x \in X\}$  and  $\chi_K^\pm(k) = \sup\{\chi_2^\pm(y) | \pi(y) = k\}$ , which  $\pi : Y \rightarrow K$  is natural function.*

Recall that in a category  $\mathcal{C}$ , the object  $F \in \mathcal{C}$  is called a free object in  $\mathcal{C}$ , if there exists  $I$  in the category  $Set$  and a function  $\varsigma : I \rightarrow [F]$  such that for any  $X \in \mathcal{C}$  and every

function  $\sigma : I \rightarrow [X]$ , there exists exactly  $\alpha \in Mor_C(F, X)$  such that  $\alpha\varsigma = \sigma$  in the category *Set*, i.e. the following diagram is commutative.

$$\begin{array}{ccc} I & \xrightarrow{\sigma} & [X] \\ \varsigma \downarrow & \nearrow \alpha & \\ [F] & & \end{array}$$

REMARK 2.6. It is clear that  $(\emptyset, \chi_0)$  is a free object, since it is an initial object in *GSet* by [?, Example 8.23].

In the concrete category, an object  $C$  is called injective if for any embedding  $m : A \rightarrow B$  and any morphism  $f : A \rightarrow C$ , there exists a morphism  $g : B \rightarrow C$  such that the following diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ f \downarrow & \nearrow g & \\ C & & \end{array}$$

LEMMA 2.7. *In the category GSet, the non-empty grey sets with constant characteristic function  $\chi_1$  are percisely injective objects.*

PROOF. Let  $f : (A, \chi_A) \rightarrow (B, \chi_B)$  be a monomorphism grey map and  $g : (A, \chi_A) \rightarrow (X, \chi_C)$  be arbitrary map in the category *GSet*, where  $\chi_1$  is constant function. We show that there exists a map  $h : (B, \chi_B) \rightarrow (X, \chi_C)$  such that  $hg = f$ . Recall that in the category *Set* any non-empty set is injective, so there exists a map  $h : B \rightarrow X$  such that extend  $g$ . We show that  $g$  is a grey map in the category **GSet**. By definition of  $X_C$ , the result  $\chi_B \leq \chi_C h$  is clear. So,  $h \in GSet$ . □

REMARK 2.8. In the category *GSet*, any non-empty object  $(A, \chi_1)$  can be embedded into injective object.

An injejective hull of  $A$  is an extension  $m : A \rightarrow B$  such that  $B$  is injective and  $m$  is essential.

LEMMA 2.9. *In the category GSet, any object has a injective hull.*

PROOF. It is clear that the injective hull of non-empty grey set, is itself. If  $A$  is empty, then  $(\{*\}, \chi_1)$  is injective hull of grey set  $A$ . □

THEOREM 2.10. *In the category GSet, there is a pullback of any morphisms  $f : (U, \chi_A) \rightarrow (W, \chi_C)$  and  $g : (V, \chi_B) \rightarrow (W, \chi_C)$ .*

PROOF. Consider grey sets  $(U, \chi_A)$ ,  $(V, \chi_B)$ ,  $(W, \chi_C)$  and morphisms  $f : (U, \chi_A) \rightarrow (W, \chi_C)$  and  $g : (V, \chi_B) \rightarrow (W, \chi_C)$ . Consider folowing diagram.

$$\begin{array}{ccc} & & V \\ & & \downarrow g \\ U & \xrightarrow{f} & W \end{array}$$

In the category *Set*, the pullback of this diagram is set  $T = \{(x, y) \in U \times V | f(x) = g(y)\}$  and functions  $\rho_U : T \rightarrow U$ ,  $\rho_V : T \rightarrow V$ , which  $\rho_U(a, b) = a$  and  $\rho_V(a, b) = b$ , for  $a, b \in U \times V$  such that  $f\rho_U = g\rho_V$ . Define characteristic function  $\chi$  such that  $\chi_D^+(a, b) = (\chi_A(a) \wedge \chi_B(b))^+$  and  $\chi_D^-(a, b) = (\chi_A(a) \wedge \chi_B(b))^-$  for any  $(a, b) \in T$ .

We claim  $\rho_U$  and  $\rho_V$  are in the category *GSet*. It is clear that for any  $a \in U$  and  $b \in V$ ,  $\chi_D^+(a, b) = (\chi_A(a) \wedge \chi_B(b))^+ \leq \chi_A^+(a) = \chi_A^+\rho_U(a, b)$  and  $\chi_D^-(a, b) = (\chi_A(a) \wedge \chi_B(b))^- = \chi_D^-(a, b)$ . Hence  $\rho_U \in \text{GSet}$ . Similarly we can show  $\rho_V \in \text{GSet}$ . Now consider grey object  $(T', \chi_{D'})$  such that  $ff' = gg'$ . From category *Set*, there exists exactly one function  $h : T' \rightarrow T$  such that  $\rho_U h = g'$  and  $\rho_V h = f'$ . It is easy to check that  $h \in \text{GSet}$ .  $\square$

**THEOREM 2.11.** *In category GSet, there is a pushout of any morphisms  $f : (U, \chi_A) \rightarrow (V, \chi_B)$  and  $g : (U, \chi_A) \rightarrow (W, \chi_C)$ .*

### References

1. J. Adámek, H. Herrlich and G. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, (1990).
2. D. Darvishi, J. Forrest and S. Liu, *A comparative analysis of grey ranking approaches*, *Grey Systems , Theory and Application*. **9** (2019), 472–487.
3. J. L. Deng, *Introduction to grey system theory*, *The Journal of Grey Systems*. **1:1** (1989), 1–24.
4. J. Harding, C. Walker and E. Walker, *Categories of fuzzy sets and relations*, *Fuzzy sets and Systems*. **265** (2014), 149–165.
5. A. R. Porselvi Agnes, D. Sivaraj and T. Tamizh chelvam, *On a new category of fuzzy sets*, *Journal of advanced research in pure mathematics*. **2:4** (2010), 73–83.
6. Y. Yang and R. John, *Grey Sets and Greyness*, *Inf. Sci.* **185:1** (2012), 249–264.



## Morphic $S$ -Acts over Commutative Monoids with Unique Zero

Zohreh Habibi

<sup>1</sup>Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

Email: Z\_habibi@pnu.ac.ir

**ABSTRACT.** In this paper, we introduce and investigate the notion of morphic  $S$ -acts over commutative monoids, focusing on the category of right  $S$ -acts that possess a unique zero element. Inspired by the concept of morphic modules introduced by El Mehdi Bouba, we define an  $S$ -act to be morphic if, for each element, its orbit corresponds to the annihilator of some element of the monoid, and the annihilator in the monoid satisfies a specific structural form. We examine how this property behaves under standard constructions, such as subacts, homomorphic images, preimages, direct sums, pullbacks, and Rees quotients. Several preservation results are established, and conditions ensuring the morphic property are identified. Throughout, we provide illustrative examples to clarify the theory and demonstrate its scope within the framework of commutative monoids.

**Keywords:** Morphic,  $S$ -act

**AMS Mathematics Subject Classification [2020]:** 06B10, 18C05, 18E20, 03E72

### 1. Introduction

The notion of morphic modules was introduced and studied by El Mehdi Bouba in [?], where modules satisfying a symmetry between images and annihilators were analyzed. Inspired by this concept, our aim in this paper is to explore an analogous structure in the category of right  $S$ -acts with a unique zero element. We investigate how the morphic property, appropriately defined for  $S$ -acts, interacts with fundamental constructions such as subacts, quotient acts, direct sums, pullbacks, and homomorphic images. A right  $S$ -act is a set  $A$  together with a map  $A \times S \rightarrow A$ , where  $(a, s) \rightarrow as$ , such that for all  $a \in A$  and  $s, t \in S$  we have  $a1 = a$  and  $a(st) = (as)t$ . Left  $S$ -acts are defined dually. Let  $S$  be a monoid and  $A$  a right  $S$ -act with a unique zero element, denoted  $0$ . For  $s \in S$ , the annihilator in  $A$  is defined as  $\text{Ann}_A(s) := \{x \in A \mid s \cdot x = 0\}$ , and for  $a \in A$ , the annihilator in the monoid is given by  $\text{Ann}_S(a) := \{t \in S \mid t \cdot a = 0\}$ . The global annihilator of the act is denoted  $\text{Ann}_S(A) := \{t \in S \mid t \cdot x = 0 \text{ for all } x \in A\}$ . A subact  $B \subseteq A$  is a subset closed under the  $S$ -action and containing the zero element. Given a subact  $B \subseteq A$  containing the zero element, the Rees quotient  $A/B$  is defined as the set of equivalence classes  $[a]$ , where  $a \sim b$  if either  $a = b$  or both  $a, b \in B$ . The action on the quotient is given by  $s \cdot [a] := [s \cdot a]$ , and the zero of  $A/B$  is  $[B]$ . For two  $S$ -acts  $A_1$  and  $A_2$ , the Cartesian product  $A_1 \times A_2$  equipped with the componentwise  $S$ -action  $s \cdot (a_1, a_2) := (s \cdot a_1, s \cdot a_2)$ .

The pullback of two  $S$ -homomorphisms  $f : A \rightarrow C$  and  $g : B \rightarrow C$  is defined as the subact  $A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$ , which inherits the  $S$ -action from the product act  $A \times B$ .

In this section, we recall the basic definitions and notations needed throughout the paper. The reader is assumed to be familiar with the general theory of  $S$ -acts over monoids. We fix a monoid  $S$  and assume all  $S$ -acts under consideration are right acts with a unique zero element, denoted by  $0$ . We also review standard constructions such as subacts, homomorphic images, pullbacks, quotients, and direct sums, as these will play a crucial role in our analysis of morphic  $S$ -acts in the subsequent sections. For more see [?], [?].

DEFINITION 1.1. An  $S$ -act  $A$  is called a *morphic  $S$ -act* if for every  $a \in A$ , there exists an element  $s \in S$  such that  $S \cdot a = \text{Ann}_A(s) := \{x \in A \mid s \cdot x = 0\}$  and  $\text{Ann}_S(a) = \text{Ann}_S(A) \cup Ss$ , where  $\text{Ann}_S(a) := \{t \in S \mid t \cdot a = 0\}$  and  $\text{Ann}_S(A) := \{t \in S \mid t \cdot x = 0 \text{ for all } x \in A\}$ . Clearly, trivial  $S$ -act is morphic.

THEOREM 1.2. *Let  $A$  be a morphic  $S$ -act and  $f : A \rightarrow B$  an injective  $S$ -homomorphism. Then  $f(A)$  is a morphic subact of  $B$ .*

PROOF. For  $b = f(a) \in f(A)$ , there exists  $s \in S$  such that

$$S \cdot a = \text{Ann}_A(s), \quad f(S \cdot a) = S \cdot f(a) = S \cdot b.$$

Since  $f$  is injective,  $S \cdot b = f(\text{Ann}_A(s)) = \text{Ann}_{f(A)}(s)$ , and the annihilator condition holds similarly.  $\square$

THEOREM 1.3. *Let  $A$  be a morphic  $S$ -act and  $B \subseteq A$  an  $S$ -subact. If for every  $b \in B$ , there exists  $s \in S$  such that  $S \cdot b = \text{Ann}_B(s)$ , then  $B$  is a morphic  $S$ -act.*

THEOREM 1.4. *Let  $B$  be a morphic  $S$ -act,  $f : A \rightarrow B$  an  $S$ -homomorphism, and  $C \subseteq B$  a morphic subact. Then the preimage  $f^{-1}(C)$  is a morphic subact of  $A$ .*

PROOF. First,  $f^{-1}(C) = \{x \in A \mid f(x) \in C\}$  is an  $S$ -subact of  $A$ , since for  $x \in f^{-1}(C)$ ,  $f(s \cdot x) = s \cdot f(x) \in C$ , and  $f(0_A) = 0_B \in C$ . For  $x \in f^{-1}(C)$ , let  $c = f(x) \in C$ . Since  $C$  is morphic, there exists  $s \in S$  such that:

$$S \cdot c = \text{Ann}_C(s), \quad \text{Ann}_S(c) = \text{Ann}_S(C) \cup Ss.$$

Compute:

$$S \cdot x = \{s \cdot x \mid s \in S\}, \quad f(S \cdot x) = \{f(s \cdot x) \mid s \in S\} = \{s \cdot f(x) \mid s \in S\} = S \cdot c = \text{Ann}_C(s).$$

Thus:

$$S \cdot x \subseteq f^{-1}(S \cdot c) = f^{-1}(\text{Ann}_C(s)).$$

For  $y \in f^{-1}(\text{Ann}_C(s))$ ,  $f(y) \in \text{Ann}_C(s)$ , so  $s \cdot f(y) = 0_B$ , i.e.,  $f(s \cdot y) = 0_B$ . If  $f$  is such that  $f(s \cdot y) = 0_B \implies s \cdot y \in f^{-1}(0_B)$ , we need:

$$\text{Ann}_{f^{-1}(C)}(s) = \{y \in f^{-1}(C) \mid s \cdot y \in f^{-1}(0_B)\}.$$

Assuming  $f^{-1}(0_B) = \{0_A\}$  (e.g.,  $f$  is injective or  $0_B$  has trivial preimage), then:

$$\text{Ann}_{f^{-1}(C)}(s) = \{y \in f^{-1}(C) \mid s \cdot y = 0_A\}.$$

Since  $f(y) \in \text{Ann}_C(s) = S \cdot c$ , there exists  $s' \in S$  such that  $f(y) = s' \cdot c = s' \cdot f(x) = f(s' \cdot x)$ , so  $y = s' \cdot x$  (if injective). Thus:

$$S \cdot x = \text{Ann}_{f^{-1}(C)}(s).$$

For the annihilator:

$$\text{Ann}_S(x) = \{t \in S \mid t \cdot x = 0_A\}, \quad t \cdot x = 0_A \implies f(t \cdot x) = t \cdot f(x) = t \cdot c = 0_B,$$

so  $\text{Ann}_S(x) \supseteq \text{Ann}_S(c) = \text{Ann}_S(C) \cup Ss$ . Conversely:

$$\text{Ann}_S(f^{-1}(C)) = \{t \in S \mid t \cdot y = 0_A \text{ for all } y \in f^{-1}(C)\}.$$

If  $t \cdot y = 0_A$ , then  $f(t \cdot y) = t \cdot f(y) = 0_B$ , so  $\text{Ann}_S(f^{-1}(C)) \subseteq \text{Ann}_S(C)$ . Thus:

$$\text{Ann}_S(f^{-1}(C)) \cup Ss \subseteq \text{Ann}_S(C) \cup Ss = \text{Ann}_S(x).$$

Equality holds under injectivity assumptions. Hence,  $f^{-1}(C)$  is morphic.  $\square$

**THEOREM 1.5.** *Let  $A$  be a morphic  $S$ -act and  $B \subseteq A$  a subact containing  $0_A$ . Then the Rees quotient  $A/B$  is a morphic  $S$ -act.*

**THEOREM 1.6.** *If  $A_1$  and  $A_2$  are morphic  $S$ -acts, then their direct sum  $A_1 \oplus A_2$  is a morphic  $S$ -act under the componentwise action.*

**THEOREM 1.7.** *Let  $A$  be a morphic  $S$ -act, and let  $I \subseteq S$  be a two-sided ideal containing  $0_S$ . Then the quotient  $A/IA$  is a morphic  $S/I$ -act, where  $IA = \{s \cdot a \mid s \in I, a \in A\}$ .*

**PROPOSITION 1.8.** *Let  $A$  and  $B$  be morphic  $S$ -acts, and let  $f : A \rightarrow C$ ,  $g : B \rightarrow C$  be  $S$ -homomorphisms. Then the pullback  $A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$  is a morphic  $S$ -act.*

## References

1. Bouba, E. M., Tamekkante, M., Tekir, Ü., Koç, S. (2024). On morphic modules over commutative rings. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 65(1), 1-11.
2. Nicholson, W.K., Sánchez Campos, E.: Morphic modules. *Comm. Algebra*. 33, 2629–2647 (2005)
3. Zhu, H., Ding, N.: Generalized morphic rings and their applications. *Comm. Algebra*. 35(9), 2820–2837 (2007)



# Application



# On Asymmetric Structures of FGM Copulas: Measures of Association and Properties

Hakim Bekrizadeh<sup>1,\*</sup>

<sup>1</sup>Department of Statistics, Payame Noor University (PNU), Tehran, Iran.

Email: Bekrizadeh@pnu.ac.ir

**ABSTRACT.** In this paper, we introduce an asymmetric structural of Farlie–Gumbel–Morgenstern (FGM) copulas and includes some of its recent extensions. Some general formulas for well-known association measures of these copulas are obtained and various properties of the proposed model are studied.

**Keywords:** FGM Copula, Associated Measures, Concepts of Dependence)

**AMS Mathematics Subject Classification [2020]:** Primary 62H05; Secondary 62H20

## 1. INTRODUCTION

One of the most popular parametric families of copulas, which were studied by Farlie (1960), Gumbel (1960) and Morgenstern (1956), is the Farlie-Gumbel-Morgenstern (*FGM*) copula. Because of their simple analytical form, *FGM* copula is widely used in modeling and studying the efficiency of nonparametric procedures. However, this copula has been shown to be somewhat limited. In detail for copula dependence parameter  $\theta \in [-1, 1]$ , the Spearman's Rho is  $[-0.33, 0.33]$ . Since the correlation domain of *FGM* copula is limited, more general copulas have been introduced with the aim of improving the correlation range. Bekrizadeh et al. (2012) proposed a new class of generalized family showing that this generalization gives negative dependence and found that  $\rho_S \in [-0.50, 0.43]$ . Bekrizadeh, et al. (2017) gave conditions for positive quadrant dependence and studied a class of bivariate uniform distribution with positive quadrant dependence property by generalizing the uniform representation of a well-known *FGM* copula. Bekrizadeh, et al. (2021) introduced an extended *FGM* family in two dimensions and studied how to approximate any distribution to this family. Bekrizadeh, et al. (2024) proposed a new class of generalized *FGM* copula and showed that their generalization can improve the correlation domain of *FGM* copula. The main contribution of this paper includes the followings: first, an extension of *FGM* copula and some fine properties are presented. Second, asymmetry properties and the general formulas for association measures of this family are studied. The main feature of this family is capability for modeling a wider range of dependence.

## 2. A New Asymmetric OF FGM Copula

The copula is mostly defined as a function  $C: [0, 1]^2 \rightarrow [0, 1]$  that satisfies the boundary conditions

\*Speaker.

A1.  $C(u, 0) = C(0, u) = 0$  and  $C(u, 1) = C(1, u) = u, \forall u \in [0, 1]$ ,

A2.  $\forall (u_1, u_2, v_1, v_2) \in [0, 1]^4$ , such that  $u_1 \leq u_2$  and  $v_1 \leq v_2$ ,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Eventually, for twice differentiable, 2-increasing property (A2) can be replaced by the condition

$$(1) \quad c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v) \geq 0,$$

where  $c(u, v)$  is the so-called copula density. A copula  $C$  is *symmetric* if  $C(u, v) = C(v, u)$ , for every  $(u, v) \in [0, 1]^2$ , otherwise  $C$  is *asymmetric*.

Regarding limitation of correlation range in *FGM* copula, along with other generalizations, Bekrizadeh, et al. (2024) proposed a new class of symmetric generalized *FGM* family whose dependence is as follows:

$$C(u, v) = uv[1 + \theta(1 - u^\alpha)(1 - v^\alpha)]^p, \alpha > 0 \text{ and } p = 0, 1, 2, \dots$$

In the following definition, an asymmetric extension of the above copula family is introduced in order to extend the *FGM* copula.

**Definition 2.1**

Suppose that the continuous functions  $A_1, A_2 : [0, 1] \rightarrow [0, 1]$  are differentiable on  $(0, 1)$ . An *asymmetric* function  $C_{\theta, p}^{A_1, A_2} : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is defined as

$$(2) \quad C_{\theta, p}^{A_1, A_2}(u, v) = uv[1 + \theta A_1(u)A_2(v)]^p, p \in [0, \infty), \forall (u, v) \in [0, 1]^2,$$

where the parameter  $\theta \in \Theta = [-1, 1]$  is called the associated parameter.

The following theorem gives sufficient and necessary conditions on  $A_1$  and  $A_2$  to ensure that  $C_{\theta, p}^{A_1, A_2}$  is a bivariate copula.

**Theorem 2.1**

Let  $A_1, A_2 : [0, 1] \rightarrow [0, 1]$  be continuously differentiable functions on  $(0, 1)$ . The function  $C_{\theta, p}^{A_1, A_2}$  is a bivariate copula if and only if  $A_1$  and  $A_2$  satisfy the following conditions:

B1.  $A_i(0) = 0$ , for  $i = 1, 2$ ,

B2.  $|xA'_i(x)| \leq 1$  and  $|A_i(x) + pxA'_i(x)| \leq 1$ , for every  $x \in [0, 1]$ , and for  $i = 1, 2$ , where  $A'_i(x) = \partial A_i(x) / \partial x$ .

**Proof:**

The proof involves two steps:

First, it is clear that  $C_{\theta, p}^{A_1, A_2}(x, 1) = C_{\theta, p}^{A_1, A_2}(1, x) = x, \forall x \in [0, 1] \Leftrightarrow (B1)$ .

Second, since  $A_1$  and  $A_2$  are continuously differentiable functions and  $[1 + \theta A_1(u)A_2(v)]^p$  is bounded on  $[0, 1]$  and 2-increasing function, by (??) the condition  $c_{\theta, p}^{A_1, A_2}(u, v) \geq 0$  hold, if and only if  $|xA'_i(x)| \leq 1$  and  $|A_i(x) + pxA'_i(x)| \leq 1$ , for every  $x \in [0, 1]$ , and  $i = 1, 2$ , where  $c_{\theta, p}^{A_1, A_2}(u, v)$  is as follows:

$$\begin{aligned} c_{\theta, p}^{A_1, A_2}(u, v) &= \partial^2 C_{\theta, p}^{A_1, A_2}(u, v) / \partial u \partial v \\ &= [1 + \theta A_1(u)A_2(v)]^{p-2} \\ &\times \{ (1 + \theta A_2(v) [A_1(u) + puA'_1(u)]) (1 + \theta A_1(u) [A_2(v) + pvA'_2(v)]) \\ &\quad + p\theta uA'_1(u)vA'_2(v) \} . \end{aligned}$$

2

2

**Remark 2.1**

The family  $C_{\theta,p}^{A_1,A_2}$  includes some known family of FGMcopulas introduced by researchers in recent years, which are as follows:

- (1) if  $A_i(x) = 1-x^\alpha, \forall x \in [0, 1]$ , for  $i= 1, 2, \alpha \geq 0$ , and  $p= 1$ , the family  $C_{\theta,p}^{A_1,A_2}$  leads to the symmetric extended FGM copula introduced by Huang and Kotz (1999)
- (2) if  $A_i(x) =x^q(1-x)^q, \forall x \in [0, 1]$ , for  $i= 1, 2, q \geq 1$  and  $p= 1$ , the family  $C_{\theta,p}^{A_1,A_2}$  leads to the symmetric extended FGM copula introduced by Lai and Xie (2000).
- (3) if  $A_i(x) = (1-x^\gamma)^\lambda, \forall x \in [0, 1]$ , for  $i= 1, 2, \gamma \geq 0, \lambda \geq 1$  and  $p= 1$ , the family  $C_{\theta,p}^{A_1,A_2}$  leads to the symmetric extended FGM copula introduced by Bairamov-Kotz (2002)
- (4) if  $A_i(x) =A(x), \forall x \in [0, 1]$ , for  $i= 1, 2$ , and  $p= 1$ , the family  $C_{\theta,p}^{A_1,A_2}$  leads to the symmetric copula introduced by Rodriguez-Lallena and Ubeda-Flores (2004).

**3. MEASURES OF DEPENDENCE**

In this section, we compute the measures of dependence for the family  $C_{\theta,p}^{A_1,A_2}$ . Since we cannot give formulas for the properties of dependence in terms of elementary functions, it is replaced by its expansion series on

$$\Omega= \{(\theta,A_1, A_2) : |\theta A_1(u)A_2(v)| < 1\}.$$

Based on  $\Omega$ , the family  $C_{\theta,p}^{A_1,A_2}$  in (2) for every  $p \in [0,\infty)$  may also be written by polynomial expansion with respect to  $A_1$  and  $A_2$  as

$$(3) \quad C_{\theta,p}^{A_1,A_2}(u, v) = uv + \sum_{k=1}^g \binom{p}{k} \theta^k u A_1^k(u) v A_2^k(v).$$

Note that, in (3), we have  $g = p$  when  $p$  is integer, otherwise,  $g$  equals to  $+\infty$ .

**Spearman’s rho**

Let  $X$  and  $Y$  be continuous random variables whose copula is  $C$ . Then the population version of Spearman’s rho for  $X$  and  $Y$  is given by

$$\rho_S = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

Note that,  $\rho_S$  coincides with correlation coefficient  $\rho$  between the uniform marginal distributions.

**Proposition 3.1**

Let  $(X, Y)$  be a pair of random variables with the family  $C_{\theta,p}^{A_1,A_2}$ . The Spearman’s rho ( $\rho_S$ ) for the family  $C_{\theta,p}^{A_1,A_2}$  is given by

$$(4) \quad \rho_S = 12 \sum_{k=1}^g \binom{p}{k} \theta^k B_1(k) B_2(k),$$

where  $B_i(k) = \int_0^1 x A_i^k(x) dx$ , for  $i = 1, 2$ .

**Proof:** By using (3), the Spearman’s  $\rho_S$  can be expanded.

**Example 3.1**

In the family  $C_{\theta,p}^{A_1,A_2}$ , let  $A_i(x) = 1 - x$ , for  $i = 1, 2$ , and  $\forall x \in [0, 1]$ . Then the family  $C_{\theta,p}^{A_1,A_2}$  leads to a new symmetric generalized FGM copula with  $-(\max\{1, p\})^{-1} \leq \theta \leq$

$p^{-1}$ . Since  $B_i(k) = \int_0^1 x A_i^k(x) dx = \frac{1}{(k+1)(k+2)}$ , we have by using (4) that

$$\rho_S = 12 \sum_{k=1}^g \binom{p}{k} \theta^k \left[ \frac{1}{(k+1)(k+2)} \right]^2,$$

where the upper bound of above Spearman's  $\rho_S$  can be increased up to approximately 0.3805 as  $p \rightarrow \infty$ , while the lower bound -0.3333 remains unchanged. Therefore, the admissible range of Spearman's  $\rho_S$  in the new symmetric generalized FGM family is  $[-0.3333, 0.3805]$ .

## REFERENCES

- (1) Amblard, C. and Girard, S. (2002). Symmetry and dependence properties within a semi parametric family of bivariate copulas. *J. Nonparamet. Stat.*, 14, 715-727.
- (2) Bekrizadeh, H., Parham, G.A. and Zadkarami, M.R. (2012). The New Generalization of Farlie Gumbel Morgenstern Copulas. *Appl. Math. Sci.*, 6(??), 3527-3533.
- (3) Bekrizadeh, H., Jamshidi, B. (2017). A new class of bivariate copulas: dependence measures and properties. *Metron* 75(??), 31-50.
- (4) Bekrizadeh, H. (2021). Generalized Family of Copulas: Definition and Properties. *Thailand Statistician* 19(??), 163-178.
- (5) Bekrizadeh, H. (2024). Generalized FGM copulas: Properties and applications. *Communications in Statistics: Simulation and Computation* 53(??), 744-755.
- (6) Cuadras, C.M. (2009). Constructing copula functions with weighted geometric means. *J. Statist. Plan. Inf.*, 139, 3766-3772.
- (7) Farlie, D.G.J. (1960). The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 47, 307-323.
- (8) Gumbel, E.J. (1960). Bivariate exponential distributions, *J. Amer. Statist. Assoc.*, 55, 698-707.
- (9) Huang, J.S. and Kotz, S. (1999). Modifications of the Farlie-Gumbel-Morgenstern distributions. A tough hill to climb. *Metrika*, 49, 135-145.
- (10) Fischer, M. and Klein, I. (2007). Constructing Symmetric Generalized FGM Copulas by means of certain Univariate Distributions. *Metrika*, 65, 243-260.
- (11) Joe, H. (1997). *Multivariate models and dependence concepts*. Chapman & Hall. London.
- (12) Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen, *Mitteilungsblatt für Mathematische. Statistik*, 8, 234-235.
- (13) Nelsen, R.B. (2006). *An introduction to copulas*. Springer Series in Statistics. Springer. New York.
- (14) Sklar, A. (1959). Fonctions de répartition à dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8, 229-231.



# Computational Algebraic Analysis of 3R Manipulators

Mahdi Dehghani Darmian<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.

Email: m.dehghanidarmian@ipm.ir

**ABSTRACT.** This paper presents an algebraic solution to the inverse kinematics of a planar 3R robot using the DETECT-CONDITION algorithm [2]. The method partitions the solution space into five distinct kinematic cases, providing explicit conditions and closed-form solutions. It automatically derives the workspace boundary and handles both regular and singular configurations, offering a robust symbolic alternative to numerical methods.

**Keywords:** 3R manipulator, DETECT-CONDITION algorithm, Inverse kinematics

**AMS Mathematics Subject Classification [2020]:** 13P10

## 1. Introduction

The planar 3R manipulator consists of three revolute joints connecting three links. Its inverse kinematics problem—finding joint angles  $\theta_1, \theta_2, \theta_3$  to achieve a desired end-effector pose  $(x, y, \phi)$ —is central to robot control and inherently algebraic, making it suitable for the DETECT-CONDITION algorithm [2]. Using the *Denavit–Hartenberg (D-H) convention* [3] with parameters  $\alpha_i = 0$ ,  $d_i = 0$ , and  $a_i = L_i$ , the robot’s geometry is systematically described. The D-H parameter table is as follows:

- **Link 1:**  $a_1 = L_1$ ,  $\alpha_1 = 0$ ,  $d_1 = 0$ ,  $\theta_1 = \theta_1$
- **Link 2:**  $a_2 = L_2$ ,  $\alpha_2 = 0$ ,  $d_2 = 0$ ,  $\theta_2 = \theta_2$
- **Link 3:**  $a_3 = L_3$ ,  $\alpha_3 = 0$ ,  $d_3 = 0$ ,  $\theta_3 = \theta_3$

*4.1.2 Homogeneous Transformation Matrices.* The transformation from frame  $\{i-1\}$  to frame  $\{i\}$  is given by the homogeneous transformation matrix  ${}^{i-1}_i T$ :

$$(1) \quad {}^{i-1}_i T = \begin{pmatrix} \cos \theta_i & -\sin \theta_i & 0 & a_{i-1} \\ \sin \theta_i \cos \alpha_{i-1} & \cos \theta_i \cos \alpha_{i-1} & -\sin \alpha_{i-1} & -d_i \sin \alpha_{i-1} \\ \sin \theta_i \sin \alpha_{i-1} & \cos \theta_i \sin \alpha_{i-1} & \cos \alpha_{i-1} & d_i \cos \alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given our parameters ( $\alpha_i = d_i = 0$ ), this simplifies to a series of planar rotations and translations: The overall transformation from the base frame  $\{0\}$  to the end-effector frame  $\{3\}$  is found by multiplying these matrices:

$$(2) \quad {}^0_3 T = {}^0_1 T \cdot {}^1_2 T \cdot {}^2_3 T = {}^0_3 T = \begin{pmatrix} \cos \phi & -\sin \phi & 0 & x \\ \sin \phi & \cos \phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

\*Speaker.

By equating the result to the desired end-effector pose matrix, we derive two equations:

$$(3) \quad x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$(4) \quad y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

The orientation  $\phi$  is the sum of the joint angles for a planar robot:

$$(5) \quad \phi = \theta_1 + \theta_2 + \theta_3$$

This is a system of three equations (for  $x$ ,  $y$ , and  $\phi$ ) in three unknowns ( $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ ). To convert this into an algebraic system suitable for the DETECT-CONDITION algorithm [2], we use the standard half-angle substitution:

$$(6) \quad t_i = \tan(\theta_i/2), \quad \sin \theta_i = \frac{2t_i}{1+t_i^2}, \quad \cos \theta_i = \frac{1-t_i^2}{1+t_i^2}$$

We now define the specific input for our algorithm. Let us fix the link lengths as symbolic parameters:  $a = L_1, b = L_2, c = L_3$ . The desired end-effector pose is also defined by parameters:  $x = P_x, y = P_y$ . The variables are the half-angle tangents:  $x_1 = t_1, x_2 = t_2, x_3 = t_3$ . After applying the half-angle substitution and clearing denominators, we obtain three very large polynomial equations  $f_1, f_2, f_3$ . The degrees of these polynomials in the variables  $x_1, x_2, x_3$  are high. For a 3R robot, the degrees are [6, 6, 2] for  $f_1$  (from the  $x$ -coordinate eq.),  $f_2$  (from the  $y$ -coordinate eq.), and  $f_3$  (from the orientation eq.) respectively. This instructs the algorithm to initialize the following three parametric polynomial equations:

- $f_1 = \sum_{i=0}^6 p_{1i}(a, b, c, P_x, P_y, \phi) \cdot x_1^i = 0$
- $f_2 = \sum_{i=0}^6 p_{2i}(a, b, c, P_x, P_y, \phi) \cdot x_1^i = 0$
- $f_3 = \sum_{i=0}^2 p_{3i}(a, b, c, P_x, P_y, \phi) \cdot x_1^i = 0$

The parameters of the system are  $\mathbf{a} = (a, b, c, P_x, P_y, \phi)$ . The variable is  $x_1$ . Using our Maple implementation of the DETECT-CONDITION algorithm [2] with input  $A = [6, 6, 2]$ , we compute the condition system for the ideal  $I = \langle f_1, f_2, f_3 \rangle$  with respect to a lexicographic order that eliminates the variable, e.g.,  $\phi \prec_{\text{lex}} P_y \prec_{\text{lex}} P_x \prec_{\text{lex}} c \prec_{\text{lex}} b \prec_{\text{lex}} a \prec_{\text{lex}} x_1$ . It is worth noting that in the following system;  $\Delta = 4a^2[(P_y - c \sin \phi)^2 + (P_x - b \cos \phi - c \sin \phi)^2 - (b^2 + c^2 - P_y^2)]$ .

$$V = \left\{ \begin{array}{l} \left( \begin{array}{l} N_1 : \{(a^2 + b^2 + c^2 - P_x^2 - P_y^2)^2 - 4a^2(b^2 + c^2 - P_y^2) \\ \quad - 4a^2(P_x - b \cos \phi - c \sin \phi)^2\} \\ W_1 : \{a, b, c, P_x^2 + P_y^2 - a^2 - b^2 - c^2 + 2a(b \cos \phi + c \sin \phi), \\ \quad P_x^2 + P_y^2 - a^2 - b^2 - c^2 - 2a(b \cos \phi + c \sin \phi)\} \\ V_1 : \left\{ t_1 = \frac{2a(P_y - c \sin \phi) \pm \sqrt{\Delta}}{P_x^2 + P_y^2 - a^2 - b^2 - c^2 + 2a(P_x - b \cos \phi - c \sin \phi)} \right\} \end{array} \right), \\ \left( \begin{array}{l} N_2 : \{P_x^2 + P_y^2 = (a + b + c)^2\} \\ W_2 : \{a, b, c\} \\ V_2 : \left\{ t_1 = \frac{P_y}{P_x + a + b + c} \right\} \end{array} \right), \\ \left( \begin{array}{l} N_3 : \{P_x^2 + P_y^2 - (a - b - c)^2\} \\ W_3 : \{a, b, c\} \\ V_3 : \left\{ t_1 = \frac{P_y}{P_x + a - b - c} \right\} \end{array} \right), \\ \left( \begin{array}{l} N_4 : \{P_x - b \cos \phi - c \sin \phi, P_y - b \sin \phi + c \cos \phi, a\} \\ W_4 : \{b, c\} \\ V_4 : \{t_1 \in \mathbb{R}\} \quad (\text{Infinite solutions}) \end{array} \right), \\ \left( \begin{array}{l} N_5 : \{\} \\ W_5 : \{(a^2 + b^2 + c^2 - P_x^2 - P_y^2)^2 - 4a^2(b^2 + c^2 - P_y^2) \\ \quad - 4a^2(P_x - b \cos \phi - c \sin \phi)^2\} \\ V_5 : \{\} \quad (\text{No solution}) \end{array} \right) \end{array} \right\}$$

## Visualization of the Five Branches in 3R Robot Inverse Kinematics

The DETECT-CONDITION algorithm [2], based on Gröbner system computation [1], partitions the parameter space of the planar 3R robot into five distinct branches, each corresponding to different kinematic configurations. These branches represent the complete solution space for the inverse kinematics problem.

**Branch 1: General Workspace Configuration.** The primary branch represents the general case where the target end-effector pose lies within the robot's workspace but is not at any singular configuration. The kinematic constraint equation:

$$(P_x^2 + P_y^2 - L_1^2 - L_2^2 - L_3^2)^2 = 4L_1^2(L_2^2 + L_3^2 - P_y^2) + 4L_1^2(P_x - L_2 \cos \phi - L_3 \sin \phi)^2$$

must be satisfied for a solution to exist. For parameters satisfying this condition, the algorithm provides the closed-form solution:

$$\theta_1 = 2 \arctan \left( \frac{2L_1(P_y - L_3 \sin \phi) \pm \sqrt{\Delta}}{P_x^2 + P_y^2 - L_1^2 - L_2^2 - L_3^2 + 2L_1(P_x - L_2 \cos \phi - L_3 \sin \phi)} \right)$$

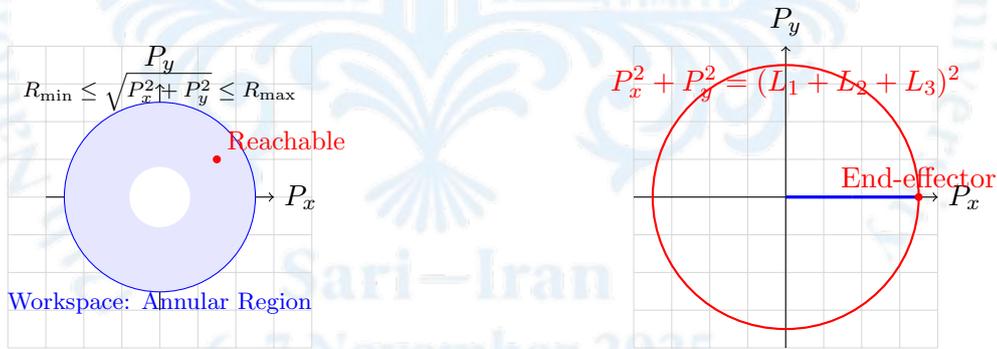
**Branch 2: Fully Extended Configuration.** This branch corresponds to the case where the end-effector is located at the maximum reachable distance from the base:

$$P_x^2 + P_y^2 = (L_1 + L_2 + L_3)^2$$

In this configuration, all links are collinear, and the robot arm is fully stretched. The solution becomes unique, with the joint angle given by:

$$\theta_1 = \arctan \left( \frac{P_y}{P_x + L_1 + L_2 + L_3} \right)$$

This represents a singular configuration where the Jacobian matrix loses rank, and the robot has reduced mobility in certain directions.



(A) Inner radius  $R_{\min} = |L_1 - L_2 - L_3|$  and outer radius  $R_{\max} = L_1 + L_2 + L_3$ .

(B) The robot links are collinear, reaching the maximum possible distance from the base.

FIGURE 1. Workspace and extended configuration of the 3R manipulator.

**Branch 3: Fully Retracted Configuration.** The third branch occurs when the end-effector is at the minimum reachable distance:  $P_x^2 + P_y^2 = (L_1 - L_2 - L_3)^2$ . In this case, the arm is completely folded back on itself. The solution is again unique:

$$\theta_1 = \arctan \left( \frac{P_y}{P_x + L_1 - L_2 - L_3} \right)$$

This configuration represents another singular point in the workspace where the robot's mobility is constrained.

**Branch 4: Degenerate Case with Infinite Solutions.** This branch represents a degenerate case where the kinematic equations become dependent, leading to infinitely many solutions. This occurs when:

$$P_x = L_2 \cos \phi + L_3 \sin \phi, \quad P_y = L_2 \sin \phi - L_3 \cos \phi, \quad L_1 = 0$$

In this configuration, the base joint coincides with the end-effector position, and any combination of joint angles that maintains the end-effector orientation is valid. This represents a workspace singularity where the inverse kinematics problem is underconstrained.

**Branch 5: Unreachable Configurations.** The final branch encompasses all parameter values for which no real solution exists. This occurs when the target pose lies outside the robot's workspace, violating the fundamental kinematic constraints. The algorithm correctly identifies these cases by the failure of all previous conditions, returning an empty solution set.

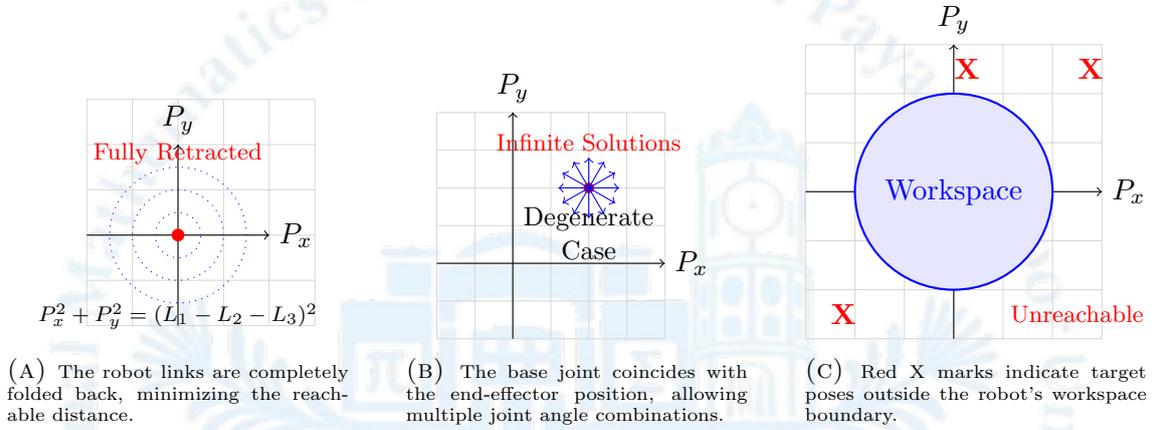


FIGURE 2. Singular and unreachable configurations of the 3R manipulator.

**Numerical Verification.** Let the link lengths be fixed:  $a = L_1 = 5$ ,  $b = L_2 = 3$ ,  $c = L_3 = 2$ . Let the desired end-effector pose be:  $P_x = 6$ ,  $P_y = 4$ ,  $\phi = \pi/4$  radians. These parameters satisfy Branch 1 conditions:

$$x_1 = t_1 = \frac{2 \cdot 5 \cdot (4 - \sqrt{2}) + \sqrt{\Delta}}{36 + 16 - 25 - 9 - 4 + 2 \cdot 5 \cdot (6 - 5\sqrt{2}/2)} \approx 0.455$$

Thus,  $\theta_1 = 2 \arctan(0.455) \approx 49.0^\circ$ . Back-substitution yields  $\theta_2 \approx 23.4^\circ$ ,  $\theta_3 \approx -27.4^\circ$ . Forward verification confirms the end-effector reaches  $(6, 4)$  with orientation  $45^\circ$ .

This result can be verified by forward kinematics: using  $\theta_1 \approx 49.0^\circ$  and solving for  $\theta_2, \theta_3$  from the remaining equations places the end-effector at  $(x, y) \approx (6, 4)$  with an orientation  $\phi \approx 45^\circ$ .

## References

- Dehghani Darmian, M. (2024) *Improvement of an Incremental Signature-Based Comprehensive Gröbner System Algorithm*, Mathematics in Computer Science, **18(12)**.
- Dehghani Darmian, M. (2024) *Automatic Identification of Conditions for Common Roots of Polynomial Equations*, 11<sup>th</sup> National Mathematics Conference of Payame Noor University, Yazd, Iran, 37–40.
- Denavit, J, Hartenberg, R. S. (1955) *A kinematic notation for lower-pair mechanisms based on matrices*, Journal of Applied Mechanics., **22(2)**, 215–221.

# Minimization of Total Mechanical Rotary Energy required for High-Pressure Natural Gas Transmission Using Thermodynamic Optimization

Syed Hossain Ebrahimi, Ph. D student

Faculty of Economics and Administrative Sciences, University of Mazandaran, Babolsar, Iran.

Email: shebrahimi1978@gmail.com / h.ebrahimi@stu.umz.ac.ir/ORCID: 0000-0002-5093-2143

Abdolhamid Safaei Ghadikolaie ,

Faculty of Economics and Administrative Sciences, University of Mazandaran, Babolsar, Iran.

Email: ab.safaei@umz.ac.ir

**Abstract:** Efficient energy utilization in high-pressure natural gas transmission system is vital for sustainable and cost-effective operation of gas supply chains. This study presents a simple thermodynamic optimization framework to minimize the total mechanical rotary energy required by turbo compressors in long-distance natural gas pipelines. The optimization is based on the analytical formulation, where the goal is to minimize the total rotary energy requirement. The proposed model accounts for real-gas behavior through the compressibility factor, adiabatic efficiency and thermo-dynamical parameters of the working gas. Parametric sensitivity analysis reveals that optimizing the pressure and maintaining high adiabatic efficiency significantly reduce energy demand. The findings provide a foundation for energy-efficient compressor station design and operation within modern natural gas supply networks.

**Keywords:** Turbo-compressor, natural gas, thermodynamic optimization, rotary energy, pipeline transmission.

## 1. Introduction

The global demand for natural gas continues to grow due to its lower carbon intensity compared with coal and oil [3]. Efficient long-distance transmission of natural gas through high-pressure pipelines is therefore a central challenge in modern energy infrastructure. Turbo compressors play a critical role in maintaining gas flow and pressure along the network, but their operation consumes substantial mechanical energy, representing one of the largest cost and emission components in gas transmission systems [4]. Minimizing the mechanical rotary energy required by compressors is therefore a key engineering goal. Conventional compressor station design is often guided by empirical rules or fixed operational schedules, neglecting opportunities for thermodynamic optimization [1,2]. This study aims to develop a rigorous analytical formulation to minimize compressor energy re-

quirements, expressed as a function of gas properties, compression ratio, and adiabatic efficiency.

## 2. Theoretical Background

For an ideal gas undergoing adiabatic compression[1,2], the specific work input is expressed as follows:

$$w = (\gamma / (\gamma - 1)) \times R \times T_1 \times [(P_2/P_1)^{((\gamma - 1)/\gamma)} - 1], \quad (1)$$

For real gases, the compressibility factor  $z$  is introduced, and compressor efficiency  $\eta_a$  is applied to account for irreversibilities. Scaling this work to volumetric flow rate yields the mechanical rotary energy  $Z_3$  (in Watt) proportional to gas throughput, inlet temperature, and pressure [4,5].

The total mechanical rotary energy [3,4] is given by:

$$Z_3 = 4.0639 \times (\gamma_1/(\gamma_1 - 1)) \times V_{tr} \times T_1 \times ((z_1 + z_2)/2) \times (1/\eta_a) \times [(P_2/P_{1\_min})^{((\gamma_1 - 1)/\gamma_1)} - 1], \quad (2)$$

where:

- $\gamma_1$ : Specific heat ratio of the natural gas mixture (typically 1.25–1.35)
- $V_{tr}$ : Volumetric gas flow rate ( $m^3/s$ )
- $T_1$ : Inlet gas temperature (K)
- $z_1, z_2$ : Compressibility factors at inlet and outlet conditions
- $\eta_a$ : Adiabatic efficiency of the compressor
- $P_2, P_{1\_min}$ : Discharge and suction pressures (Pa)

## 3. Mathematical modeling

In this study, It is intended to minimize the mechanical rotary energy required to transmit natural gas through the high pressure pipeline. So, the minimization problem can be expressed as:

$$\text{Minimize } Z_3(P_2, \eta_a) = dZ_3/dP_2 = 4.0639 \times V_{tr} \times T_1 \times ((z_1 + z_2)/2) \times (1/\eta_a) \times (1/P_{1\_min}) \times (P_2/P_{1\_min})^{((\gamma_1 - 1)/\gamma_1 - 1)}$$

subject to:

$$P_{1\_min} \leq P_2 \leq P_{2\_max}, \text{ and } 0 < \eta_a \leq 1.$$

## 4. Numerical Case Study

A detailed numerical case study was conducted to illustrate the behavior of the optimization function and quantify the influence of the operating parameters on the total mechanical rotary energy  $Z_3$ .

Baseline parameters (typical for a high-pressure natural gas transmission system) were used:

$$\gamma_1 = 1.3, V_{tr} = 50 \text{ m}^3/\text{s}, T_1 = 300 \text{ K}, z_1 = z_2 = 0.9, \eta_a = 0.85, \text{ and } P_{1\_min} = 5 \text{ MPa.}$$

The pressure ratio  $P_2/P_{1\_min}$  was varied between 1.05 and 2.50 to examine its influence on total rotary energy consumption.

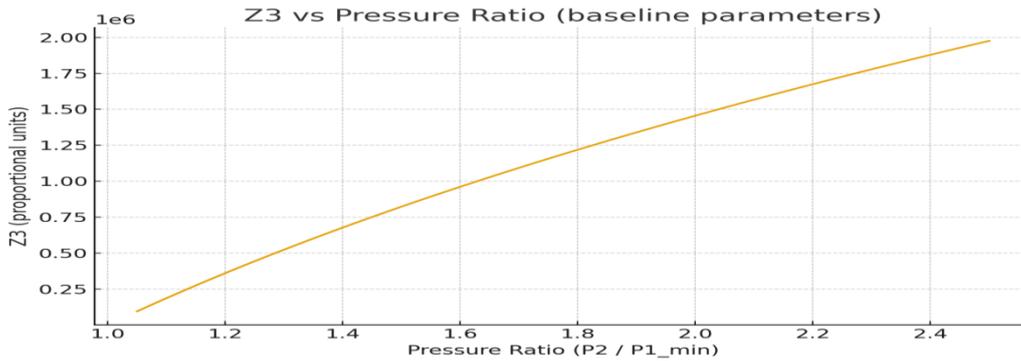


Figure 1. Variation of total rotary energy  $Z_3$  with compressor pressure ratio for baseline parameters.

The figure above demonstrates that the total mechanical rotary energy  $Z_3$  increases monotonically and nonlinearly with the compressor pressure ratio. For instance,  $Z_3 \approx 9.5 \times 10^4$  at  $P_2/P_1 = 1.05$  and grows to  $\approx 2.0 \times 10^6$  at  $P_2/P_1 = 2.5$  under the given conditions.

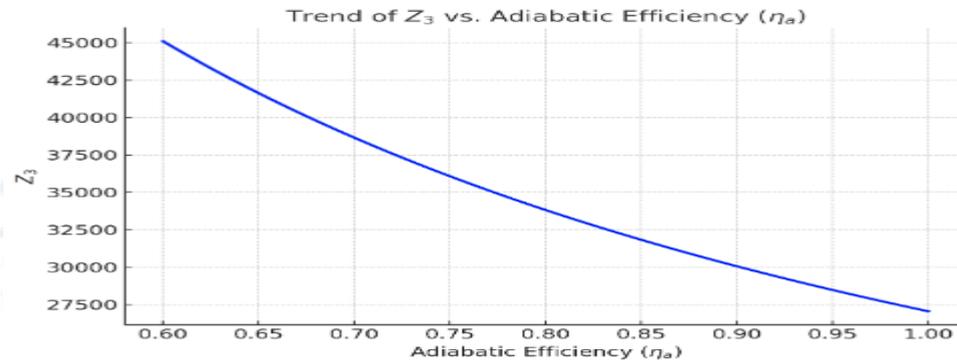


Figure 2. Variation of total rotary energy  $Z_3$  with adiabatic efficiency  $\eta_a$  for baseline parameters.

Besides raising adiabatic efficiency  $\eta_a$  from 0.80 to 0.90 yields about 11% reduction in rotary energy due to its inverse proportionality ( $1/\eta_a$ ). This issue is obviously shown in figure 2.

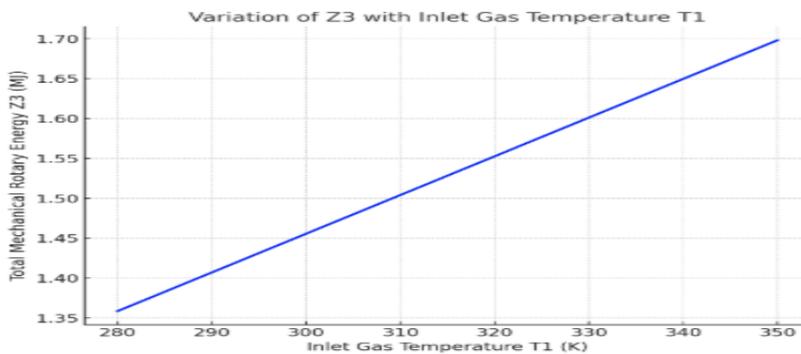


Figure 3. Variation of total rotary energy  $Z_3$  with Inlet gas temperature  $T_1$  for baseline parameters.

As expected and depicted in figure 3,  $Z_3$  increases linearly with  $T_1$ , reflecting the direct proportionality in the formula.

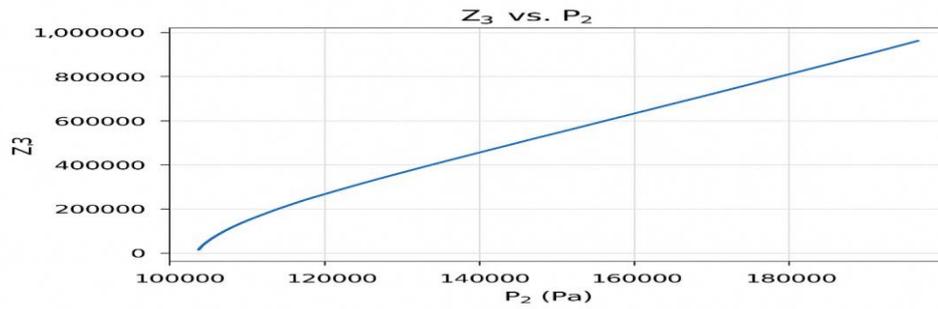


Figure 4. Variation of total rotary energy  $Z_3$  with outlet pressure  $P_2$  for baseline parameters.

Moreover,  $Z_3$  increases while  $P_2$  becomes greater in value. Analytically, differentiating the  $Z_3$  expression with respect to  $P_2$  shows  $\partial Z_3 / \partial P_2 > 0$  for all positive  $P_2$ . This confirms that  $Z_3$  is a monotonically increasing function of discharge pressure, and minimum energy occurs at the lowest allowable discharge pressure constrained by system hydraulics or downstream delivery requirements. These findings provide clear operational guidance for compressor optimization:

1. Maintain high adiabatic efficiency through advanced blade design and maintenance.
2. Minimize unnecessary pressure ratios by optimizing pipeline control and staging.
3. Employ gas cooling systems before compression to lower inlet temperature and energy consumption.

## 5. Conclusion

A closed-form thermodynamic optimization model has been developed to minimize the total mechanical rotary energy in natural gas pipeline compressors. The derived formula captures real-gas, thermodynamical, and mechanical characteristics influencing energy consumption. The analysis confirms that reducing compression ratio (to the minimum acceptable level) and improving adiabatic efficiency are the most effective levers for energy savings. The simplified model complements more advanced approaches (network optimization, transient control, exergy or economic coupling) and can act as a baseline or a module in larger frameworks. Future work should embed this into dynamic simulations, digital twin systems, or full network optimizations to manage energy use across realistic transmission grids.

## References

1. Çengel, Y. A., & Boles, M. A. (2020). *Thermodynamics: An Engineering Approach*. McGraw-Hill.
2. Moran, M. J., & Shapiro, H. N. (2018). *Fundamentals of Engineering Thermodynamics*. Wiley.
3. Mokhtab, S., Poe, W. A., & Speight, J. G. (2019). *Handbook of Natural Gas Transmission and Processing*. Gulf Professional Publishing.
4. Tarbell, J. M. (2015). Energy efficiency analysis of gas pipeline compressor stations. *Journal of Energy Engineering*, 141(4), 04014029.
5. Ghilardi, L. M. P. et al. (2025). Optimal Operation of Gas Transport Pipelines with Detailed Compressor Modeling. *ACS Industrial & Engineering Chemistry Research*.



## Mathematical SIV epidemic model with standard incidence rate and temporary–perfect vaccination

Mahmood Parsamanesh<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.

Email: mparsamanesh@tvu.ac.ir

**ABSTRACT.** The spread of an infection is considered by a compartmental model including three sub-populations as susceptible, infectious, and vaccinated. The incidence rate is of standard type, the vaccination is temporary, with perfect impact, and includes not only susceptible but also the new-comers. Two equilibria of the model, and the basic reproduction number are given and then the stability of the model is studied.

**Keywords:** Epidemic model, Equilibrium, Basic reproduction number, Stability

**AMS Mathematics Subject Classification [2020]:** 92D25, 34D23

### 1. Introduction

The susceptible-infected-susceptible (SIS) epidemic models are one of the well known type of epidemic models. In this model infectious individuals become susceptible after recovery without permanent immunity [2-4]. To control and eliminate the infectious diseases, vaccination is usually preferred because of its efficiency compared with other measures. So, we consider the vaccinated individuals as a separate sub-population in the model. In this paper, we investigate an SIS epidemic model with vaccination and standard incidence.

### 2. The model

Suppose that the interactions and transmissions of individuals in a population follow a dynamical system as:

$$(1) \quad \begin{aligned} S' &= (1 - \nu)\Lambda + \gamma I + \theta V - \frac{\beta SI}{S+I+V} - (\mu + \rho)S, \\ I' &= \frac{\beta SI}{S+I+V} - (\mu + \gamma + \alpha)I, \\ V' &= \nu\Lambda + \rho S - (\mu + \theta)V. \end{aligned}$$

Here, a population with  $N$  individuals has been divided to two sub-populations as susceptible, infectious, and vaccinated individuals with  $S$ ,  $I$ , and  $V$  individuals in each one.  $\Lambda$  and  $\mu$  are respectively recruitment rate and natural death rate, while  $\alpha$  denotes the rate of deaths due to infection. Susceptible individuals become infectious at standard incidence

\*Speaker.

rate  $\beta SI/N$  where  $\beta$  is transmission coefficient, and recover from infection with rate  $\gamma$ . The vaccination program is applied on both new members (with rate  $\nu$ ) and susceptible individuals (with rate  $\rho$ ). Although it assumed completely effective, its immunity is lost by passing time (with rate  $\theta$ ). All parameters are assumed to be non-negative in addition to  $\Lambda \neq 0$  and  $\mu \neq 0$ .

The equilibria of the model (1) are obtained by solving the following equations:

$$(2) \quad \begin{aligned} (1 - \nu)\Lambda - \frac{\beta S \bar{I}}{S + \bar{I} + \bar{V}} - (\mu + \rho)\bar{S} + \gamma\bar{I} + \theta\bar{V} &= 0, \\ \left(\frac{\beta \bar{S}}{S + \bar{I} + \bar{V}} - (\mu + \gamma + \alpha)\right)\bar{I} &= 0, \\ \nu\Lambda + \rho\bar{S} - (\mu + \theta)\bar{V} &= 0. \end{aligned}$$

System (1) has two equilibria in the form  $(\bar{I}, \bar{S}, \bar{V})$ . The infection-free equilibrium of the model is

$$EV^0 = \left(0, \frac{\Lambda(\mu(1 - \nu) + \theta)}{\mu(\mu + \theta + \rho)}, \frac{\Lambda(\mu\nu + \rho)}{\mu(\mu + \theta + \rho)}\right),$$

and the infected equilibrium is

$$EV^* = \left(\frac{\Lambda(\mathcal{R}_0^v - 1)}{\mu\mathbf{R} + \alpha(\mathbf{R} - 1)}, \frac{\Lambda(\mu + \gamma + \alpha)}{\beta\mu} \left(1 - \frac{\alpha(\mathcal{R}_0^v - 1)}{\mu\mathbf{R} + \alpha(\mathbf{R} - 1)}\right), \frac{\nu\Lambda}{\nu + \theta} \left(1 + \frac{\rho(\mu + \alpha\nu + \theta)/\nu(\mu + \rho + \theta)}{\mu\mathbf{R} + \alpha(\mathbf{R} - 1)}\right)\right),$$

in which

$$\mathcal{R}_0^v = \frac{\beta(\mu(1 - \nu) + \theta)}{(\mu + \gamma + \alpha)(\mu + \rho + \theta)},$$

is the basic reproduction number of model (1) and

$$\mathbf{R} = \frac{\beta(\mu + \theta)}{(\mu + \gamma + \alpha)(\mu + \rho + \theta)}.$$

It is seen that  $EV^*$  exists if  $\mathcal{R}_0^v > 1$ . Therefore we can state the following lemma:

LEMMA 2.1. *System (1) has only the infection-free equilibrium  $EV^0$  when  $\mathcal{R}_0^v \leq 1$  and it also has a unique infected equilibrium  $EV^*$  if  $\mathcal{R}_0^v > 1$ .*

### 3. Stability of the model

In the following theorem the local asymptotic stability of equilibria is considered:

THEOREM 3.1. *For model (1),*

- (I) *the infection-free equilibrium  $EV^0$  is stable if  $\mathcal{R}_0^v < 1$  and it is unstable if  $\mathcal{R}_0^v > 1$ ,*
- (II) *the infected equilibrium  $EV^*$  is stable if  $\mathcal{R}_0^v > 1$ .*

PROOF. Part (I) can be easily proven by Theorem 2 of [5]. For proving Part (II), letting  $F = \frac{\beta SI}{N}$ , the Jacobian matrix of the model (1) at  $EV^*$  has the following form:

$$(3) \quad J(EV^*) = \begin{pmatrix} F_I - (\mu + \gamma + \alpha) & F_S & F_V \\ -F_I + \gamma & -F_S - (\mu + \rho) & -F_V + \theta \\ 0 & \rho & -(\mu + \theta) \end{pmatrix},$$

in which

$$\begin{aligned} F_I &= \left. \frac{\partial F}{\partial I} \right|_{EV^*} = \frac{\beta S^*}{(S^* + I^* + V^*)^2} (S^* + V^*), \\ F_S &= \left. \frac{\partial F}{\partial S} \right|_{EV^*} = \frac{\beta I^*}{(S^* + I^* + V^*)^2} (S^* + V^*), \\ F_V &= \left. \frac{\partial F}{\partial V} \right|_{EV^*} = -\frac{\beta S^* I^*}{(S^* + I^* + V^*)^2}. \end{aligned}$$

The characteristic equation of matrix  $J(EV^*)$  is [1]

$$(4) \quad p(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3,$$

where

$$\begin{aligned} a_1 &= -\text{tr}(J(EV^*)) = -F_I + F_S + (\mu + \gamma + \alpha) + (2\mu + \rho + \theta), \\ a_2 &= -\frac{1}{2}(\text{tr}(J(EV^*))^2 - \text{tr}^2(J(EV^*))) = -F_I(2\mu + \rho + \theta) + F_S(2\mu + \alpha + \theta) + F_V \rho \\ &\quad + \mu(\mu + \rho + \theta) + (\mu + \gamma + \alpha)(2\mu + \rho + \theta), \\ a_3 &= -\det(J(EV^*)) = -F_I \mu(\mu + \rho + \theta) + F_S(\mu + \alpha)(\mu + \theta) + F_V \rho(\mu + \alpha) \\ &\quad + \mu(\mu + \gamma + \alpha)(\mu + \rho + \theta). \end{aligned}$$

We have

$$(5) \quad \begin{aligned} F_I &= (\mu + \gamma + \alpha) \left( 1 - \frac{\mu(\mathcal{R}_0 - 1)}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right), \\ F_S &= \mu(\beta - (\mu + \gamma + \alpha)) \left( \frac{\mathcal{R}_0 - 1}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right), \\ F_V &= -\mu(\mu + \gamma + \alpha) \left( \frac{\mathcal{R}_0 - 1}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right). \end{aligned}$$

Obviously  $F_V < 0$  and  $F_I = (\mu + \gamma + \alpha) \left( \frac{(\mu + \alpha)(\mathbf{R} - \mathcal{R}_0) + \mu}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) > 0$ . Moreover, notice that the equilibrium  $E^*$  exists if  $\mathcal{R}_0 > 1$  and this implies

$$\beta > \beta \left( \frac{\mu(1 - \nu) + \theta}{\mu + \rho + \theta} \right) > (\mu + \gamma + \alpha),$$

and thus  $F_S > 0$ .

Using (5) we obtain

$$(I) \quad -F_I + F_S = -(\mu + \gamma + \alpha) + \mu\beta \left( \frac{\mathcal{R}_0 - 1}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) \text{ thus}$$

$$a_1 = \mu\beta \left( \frac{\mathcal{R}_0 - 1}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) + (2\mu + \rho + \theta) > 0,$$

(II) We can see that

$$\begin{aligned} -F_I(2\mu + \rho + \theta) + F_V \rho + (\mu + \gamma + \alpha)(2\mu + \rho + \theta) \\ = (\mu + \gamma + \alpha) \left( \frac{\mu(\mathcal{R}_0 - 1)}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) (2\mu + \theta), \end{aligned}$$

therefore

$$\begin{aligned} a_2 &= (\mu + \gamma + \alpha) \left( \frac{\mu(\mathcal{R}_0 - 1)}{\mu \mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) (2\mu + \theta) \\ &\quad + F_S(2\mu + \alpha + \theta) + \mu(\mu + \rho + \theta) > 0. \end{aligned}$$

(III) We have

$$\begin{aligned} a_3 &= \mu(\mu + \rho + \theta)(\mu + \gamma + \alpha) \left( \frac{\mu(\mathcal{R}_0 - 1)}{\mu\mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) \\ &+ \left( \beta(\mu + \theta) - (\mu + \rho + \theta)(\mu + \gamma + \alpha) \right) \mu(\mu + \alpha) \left( \frac{\mathcal{R}_0 - 1}{\mu\mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) \\ &+ \mu(\mu + \rho + \theta)(\mu + \gamma + \alpha). \end{aligned}$$

Besides,  $\mathbf{R} > \mathcal{R}_0$  and  $\mathcal{R}_0 > 1$  implies  $\beta(\mu + \theta) > (\mu + \rho + \theta)(\mu + \gamma + \alpha)$ . Therefore

$$a_3 > \mu(\mu + \rho + \theta)(\mu + \gamma + \alpha) \left( 1 + \frac{\mu(\mathcal{R}_0 - 1)}{\mu\mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) > 0.$$

(IV)

$$\begin{aligned} a_1 a_2 - a_3 &= (\mu + \theta)(a_2 - F_S(\mu + \alpha)) + (\mu + \gamma + \alpha)(a_2 - \mu(\mu + \rho + \theta)) \\ &+ (-F_I + F_S + \mu + \rho)a_2 + F_I\mu(\mu + \rho + \theta) - F_V\rho(\mu + \alpha) \\ &= (\mu + \theta)(a_2 - F_S(\mu + \alpha)) + ((\mu + \gamma + \alpha) - F_I)(a_2 - \mu(\mu + \rho + \theta)) \\ &+ (F_S + \mu + \rho)a_2 - F_V\rho(\mu + \alpha). \end{aligned}$$

We see that  $F_I = (\mu + \gamma + \alpha) \left( 1 - \frac{\mu(\mathcal{R}_0 - 1)}{\mu\mathbf{R} + \alpha(\mathbf{R} - \mathcal{R}_0)} \right) < (\mu + \gamma + \alpha)$  and as a result

$$a_1 a_2 - a_3 > (\mu + \theta)(a_2 - F_S(\mu + \alpha)) + (F_S + \mu + \rho)a_2 - F_V\rho(\mu + \alpha) > 0.$$

In preceding relations we got  $a_1 > 0$ ,  $a_2 > 0$ ,  $a_3 > 0$  and  $a_1 a_2 - a_3 > 0$ , thus by using Routh-Hurwitz criterion [1] the real part of all eigenvalues of the Jacobian matrix  $J(EV^*)$  must be negative. Therefore the theorem has been proved.  $\square$

#### 4. Conclusion

An SIS epidemic model with vaccination program was investigated. Two equilibria for model, the infection-free equilibrium and the infected equilibrium, were found. The basic reproduction of model was also obtained and it was shown that when  $\mathcal{R}_0 < 1$  the infection-free equilibrium is stable while, when  $\mathcal{R}_0 > 1$  this equilibrium becomes unstable and the infected equilibrium becomes stable.

#### References

1. Allen, L.J. (2007) *Introduction to mathematical biology*, Pearson/Prentice Hall.
2. Brauer, F., Castillo-Chavez, C. (2001) *Mathematical Models in Population Biology and Epidemiology*, vol. 1. Springer, Berlin.
3. Parsamanesh, M., Erfanian, M. (2018). *Global dynamics of an epidemic model with standard incidence rate and vaccination strategy*, *Chaos, Solitons & Fractals*, **117**, 192–199.
4. Parsamanesh, M., Erfanian, M., Akrami, A. (2021). *Modeling of the propagation of infectious diseases: mathematics and population*, *EBNESINA Journal*, **22** (4), 60–74
5. Van den driessche, P., Watmough, J. (2002) *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, *Math Biosci*, **180**(1), 29–48.



# An Adaptive Perona–Malik Model with Coupled Gradient Weight and Gradient-Dependent Fidelity

Neda Mohamadi<sup>1,\*</sup>

<sup>1</sup>Department of Mathematics and Computer Science, Ma.C., Islamic Azad University, Mashhad, Iran.

Email: nedamohamadi@iau.ac.ir

---

**ABSTRACT.** We propose an image denoising model based on partial differential equations that extends the classical Perona–Malik anisotropic diffusion method. The diffusion coefficient depends on the image gradient and controls smoothing, while a gradient-dependent fidelity term adjusts adherence to the noisy image. This approach reduces noise in flat regions and preserves edges and fine details. Experiments on standard test images show improvements in peak signal-to-noise ratio and structural similarity index, demonstrating effective edge-preserving denoising.

**Keywords:** image denoising, anisotropic diffusion, Perona–Malik method, gradient-dependent fidelity

**AMS Mathematics Subject Classification [2020]:** 68U10, 65M06, 94A08

---

## 1. Introduction

Image denoising, a fundamental problem in image processing, seeks to reconstruct a clean image from noisy observations while preserving important structures such as edges and textures. Partial Differential Equation (PDE)-based approaches have been widely explored for this purpose due to their solid mathematical foundation and flexibility. Among them, the Perona–Malik (PM) anisotropic diffusion model [1] introduced a spatially varying diffusion process that effectively suppresses noise in homogeneous areas while preserving edges. Various extensions, including adaptive PM models, have been developed to overcome the well-known drawbacks of the original PM formulation, such as staircasing artifacts and contrast loss, by adjusting diffusion or fidelity parameters according to local image features.

Recent high-quality denoising methods, including deep neural networks and diffusion-based probabilistic models, achieve impressive results [2, 5]. However, these approaches often require large datasets and high computational cost. In contrast, PDE-based methods such as adaptive PM [3] remain attractive due to their interpretability, stability, and efficiency. Motivated by these advantages, this paper proposes a modified PM model with a coupled gradient-weight and gradient-dependent fidelity term that adaptively balances

---

\*Speaker.

diffusion and data fidelity. The proposed approach achieves enhanced noise suppression and edge preservation without heavy computation.

## 2. Proposed Model

Let  $u(x, y, t)$  denote the evolving image on a domain  $\Omega \subset \mathbb{R}^2$  with initial condition  $u_0(x, y)$ . We propose the following PDE for image denoising:

$$(1) \quad \frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|)\nabla u) - \lambda(x, y)(u - u_0),$$

with Neumann (reflective) boundary conditions.

The first term in (1) represents an anisotropic diffusion process governed by the diffusion coefficient  $g(|\nabla u|)$ . When the image gradient is small, diffusion is strong, efficiently smoothing flat regions. Conversely, large gradients corresponding to edges yield smaller  $g$ , preserving discontinuities. This mechanism generalizes the Perona–Malik diffusion [1] and provides adaptive smoothing guided by local contrast.

The fidelity coefficient  $\lambda(x, y)$  is defined as

$$\lambda(x, y) = \lambda_0 e^{-\alpha|\nabla u|^2} + \mu \frac{1}{\sqrt{1 + |\nabla u|^2}},$$

where  $\lambda_0$  and  $\mu$  are positive parameters.

The first exponential term dominates in flat regions ( $|\nabla u|$  small) to enforce stronger adherence to the noisy image  $u_0$ , accelerating denoising in homogeneous areas. The second term provides a minimal fidelity contribution near edges ( $|\nabla u|$  large), ensuring that the fidelity term never vanishes completely. This mechanism helps to preserve edge structures while improving numerical stability and smooth convergence of the iterative scheme.

Overall, this gradient-dependent formulation adaptively balances the diffusion and fidelity terms, achieving effective noise suppression in flat regions while preventing edge blurring [4].

## 3. Numerical Scheme and Stability

We discretize the spatial domain using grid step  $h$  and time step  $\Delta t$ . Gradients are approximated by central differences:

$$(\nabla_x u)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad (\nabla_y u)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}.$$

The gradient magnitude is computed as

$$|\nabla u|_{i,j} = \sqrt{(\nabla_x u)^2 + (\nabla_y u)^2 + \varepsilon},$$

where  $\varepsilon > 0$  avoids division by zero.

**Discrete Divergence Operator.** To approximate the divergence term  $\nabla \cdot (g\nabla u)$  with variable coefficient  $g$ , we use the central difference formula on cell faces:

$$\nabla \cdot (g\nabla u)_{i,j} = \frac{1}{h^2} \left[ g_{i+\frac{1}{2},j} (u_{i+1,j} - u_{i,j}) - g_{i-\frac{1}{2},j} (u_{i,j} - u_{i-1,j}) + g_{i,j+\frac{1}{2}} (u_{i,j+1} - u_{i,j}) - g_{i,j-\frac{1}{2}} (u_{i,j} - u_{i,j-1}) \right],$$

where  $g_{i+\frac{1}{2},j}$  denotes the diffusivity at the midpoint between nodes  $i$  and  $i+1$ ,

$$g_{i+\frac{1}{2},j} = \frac{1}{2}(g_{i,j} + g_{i+1,j}), \quad g_{i,j+\frac{1}{2}} = \frac{1}{2}(g_{i,j} + g_{i,j+1}).$$

This symmetric averaging ensures both stability and consistency with the continuous operator, yielding second-order spatial accuracy.

The explicit time-marching scheme is then:

$$u_{i,j}^{n+1} = u_{i,j}^n + \Delta t [\nabla \cdot (g \nabla u)_{i,j}^n - \lambda_{i,j}^n (u_{i,j}^n - u_{0,i,j})].$$

**Stability Condition.** For explicit schemes, stability follows the Courant-Friedrichs-Lewy (CFL) condition:

$$\Delta t \leq \frac{h^2}{4 \max_{i,j} \{g_{i,j}\} + \max_{i,j} \{\lambda_{i,j}\}}.$$

Since  $0 < g(|\nabla u|) \leq 1$ , this implies  $\Delta t \lesssim h^2/4$ , consistent with classical PM-type diffusion schemes. This ensures monotonicity and prevents oscillations or divergence.

**Existence and Uniqueness of Solution.** Equation (1) belongs to the class of quasi-linear parabolic PDEs. Following [4], given  $u_0 \in L^2(\Omega)$  and smooth, positive coefficients  $g$  and  $\lambda$ , there exists a unique weak solution  $u(x, y, t) \in L^2(0, T; H^1(\Omega))$ .

Moreover, the solution satisfies the energy dissipation property:

$$\frac{d}{dt} \int_{\Omega} (u - u_0)^2 dx dy \leq 0,$$

which guarantees convergence to a steady-state denoised image and ensures numerical stability of the discretization.

#### 4. Numerical Results

We tested the model on  $256 \times 256$  grayscale images (*Lena*, *Boat*, *Rice*) with Gaussian noise  $\sigma_n = 15, 25$  and speckle noise. Parameters were chosen empirically as  $\lambda_0 = 0.2$ ,  $\alpha = 20$ ,  $\mu = 0.05$ ,  $k = 0.15$ , and  $\Delta t = 0.15h^2$  to balance edge preservation and noise removal. Figure 1 illustrates the visual denoising results, showing from left to right the original image, the noisy observation, and the denoised output produced by the proposed model.

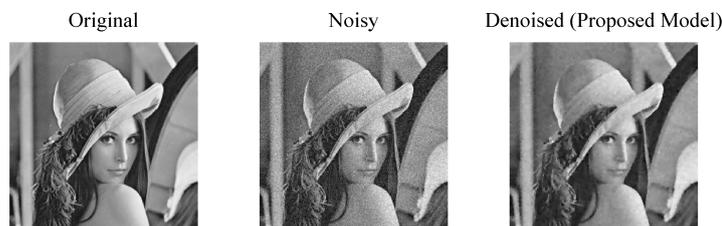


FIGURE 1. Denoising of Lena with Gaussian noise ( $\sigma_n = 15$ ).

It can be seen that the proposed method effectively removes noise while preserving edges and fine details, consistent with the quantitative improvements reported in Table 1.

TABLE 1. Peak Signal-to-Noise Ratio (PSNR)/Structural Similarity Index (SSIM) for Lena ( $\sigma_n = 15$ )

Method	PSNR (dB)	SSIM
PM	29.12	0.845
PM + Adaptive $\lambda$	30.06	0.872
Proposed (Coupled)	<b>31.03</b>	<b>0.891</b>

## 5. Discussion

The proposed model demonstrates clear improvements in both quantitative and visual metrics. The gradient-dependent fidelity term with parameter  $\mu$  allows strong noise suppression in homogeneous regions while preserving edges. Parameter selection was empirical:  $\lambda_0, \alpha$  control fidelity in flat regions,  $\mu$  prevents over-smoothing at edges, and  $\Delta t$  satisfies the CFL stability condition. Future studies could explore systematic parameter optimization using cross-validation or automated selection techniques.

Limitations include the current restriction to grayscale images and modest resolution. Extension to color and hyperspectral images is planned. Additionally, comparisons with deep learning-based denoising methods could provide further insight into relative performance.

## 6. Conclusion

We introduced a coupled gradient-weight and gradient-dependent fidelity PM model for image denoising. The method is interpretable, efficient, and maintains edge structures while achieving high PSNR and SSIM. Future work includes semi-implicit solvers, color/hyperspectral extension, and deeper comparisons with modern neural denoising approaches.

## References

1. P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 12, no. 7, pp. 629–639, 1990.
2. Q. Li, Adaptive PeronaMalik model based on dynamical threshold, *Computers & Mathematics with Applications*, vol. 137, 2023.
3. Z. Guo, J. Sun, D. Zhang, B. Wu, Adaptive PeronaMalik model based on the variable exponent for image denoising, *IEEE Trans. Image Process.*, 21(3):958–967, 2012.
4. S. Esedoglu, Stability Properties of the PeronaMalik Scheme, *SIAM J. Numer. Anal.*, 2004.
5. M. Vasilyeva, A. Krasnikov, K. Gajamannage, M. Mehrubeoglu, Multiscale method for image denoising using nonlinear diffusion process, *arXiv preprint arXiv:2409.15952*, 2024.

# Performance analysis of Mamdani and Takagi-Sugeno fuzzy systems for estimating the systolic blood pressure gradient in women using Persian medicine pulsology

Mohammad Dehghandar<sup>1\*</sup>, Hadi Shokohi Amiri<sup>2</sup>, Maliheh Rouhani Bidgoli<sup>3</sup>

<sup>1</sup>Assistant Professor, Department of Mathematics, Payame Noor University, PO Box 3697-19395, Tehran, Iran

<sup>2</sup> Department of Mathematics, Payame Noor University, Tehran, Iran.

<sup>3</sup> Mathematics Teacher, Alborz Province Education Department, District 4, Karaj

\* **Corresponding Authors:** Mohammad Dehghandar, Ph.D., Assistant Professor, Department of Mathematics, Payame Noor University, PO Box 3697-19395, Tehran, Iran, [m\\_dehghandar@pnu.ac.ir](mailto:m_dehghandar@pnu.ac.ir)

## Abstract

**Background:** The systolic blood pressure gradient (SBPG) has shown significant diagnostic and prognostic value in assessing overall cardiac function. Persian medicine pulsology (PM) offers a time-tested diagnostic framework; however, its integration with modern fuzzy-based SBPG analysis lacks systematic performance comparisons between inference architectures. **Objective:** This study presents a comparative evaluation of the performance of Mamdani and Takagi-Sugeno (TS) fuzzy inference systems for predicting systolic blood pressure gradient, considering diagnostic parameters from Persian Medicine (PM) pulsology. **Methods:** Both architectures were implemented in MATLAB R2021b and validated using clinical PM pulse data, including speed, frequency, strength, and length as input variables and SBPG as the output variable. Mamdani and TS fuzzy systems were used, and their performance was evaluated using MAE, RMSE, and MRE criteria. **Key Findings:** A comparative analysis revealed a significant accuracy advantage for the TS system (MAE = 0.1838, RMSE = 0.2495, MRE = 2.96%) over the Mamdani system (MAE = 0.2700, RMSE = 0.3337, MRE = 4.11%). **Innovative Contributions:** The study's primary contributions are threefold: (1) pioneering the use of PM pulsology in fuzzy-based systolic BP gradient forecasting, (2) establishing a clinically-validated and structurally stable framework for both models, and (3) open-sourcing the complete Mamdani and TS model implementations.

**Keywords:** performance, Fuzzy logic systems, systolic blood pressure gradient, Persian Medicine, Takagi-Sugeno

## 1. Introduction

The growing emphasis on intelligent systems in computational medicine has underscored their value in tackling intricate health issues, especially by bridging conventional diagnostics with cutting-edge technology. A notable example is Persian Medicine (PM) pulsology—an ancient diagnostic framework using



multidimensional pulse assessment—which is increasingly recognized for its ability to enhance modern cardiovascular evaluation. [1-3]. Hypertension, specifically high systolic blood pressure, represents a considerable health burden that diminishes life quality and commonly precipitates grave complications such as cerebral strokes and cardiac events. [4,5]. Due to its frequently asymptomatic and perilous nature, high blood pressure has earned the nickname "silent killer" in medical circles. [6]. Thus, the ongoing tracking of blood pressure is considered the foremost approach for managing the disease burden of hypertension. [7]. However, the traditional practice of PM pulsology faces significant challenges in standardization and training, as its finger-based diagnostic method is inherently subjective and its complex rule system is difficult to master consistently. These limitations have prompted the exploration of advanced computational tools, particularly fuzzy logic systems, which are uniquely suited to model the imprecise, multi-dimensional nature of pulse data. By providing a framework to quantify subjective assessments and formalize diagnostic reasoning, these intelligent systems are now enabling the development of standardized protocols that can effectively bridge traditional PM with modern evidence-based medical practice. [1, 3, 8]. Fuzzy logic is highly effective for creating medical knowledge-based systems, diagnosing diseases, and enabling real-time patient data monitoring. [2,9]. This research aimed to evaluate the usability of Persian medicine pulsology to estimate systolic blood pressure gradient (SBPG) with the help of Mamdani and Sugeno fuzzy systems design. Thus, fuzzy systems were designed to estimate the SBPG using selected pulse parameters of PM. The significant and innovative aspect of such a study is, first, to Application of Takagi-Sugeno fuzzy systems: Previously, only Mamdani fuzzy systems have been used in Persian medicine pulsology, and second, Guaranteed stability modeling: Previous works (e.g., [10]) lack formal stability analysis for PM with SBPG integration. This will help integrate the approved statements of centuries of traditional knowledge and clinical experience of PM physicians into conventional medicine, benefiting human health. Below, the concepts related to the fuzzy inference systems and the SBPG, as well as the pulse of PM, are reviewed, and then the method of design and analysis of the results of the fuzzy systems is stated.

## 2. Literature Review

### 2.1. Persian Medicine Pulseology in Modern Diagnostics

Persian Medicine's (PM) unique approach to pulse diagnosis is well-documented [11,12]. Initial

efforts to computationally align this traditional knowledge with modern methods were undertaken by Dehghandar et al. [8,10,12], focusing on blood pressure and photoplethysmogram analysis. Nevertheless, these pioneering implementations omitted crucial formal stability assessments and lacked comprehensive validation against established medical standards.

## 2.2. Comparative Studies of Fuzzy Architectures

The comparative performance of Mamdani and Takagi-Sugeno (TS) fuzzy systems has been studied in various fields. Pop et al. [13] found TS models more effective for real-time calibration in transportation, whereas Ouifak and Idri [14] emphasized the superior interpretability of Mamdani systems in medical diagnostics. However, a focused comparison within physiological signal analysis, especially for traditional medicine applications, remains absent. Given the established diagnostic relevance of systolic blood pressure gradient (SBPG) for cardiac function and its intrinsic relationship with PM pulse parameters [15], this study utilizes a fuzzy intelligent system to estimate SBPG from PM pulse parameters and conducts a comparative evaluation of both fuzzy architectures.

## 3. MATERIALS AND METHODS

The present study is a diagnostic that estimates the brachial systolic blood pressure gradient (SBPG) at the input centers. The statistical population is patients who were examined at the Iranian Medicine Clinic of Tehran University of Medical Sciences. There were 32 women who participated in this study. The questionnaire also contains information about pulse items such as: speed, frequency, strength, and length. It can also be used. Received. 4 Changes in speed, frequency, strength, length, and SBPG are considered as outputs.

Since there are considerations about the experience and analysis of pulse results by physicians and the education of Persian medicine students, which are associated with errors, and the inference from pulse rules is complex due to the large number of them, this problem can be managed using appropriate fuzzy systems. [8,16]. Thus, it is necessary to evaluate the performance quality of fuzzy systems and the amount of their errors to use more appropriate systems for each study. This evaluation implies the existence of a set of actual data, predicted data, and a performance evaluation system responsible for identifying the degree to which the predicted values are close to the actual values. In the following sections, the features of each of the systems discussed in this article will be described in detail.

Table 1 shows the input and output changes.

Table 1. Input and output data values for 32 healthy female volunteers

I/O	Variables	Max	Min	Mean	Standard deviation
I <sub>1</sub>	Pulse Speed	3	1	1.94	0.56
I <sub>2</sub>	Pulse Frequency	3	1	1.97	0.69
I <sub>3</sub>	Pulse Strength	3	2	2.19	0.40
I <sub>4</sub>	Pulse Length	2	0	1.03	0.31
O <sub>1</sub>	Systolic Blood Pressure Gradient	12.32	2.8	7.09	2.27

Now, the design steps of Mamdani and Sugino fuzzy systems with 4 input variables and one

output variable are described using the data in Table 1.

### 3.1. Mamdani fuzzy system

As mentioned in [10], based on 32 healthy female volunteers data, this Mamdani fuzzy system also obtained fuzzy rules related to fuzzy sets and membership functions. The Mamdani fuzzy system was designed with product inference engine, singleton fuzzifier, centroid defuzzifier, and triangular membership functions using MATLAB R2021b software.

### 3.2. Takagi–Sugeno fuzzy system

The structure of the Takagi-Sugeno fuzzy system is such that after receiving the inputs, it generates a first-order polynomial from the input variables. In this system, to approximate the function  $f$ , a set of fuzzy if-then rules of the Takagi-Sugeno (TS) type is used for  $m$  vectors consisting of  $n$  inputs and one output. Therefore, if the output of a fuzzy system is a combination of inputs, it is called a TS system, and according to Equation (1), its rules are as follows [2,14]:

Rule: IF  $x_1$  is  $A_1^L$  and  $x_2$  is  $A_2^L$  and, ...,  $x_n$  is  $A_n^L$  then  $y^L = \alpha_0^L + \alpha_1^L x_1 + \dots + \alpha_n^L x_n$  (1)

wherein  $L = \{1, 2, \dots, r\}$ ,  $\alpha^L = \{\alpha_0^L, \alpha_1^L, \alpha_2^L, \dots, \alpha_n^L\}$

And the fuzzy sets are expressed as:

$$A_i = \{A_i^1, A_i^2, \dots, A_i^L\}$$

And if  $x = (x_1, x_2, \dots, x_n)^T$  then for the number of rules  $r$ , the weighted average of the output will be as follows:

$$f(x) = \sum_{L=1}^r y^L \bar{w}^L \quad (2)$$

wherein  $w^L = \prod_{i=1}^n \mu_{A_i^L}(x_i)$ ,  $\bar{w}^L = \frac{w^L}{\sum_{L=1}^r w^L}$ .

If the fuzzy sets are in the interval  $[-\alpha_i, \beta_i]$  and as a Gaussian membership function, then each  $x_i \in [-\alpha_i, \beta_i]$  of the domains are defined as  $A_i$  existing in Equation (2) and the degree of the membership function is non-zero, that is  $\mu_i^L(x_i) \neq 0$ . Equation (3) represents the Gaussian membership functions of any fuzzy set  $A^L$ ,  $L \in \{1, 2, \dots, r\}$  [2]:

$$\mu_{A_i}(x_i) = e^{-\frac{1}{2} \left( \frac{x_i - m_i}{\sigma_i} \right)^2} \quad (3)$$

Where  $\sigma_i$  and  $m_i$  are the variances and centers of the adjustable and the set of parameters of the antecedent section, respectively. The design of the Takagi-Sugeno fuzzy system is explained below.

Using data from 32 healthy female volunteers, a TS fuzzy system was designed using MATLAB R2021b software. First, fuzzy sets related to pulse parameters in the corresponding intervals, and fuzzy sets related to the systolic blood pressure gradient (SBPG) in the interval [2.8, 12.32] were defined. Subsequently, information on pulse parameters was considered as input variables, and the SBPG was considered as the output variable. Then, with the information of 32 healthy female volunteers TS fuzzy system was designed

### 3.3. Performance Evaluation

Performance analysis considers the error of Mamdani and Takagi-Sugeno (TS) fuzzy systems as the main evaluation method; this can also be observed based on the visual analysis of the

obtained results. To better evaluate performance and investigate prediction accuracy, this study used the evaluation indices of mean absolute error (MAE), root mean square error (RMSE), and Mean Relative Error (MRE), which can be calculated using Equations (3), (4), and (5), respectively. These metrics are calculated as follows [17,18]:

$$MAE = \frac{1}{N} \sum_{i=1}^N |SBPG_O^i - SBPG_{FIS}^i| \quad (3)$$

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^N (SBPG_O^i - SBPG_{FIS}^i)^2} \quad (4)$$

$$MRE = \frac{1}{N} \sum_{i=1}^N \frac{|SBPG_O^i - SBPG_{FIS}^i|}{|SBPG_O^i|} \quad (5)$$

where  $SBPG_O^i$  and  $SBPG_{FIS}^i$  are the  $i$ -th recordings of observed and predicted values from a total of  $N$  recorded data. The calculation of these metrics for the systolic blood pressure gradient  $SBPG_O^i$  corresponding to observed values and for  $SBPG_{FIS}^i$  corresponding to predicted values is available in Section 5.

#### 4. Results

The relative error calculation results in Figure 1 also show that both the Mamdani and TS fuzzy recognition systems performed successfully in predicting the values of the systolic blood pressure gradient (SBPG) for the 32 healthy female volunteers used in this study. The TS fuzzy inference system performs with a much lower relative error in predicting the SBPG value than the Mamdani fuzzy inference system, and these relative error values in the TS system are much lower than those in the Mamdani system in most volunteers, except in a few cases.

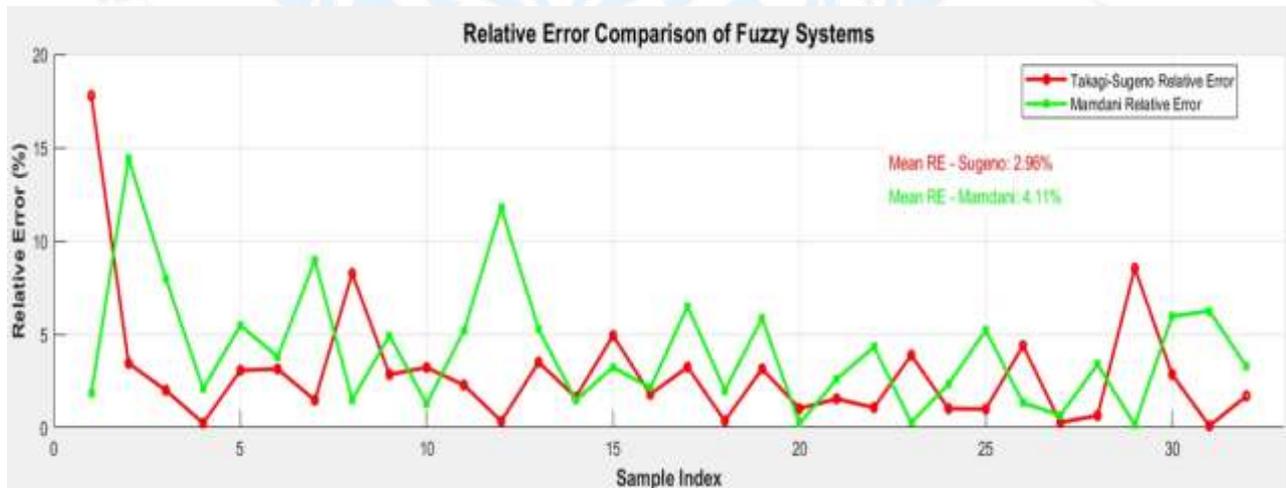


Figure 1. Comparison of relative errors of Mamdani and Takagi-Sugeno systems (%) for predicting the systolic blood pressure gradient

Visual analysis shows that although the Mamdani fuzzy system has better predictions for the SBPG of some volunteers, the Takagi-Sugeno fuzzy system provides better predictions for the SBPG and performs more accurately for most volunteers. A full review of these predictions and their results is presented in Section 5.

## 5. Discussion

The simulation results demonstrate that both the Mamdani and Takagi-Sugeno (TS) fuzzy inference systems (FISs) successfully estimated systolic blood pressure gradient (SBPG) from pulse speed, pulse frequency, pulse strength, and pulse length parameters in Persian medicine (PM), as evidenced by the high accuracy metrics (Section 4) and stability conditions (Section 3). The graph of predicted values of the SBPG for 32 healthy female volunteers by Mamdani and Takagi-Sugeno fuzzy systems can be seen in Figure 2.

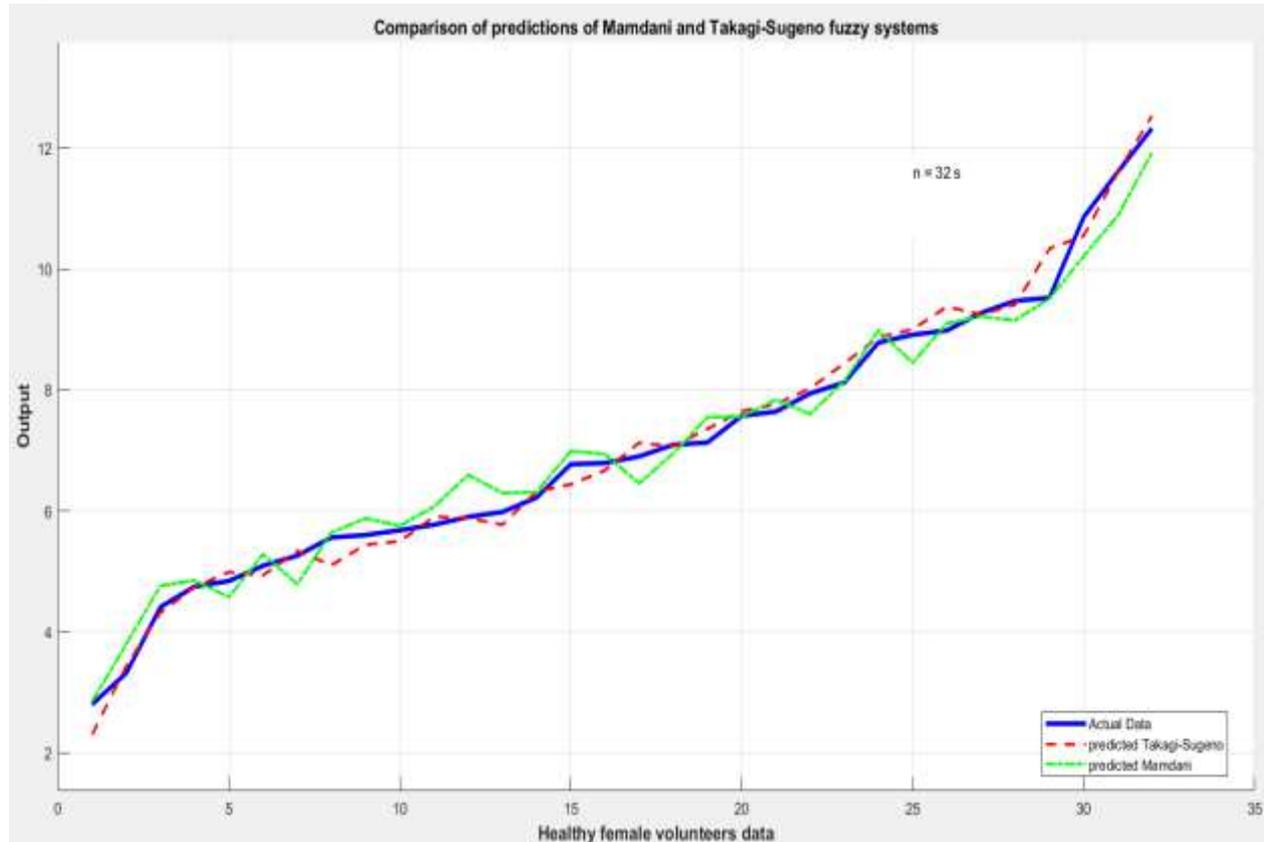


Figure 2. Comparison of predictions of Mamdani and Takagi-Sugeno systems for the systolic blood pressure gradient

However, key differences emerged in their performance and applicability; the existence of errors and simulated values for predicting the systolic blood pressure gradient (SBPG) has been explained by the existing literature. Mamdani fuzzy inference system identifies fuzzy output values by aggregating several fuzzy rule outputs [12,19,20], like the Centroid technique applied in this study, while each fuzzy rule in the case of the TS fuzzy inference system [2,13,14] adds a weighted linear function to the overall output. Table 2 shows the calculated values for the performance measures used in this research (i.e., MAE, RMSE, Mean Relative Error (MRE)) based on the error obtained to verify the applicability of Mamdani and TS fuzzy recognition systems in predicting the SBPG.

Table 1. Monitoring data prediction errors for SBPG between Mamdani and Takagi-Sugeno systems

fuzzy inference system	MAE	RMSE	MRE
Mamdani	0.2700	0.3337	4.11%
Takagi-Sugeno	0.1838	0.2495	2.96%

### 5.1. Performance Analysis

The results show that in this study, the Mamdani system outperformed the TS system in all key metrics: Lower Errors: TS achieved significantly better MAE (0.1838 vs. 0.2700) and RMSE (0.2495 vs. 0.3337), indicating higher precision in predicting SBPG from pulse parameters. Superior Explanatory Power: The Mean Relative Error (MRE) value (2.96%) for TS suggests its rules align almost perfectly with observed physiological relationships between pulse parameters and SBPG.

### 5.2. Interpretability and Clinical Utility

The Mamdani system (e.g., Gaussian membership functions) allows physicians to trace how input pulse variables map to SBPG estimates. This aligns with PM's emphasis on interpretable diagnostics. In contrast, the clear linear outputs of the TS system, although they obscure the underlying logic, are computationally efficient, making it more suitable for clinical validation.

### 5.3. Limitations

Data Scope: Results are based on 32 healthy female volunteers; performance on pathological cases (e.g., cardiovascular diseases) requires validation. Computational Load: Mamdani's defuzzification is slower than TS's weighted averaging, though this is negligible for offline clinical analysis.

### Future Directions

Hybrid Systems: Combine Mamdani's interpretability with TS's efficiency for real-time monitoring (e.g., wearable PPG devices). Pathological Validation: Test the system on patients with vascular disorders to assess generalizability.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Acknowledgements

Here, we would like to express our gratitude and appreciation for the cooperation and assistance of Dr. Mehdi Alizadeh Vaghasloo, Assistant Professor of Persian Medicine at Tehran University of Medical Sciences. We also sincerely thank the cooperation of the Ahmadiyah Persian Medicine Clinic at Tehran University of Medical Sciences.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### REFERENCES

- [1]. [Alizadeh Vaghasloo M, Keshavarz M, Ebadiani M, Nazem E, Isfahani MM. Complexity and rationality of Avicenna's pulsology: a step towards understanding the past for today's applications. Int J Cardiol. 2012;157\(3\):434-435.](#)
- [2]. [Dehghandar M, Rezvani S. Classification of COVID-19 individuals using adaptive neuro-fuzzy inference system. J Med Signals Sens. 2022;12\(4\):334-340.](#)
- [3]. [Dehghandar M. Investigating the effect of pulse traditional medicine on the factors of blood pressure by fuzzy calculations \[Dissertation\]. Payame Noor University; 2016.](#)

- [4]. [De Oliveira HS, Pinto RA, Souto EJP, Giusti R. Estimation of Systolic and Diastolic Blood Pressure for Hypertension Identification from Photoplethysmography Signals. Applied Sciences. 2024; 14\(6\):2470. <https://doi.org/10.3390/app14062470>](#)
- [5]. [Athaya, T.; Choi, S. An Estimation Method of Continuous Non-Invasive Arterial Blood Pressure Waveform Using Photoplethysmography: A U-Net Architecture-Based Approach. Sensors 2021, 21, 1867.](#)
- [6]. [Campbell, N.R.; Burnens, M.P.; Whelton, P.K.; Angell, S.Y.; Jaffe, M.G.; Cohn, J.; Brito, A.E.; Irazola, V.; Brettler, J.W.; Roccella, E.J.; et al. 2021 World Health Organization guideline on pharmacological treatment of hypertension: Policy implications for the region of the Americas. Lancet Reg. Health 2022, 9, 1–10](#)
- [7]. [Elgendi, M.; Fletcher, R.; Liang, Y.; Howard, N.; Lovell, N.H.; Abbott, D.; Lim, K.; Ward, R. The use of photoplethysmography for assessing hypertension. Digit. Med. 2019, 2, 60.](#)
- [8]. [Dehghandar M, Mirhosseini-Alizamini M, Alizadeh Vaghasloo M, Khosravi Najaf Abadi A. Design and Implementation of a Fuzzy Intelligent System to Estimate the Photoplethysmogram Systolic Features by Persian Medicine Pulsology. Health Man & Info Sci 2024 Vol 10, No. 4, pp.226-234.](#)
- [9]. [Ramzan M, Hamid M, Alhussan AA, AlEisa HN, Abdallah HA. Accurate prediction of anxiety levels in Asian countries using a fuzzy expert system. Healthcare. 2023;11\(11\):1594.](#)
- [10]. [Dehghandar M, Alizadeh Vaghasloo M, Moradi B. Estimation of Men's brachial blood pressure gradient using fuzzy system by pulse parameters in Persian medicine. 4th National Seminar on control and optimization, Behshahr,2022.](#)
- [11]. [Alizadeh Vaghasloo M, Naghizadeh A, Keshavarz M. The Concept of Pulse. Traditional and integrative medicine. Trad Integr Med. 2017; Vol. 2, No. 2, pp. 54-60.](#)
- [12]. [Dehghandar M, Nikbakht M, Yousefi Nadar Abadi F. Designing a Fuzzy Stable System to Estimate Peripheral Resistance Using Frequency and Pulse Strength Parameters of Persian Medicine. Journal of Health and Biomedical Informatics 2025; 11\(4\): 291-305. \[In Persian\]](#)
- [13]. [Pop, M.-D.; Pescara, D'Marco, M.V. Mamdani vs. Takagi–Sugeno Fuzzy Inference Systems in the Calibration of Continuous-Time Car-Following Models. Sensors 2023, 23, 8791. <https://doi.org/10.3390/s23218791>](#)
- [14]. [Ouifak H, Idri A. On the performance and interpretability of Mamdani and Takagi-Sugeno-Kang based neuro-fuzzy systems for medical diagnosis. Scientific African 2023, Vol 20, <https://doi.org/10.1016/j.sciaf.2023.e01610>](#)
- [15]. [Ghods R, Gharooni M, Amin G, Nazem E, Nikbakht Nasrabadi A. Hypertension from the perspective of Iranian traditional medicine. Iran Red Crescent Med J. 2014 Mar;16\(3\):e16449. doi: 10.5812/ircmj.16449. Epub 2014 Mar 5. PMID: 24829789; PMCID: PMC4005451.](#)
- [16]. [Dehghandar M, Khaloozadeh H, Soltanian F, Keshavarz M. Application of Fuzzy logic to determine the retentive causes of pulse body by the pulse parameters in Iranian Traditional Medicine. Journal of Multidisciplinary Engineering Science and Technology \(JMEST\). 2016; Vol. 3, No. 2, pp 3881-3884.](#)
- [17]. [Pop, M.-D.; Pescara, D'Marco, M.V. Mamdani vs. Takagi–Sugeno Fuzzy Inference Systems in the Calibration of Continuous-Time Car-Following Models. Sensors 2023, 23, 8791. <https://doi.org/10.3390/s23218791>](#)



- [18]. [Xiao, X.; Wang, Z.; Zhang.; Luo, Y.; Chen, F.; Deng, Y.; Lu, N.; Chen, Y. A Novel Method of Bridge Deflection Prediction Using Probabilistic Deep Learning and Measured Data. Sensors 2024, 24, 6863. <https://doi.org/10.3390/s24216863>](#)
- [19]. [Wang L. A Course in Fuzzy Systems and Control. NJ United States, Prentice-Hall, Inc. Division of Simon and Schuster, One Lake Street, Upper Saddle River. 1996, ISBN:978-0-13-540882-7.](#)
- [20]. [Dehghandar M, Pabasteh M, Heydari R. Diagnosis of COVID-19 disease by fuzzy expert system designed based on input-output. J Control. 2021;14\(5\):71-78.](#)





## New perspectives on differential and integral calculus

Mohammad Molaie<sup>1,\*</sup>, Mohammad Jafari<sup>2</sup>

<sup>1,2</sup>Department of Science, Payame Noor University, Tehran, Iran.

Email: m.mola@pnu.ac.ir

m.jafari@pnu.ac.ir

**ABSTRACT.** In this paper, we aim to solve a series of ordinary and fractional differential equations with a new perspective, using the fractional Laplace transform of order  $\alpha + \beta$ . In this method, the second-order differential equation can be solved without the need for initial conditions, and its general solution can be obtained.

**Keywords:** conformable derivative, Laplace transform of order  $\alpha + \beta$ , initial condition

**AMS Mathematics Subject Classification [2020]:** 34A08, 44A10, 26A33

### 1. Introduction

Fractional differential equations have gained significant attention from researchers in the past decade. The primary reason for this is that previous fractional derivatives often do not align with classical derivatives in many cases. However, since calculus is fundamentally based on classical derivatives, it is crucial for researchers to focus on developing a new definition of fractional derivatives that is compatible with classical derivatives. To address this issue, in 2014, Khalil et al. [2] introduced a new definition of compatible derivatives in a paper, which is presented as follows.

**DEFINITION 1.1.** The conformable fractional derivative of order  $0 < \alpha \leq 1$  starting from  $a \in \mathbb{R}$  the function  $y : [a, +\infty) \rightarrow \mathbb{R}$ , is defined by

$$(1) \quad ({}_t T_\alpha^a y)(t) = \lim_{\epsilon \rightarrow 0} \frac{y(t + \epsilon(t-a)^{1-\alpha}) - y(t)}{\epsilon}.$$

By choosing  $a = 0$ , the following result can be easily obtained through this definition.

$$(2) \quad {}_t T_\beta [{}_t T_\alpha y(t)] = (1 - \alpha)t^{1-(\alpha+\beta)} y'(t) + t^{2-(\alpha+\beta)} y''(t).$$

**DEFINITION 1.2.** [1]. The conformable fractional Laplace transform (CFLT) of function  $y : [0, \infty) \rightarrow \mathbb{R}$  for  $t > 0$ , of order  $0 < \alpha \leq 1$ , starting from  $a$  of  $u$  is defined by

$$(3) \quad L_\alpha^a \{y(t)\} = \int_a^\infty e^{-s \frac{(t-a)^\alpha}{\alpha}} y(t) (t-a)^{\alpha-1} dt = Y_\alpha^a(s).$$

\*Mohammad Molaie.

The following results can be easily proved through the above definitions. See [5] and [6].

$$(4) \quad L_{\alpha}^0 \{t^{n\alpha} y(t)\} = (-1)^n \alpha^n \frac{d^n}{ds^n} [Y_{\alpha}^0(s)], \quad n \in \mathbb{N}.$$

$$(5) \quad L_{(\alpha+\beta)}^0 \left\{ {}_t T_{\beta} \left( {}_t T_{\alpha} y(t) \right) \right\} = \alpha y(0) - (2\alpha + \beta) s Y_{(\alpha+\beta)}(s) - (\alpha + \beta) s^2 Y'_{(\alpha+\beta)}(s).$$

$$(6) \quad L_{\alpha+\beta}^0 \left\{ t^{1-(\alpha+\beta)} y'(t) \right\} = L_{\alpha+\beta}^0 \{T_{\alpha+\beta} y(t)\} = -y(0) + s Y_{\alpha+\beta}(s).$$

$$(7) \quad L_{\alpha+\beta}^0 \left\{ t^{2-(\alpha+\beta)} y''(t) \right\} = (-1 - (\alpha + \beta)) s Y_{\alpha+\beta}(s) + y(0) - (\alpha + \beta) s^2 Y'_{\alpha+\beta}(s).$$

$$(8) \quad L_{(\alpha+\beta)}^0 \left\{ {}_t T_{\beta} \left( {}_t T_{\alpha} y(t) \right) + {}_t T_{\alpha} \left( {}_t T_{\beta} y(t) \right) \right\} \\ = (\alpha + \beta) y(0) - (3\alpha + 3\beta) s Y_{(\alpha+\beta)}(s) - (2\alpha + 2\beta) s^2 Y'_{(\alpha+\beta)}(s).$$

## 2. results and applications

In this section, we use the fractional Laplace transform to solve some second-order ODEs.

PROPOSITION 2.1. *The solution of the second-order ODE*

$$(9) \quad 2ty''(t) + (t+1)y'(t) + y(t) = q(t)$$

is given by

$$y(t) = L^{-1} \left\{ \frac{1}{(2s+1)^{\frac{3}{2}}} \left( \int \frac{y(0) - Q(s)}{s} (2s+1)^{\frac{1}{2}} ds + C \right) \right\}.$$

PROOF. Equation (9) can be rewritten as

$$\frac{2}{3}y'(t) + ty''(t) + \frac{1}{3}y'(t) + ty''(t) + ty'(t) + y(t) = q(t),$$

or equivalently,

$$\left(1 - \frac{1}{3}\right)t^{1-(\frac{1}{3}+\frac{2}{3})}y'(t) + t^{2-(\frac{1}{3}+\frac{2}{3})}y''(t) + \left(1 - \frac{2}{3}\right)t^{1-(\frac{2}{3}+\frac{1}{3})}y'(t) + t^{2-(\frac{2}{3}+\frac{1}{3})}y''(t) + ty'(t) + y(t) = q(t).$$

Using 2 we can write

$${}_t T_{\frac{2}{3}} \left( {}_t T_{\frac{1}{3}} y(t) \right) + {}_t T_{\frac{1}{3}} \left( {}_t T_{\frac{2}{3}} y(t) \right) + 1ty'(t) + 1y(t) = q(t).$$

Now, 8 gives us

$$y(t) = (L_1^0)^{-1} \left\{ \frac{1}{(2s+1)^{\frac{3}{2}}} \left( \int \frac{y(0) - Q_1(s)}{s} (2s+1)^{\frac{1}{2}} ds + C \right) \right\} \\ = L^{-1} \left\{ \frac{1}{(2s+1)^{\frac{3}{2}}} \left( \int \frac{y(0) - Q(s)}{s} (2s+1)^{\frac{1}{2}} ds + C \right) \right\}.$$

□

For example, the solution of the differential equation

EXAMPLE 2.2.

$$2ty''(t) + (t + 1)y'(t) + y(t) = te^{-t}, \quad y(0) = 1$$

is given by

$$\begin{aligned} y(t) &= (L_1^0)^{-1} \left\{ \frac{1}{(2s+1)^{\frac{3}{2}}} \left( \int \frac{1 - \frac{1}{(s+1)^2}}{s} (2s+1)^{\frac{1}{2}} ds + C \right) \right\} \\ &= L^{-1} \left\{ \frac{1}{(2s+1)^{\frac{3}{2}}} \left( \frac{(2s+1)^{\frac{3}{2}}}{s+1} + C \right) \right\} \\ &= L^{-1} \left\{ \frac{1}{s+1} \right\} + L^{-1} \left\{ \frac{C}{(2s+1)^{\frac{3}{2}}} \right\} \\ &= e^{-t} + C \left( \frac{\sqrt{2}}{2} \sqrt{\frac{t}{\pi}} e^{-\frac{1}{2}t} \right). \end{aligned}$$

### 3. Conclusion

By utilizing conformable differential equations, many second-order non-homogeneous differential equations can be solved analytically, and the results can be applied in numerical methods.

### References

1. Abdeljawad, T. (2015) *On conformable fractional calculus*, Journal of computational and Applied Mathematics, **279**, 57–66.
2. Roshdi, Kh. Mohammed, A.H. Abdelrahman Y. and Mohammad, S. (2014) *A new definition of fractional derivative*, Journal of computational and applied mathematics, **264**, 65–70.
3. Hashemi, MS. (2018) *Invariant subspaces admitted by fractional differential equations with conformable derivatives*, Chaos, Solitons & Fractals, **107**, 161–169.
4. Mingarelli, AB. (2018) *On generalized and fractional derivatives and their applications to classical mechanics*, Journal of Physics A: Mathematical and Theoretical, **51**, no. 36 365204.
5. Molaei, M. Dastmalchi Saei, F. Javidi, M. and Mahmoudi, Y. (2022) *New analytical methods for solving a class of conformable fractional differential equations by fractional Laplace transform*, Computational Methods for Differential Equations, **10**, 396–407.
6. Molaei, M. Dastmalchi Saei, F. Javidi, M. and Mahmoudi, Y. (2022) *Solving a class of ordinary differential equations and fractional differential equations with conformable derivative by fractional Laplace transform*, Turkish Journal of Mathematics, **46**, 3025–3044.
7. Molaei, M. Dastmalchi Saei, F. Javidi, M. and Mahmoudi, Y. (2024) *FRACTIONAL LAPLACE TRANSFORM TO SOLVE CONFORMABLE DIFFERENTIAL EQUATIONS*, Facta Universitatis, Series: Mathematics and Informatics, ., 621–642.



## Discrete-time SIV epidemic model with constant recruitment rate and population size

Mahmood Parsamanesh<sup>1,\*</sup>

<sup>1</sup>Department of Basic Sciences, Technical and Vocational University (TVU), Tehran, Iran.

Email: mparsamanesh@tvu.ac.ir

**ABSTRACT.** A discrete-time epidemic model with compartments as Susceptible, Infectious, and Vaccinated, namely SIV model, is studied. After obtaining some properties of this model such as the boundedness of solutions and equilibria of the model, the basic reproduction number  $\mathcal{R}_0$  is given. Then stability and bifurcations of the model is given in terms of  $\mathcal{R}_0$ .

**Keywords:** Discrete model, Jury conditions, Vaccination, Bifurcation

**AMS Mathematics Subject Classification [2020]:** 92D25, 39A33

### 1. Introduction

Discrete-time models have some advantages in comparison to the continuous-time models and have also gotten attention in mathematical epidemiology. Collecting epidemic data in separate time intervals, using discretization by numerical methods for solving differential equations, and presenting more complex dynamics than continuous models (such as bifurcations and chaos), encourage researchers to use discrete-time modeling. [3, 5] Let  $N_n$  represent the total population size at the  $n$ -th time interval. The population is divided into three sub-populations as susceptible, infectious, and vaccinated which are denoted by  $S_n$ ,  $I_n$ , and  $V_n$  at the same time, respectively. We consider the following model for the interaction between the compartments:

$$\begin{aligned} S_{n+1} &= (1-q)\mu N_n - \beta S_n I_n / N_n + [1 - (\mu + p)]S_n + \gamma I_n + \epsilon V_n, \\ I_{n+1} &= \beta S_n I_n / N_n + [1 - (\mu + \gamma)]I_n, \\ V_{n+1} &= q\mu N_n + pS_n + [1 - (\mu + \epsilon)]V_n. \end{aligned}$$

By adding the equations of the above system we see  $N_{n+1} = N_n$  and then the population size will remain a constant value  $N$ . Thus letting  $V_n = N - S_n - I_n$ , the corresponding difference equation is deleted and the following system of two difference equations is obtained:

$$(1) \quad \begin{aligned} S_{n+1} &= [(1-q)\mu + \epsilon]N - \beta S_n I_n / N + [1 - (\mu + p + \epsilon)]S_n + (\gamma - \epsilon)I_n, \\ I_{n+1} &= \beta S_n I_n / N + [1 - (\mu + \gamma)]I_n, \end{aligned}$$

\*Speaker.

The conditions for non-negativity of the solutions of system (1) are as the follow:

$$(2) \quad \begin{aligned} \mu + p + \epsilon + \beta &< 1, \\ \mu + \gamma &< 1. \end{aligned}$$

The equilibria of the model are solutions of the following system:

$$\begin{aligned} \bar{I} \left[ \beta \bar{S} / N - (\mu + \gamma) \right] &= 0, \\ [(1 - q)\mu + \epsilon]N - \beta \bar{S} \bar{I} / N - (\mu + p + \epsilon)\bar{S} + (\gamma - \epsilon)\bar{I} &= 0. \end{aligned}$$

When  $\bar{I} = 0$ , the disease-free equilibrium is obtained as

$$Q^0 = (I^0, S^0) = \left( 0, \frac{[(1 - q)\mu + \epsilon]N}{\mu + p + \epsilon} \right),$$

while if  $I \neq 0$ , we obtain the endemic equilibrium is obtained as

$$Q^* = (I^*, S^*) = \left( \frac{[(1 - q)\mu + \epsilon]\beta N - (\mu + p + \epsilon)(\mu + \gamma)N}{\beta(\mu + \epsilon)}, \frac{(\mu + \gamma)N}{\beta} \right).$$

Notice that  $I^* > 0$  if and only if  $[(1 - q)\mu + \epsilon]\beta - (\mu + p + \epsilon)(\mu + \gamma) > 0$  if and only if

$$\mathcal{R}_0 = \frac{\beta[(1 - q)\mu + \epsilon]}{(\mu + p + \epsilon)(\mu + \gamma)} > 1.$$

The quantity  $\mathcal{R}_0$  is referred to as the basic reproduction number of the model (1). Also we have

$$I^* = \frac{(\mu + p + \epsilon)(\mu + \gamma)N}{(\mu + \epsilon)\beta} (\mathcal{R}_0 - 1),$$

LEMMA 1.1. *For SIV epidemic model (1) the disease-free equilibrium  $Q^0$  always exists and the endemic equilibrium  $Q^*$  also exists if  $\mathcal{R}_0 > 1$ .*

## 2. Stability of the equilibria

THEOREM 2.1. *The disease-free equilibrium is stable if and only if  $\mathcal{R}_0 < 1$ .*

The Jacobian matrix at  $Q^0$  is given by

$$J(Q^0) = \begin{pmatrix} 1 - (\mu + \gamma) + (\mu + \gamma)\mathcal{R}_0 & 0 \\ (\mu + \gamma)\mathcal{R}_0 + (\gamma - \epsilon) & 1 - (\mu + p + \epsilon) \end{pmatrix}.$$

The eigenvalues of  $J(Q^0)$  are  $\lambda_1 = 1 - (\mu + \gamma) + (\mu + \gamma)\mathcal{R}_0$  and  $\lambda_2 = 1 - (\mu + p + \epsilon)$ . obviously,  $|\lambda_2| < 1$  by positivity assumptions on parameters and  $|\lambda_1| < 1$  if and only if  $\mathcal{R}_0 < 1$ .

THEOREM 2.2. *When  $\mathcal{R}_0 > 1$  the endemic equilibrium  $Q^*$  is stable and otherwise is unstable.*

At  $Q^*$  we have  $\beta S^*/N = (\mu + \gamma)$  and so

$$J^* = J(Q^*) = \begin{pmatrix} 1 & \beta I^*/N \\ -(\mu - \epsilon) & 1 - (\mu + p + \epsilon) - \beta I^*/N \end{pmatrix}.$$

Thus we get

$$\begin{aligned} tr(J^*) &= 2 - (\mu + p + \epsilon) - \beta I^*/N, \\ det(J^*) &= 1 - (\mu + p + \epsilon) - \beta I^*/N + (\mu + \epsilon)\beta I^*/N, \end{aligned}$$

and by assuming

$$\begin{aligned} b_1 &= (\mu + p + \epsilon) + \beta I^*/N, \\ b_2 &= (\mu + \epsilon)\beta I^*/N, \end{aligned}$$

we can rewrite them as

$$\begin{aligned} \text{tr}(J^*) &= 2 - b_1, \\ \det(J^*) &= 1 - b_1 + b_2. \end{aligned}$$

The characteristic equation of  $J^*$  is of the form  $P(\lambda) = \lambda^2 - \text{tr}(J^*)\lambda + \det(J^*)$  and according to the Jury conditions all eigenvalues of  $J^*$  are from module less than one if and only if [2]

$$(3) \quad |\text{tr}(J^*)| < 1 + \det(J^*) < 2.$$

Firstly,  $1 + \det(J^*) < 2$  holds if and only if  $-b_1 + b_2 < 0$ . Besides,  $\beta I^*/N > (\mu + \epsilon)\beta I^*/N$  and so  $(\mu + p + \epsilon) + \beta I^*/N > (\mu + \epsilon)\beta I^*/N$ , that is  $b_1 > b_2$  and thus condition  $1 + \det(J^*) < 2$  holds.

Secondly, if  $\text{tr}(J^*) > 0$  we must show that  $\text{tr}(J^*) < 1 + \det(J^*)$  which holds since it is equivalent to  $b_2 > 0$ . If  $\text{tr}(J^*) < 0$  we have to prove  $-\text{tr}(J^*) < 1 + \det(J^*)$  which holds if and only if  $4 - 2b_1 + b_2 > 0$ . But we have

$$\begin{aligned} 4 - 2b_1 + b_2 &= 4 + (\mu + \epsilon)\beta I^*/N - 2[(\mu + p + \epsilon) + \beta I^*/N] \\ &> 2 + (\mu + \epsilon)\beta I^*/N - 2\beta I^*/N \\ &> (\mu + \epsilon)\beta I^*/N > 0, \end{aligned}$$

since  $\mu + p + \epsilon < 1$  and  $\beta I^*/N < 1$ . Therefore, when  $\mathcal{R}_0 > 1$  the Jury conditions are satisfied and the theorem is proven:

### 3. Bifurcations of the model

In a discrete-time system bifurcations occur at the equilibria of the under study system when there exist some eigenvalues of Jacobian matrix with module one [1, 4].

**THEOREM 3.1.** *At disease-free equilibrium  $Q^0$  of SIVS epidemic model (1), transcritical bifurcations happens if  $\mathcal{R}_0 = 1$  while period-doubling bifurcation and Neimark-Sacker bifurcation don't take place.*

As we saw, the eigenvalues of  $J(Q^0)$  are  $\lambda_1 = 1 - (\mu + \gamma) + (\mu + \gamma)\mathcal{R}_0$  and  $\lambda_2 = 1 - (\mu + p + \epsilon)$ .  $\lambda_1 = 1$  if and only if  $\mathcal{R}_0 = 1$  and thus transcritical bifurcation occurs at  $Q^0$  when  $\mathcal{R}_0 = 1$ . On the other hand,  $\lambda_1 = -1$  if and only if  $\mathcal{R}_0 = 1 - \frac{2}{\mu + \gamma}$ . But this is impossible because  $\mu + \gamma < 1$  and  $\mathcal{R}_0$  becomes a negative value. This shows that period-doubling bifurcation doesn't occur at  $Q^0$ . In addition, the eigenvalues of the  $J(Q^0)$  are both real and therefore Neimark-Sacker bifurcation doesn't take place, too.

**THEOREM 3.2.** *At endemic equilibrium  $Q^*$  for SIVS model (1), transcritical bifurcation happens if  $\mathcal{R}_0 = 1$ , while period-doubling bifurcation and Neimark-Sacker bifurcation don't occur.*

$\lambda = 1$  is an eigenvalue of Jacobian matrix  $J(Q^*)$  if it is a root of the corresponding characteristic equation,  $1 - \text{tr}(J^*) + \det(J^*) = 0$ . This holds if and only if  $b_2 = 0$  if and only if  $\beta I^*/N = 0$  if and only if  $\mathcal{R}_0 = 1$ , since

$$\beta I^*/N = \frac{(\mu + p + \epsilon)(\mu + \gamma)}{(\mu + \epsilon)}(\mathcal{R}_0 - 1).$$

However,  $\lambda = -1$  is an eigenvalue of  $J(Q^*)$  if  $P(-1) = 0$ . This is satisfied if and only if  $4 - 2b_1 + b_2 = 0$  that can be written as

$$4 - 2(\mu + p + \epsilon) - \beta I^*/N[2 - (\mu + \epsilon)] = 0,$$

or equivalently

$$(4) \quad 2[2 - (\mu + p + \epsilon)] - \beta I^*/N[2 - (\mu + \epsilon)] = 0.$$

Now, notice that as we concluded previously,  $P(-1) > 0$  when  $\mathcal{R}_0 > 1$ . Also,  $\mathcal{R}_0 = 1$  implies  $\beta I^*/N = 0$  and this results in  $2 - (\mu + p + \epsilon) = 0$  which is impossible. These discussions state that period-doubling bifurcation doesn't happen at  $Q^*$ .

If we write the characteristic equation of  $J^*$  as  $P(\lambda) = \lambda^2 + a_1\lambda + a_2$ , we see that

$$\begin{aligned} a_1^2 - 4a_2 &= (-2 + b_1)^2 - 4(1 - b_1 + b_2) \\ &= b_1^2 - 4b_2 \\ &= (\mu + p + \epsilon)^2 + 2(\mu + p + \epsilon)\beta I^*/N + (\beta I^*/N)^2 - 4(\mu + \epsilon)\beta I^*/N \\ &> (\mu + p + \epsilon)^2 - 2(\mu + p + \epsilon)\beta I^*/N + (\beta I^*/N)^2 \\ &= [(\mu + p + \epsilon) + \beta I^*/N]^2 > 0. \end{aligned}$$

Hence, the roots of  $P(\lambda)$  are both real and thus Neimark-Sacker bifurcation can not be appeared at  $Q^*$ .

#### 4. Summary

In this paper, we introduced and studied an SIS epidemic model includes a vaccination program. The equilibria of the model were detected: The disease-free equilibrium  $Q^0$  in which the infection will be extinct, and the endemic equilibrium  $Q^*$  in which the disease will persist in population. It was proven that  $Q^0$  and  $Q^*$  are stable if  $\mathcal{R}_0 < 1$  and  $\mathcal{R}_0 > 1$ , respectively. Furthermore, the bifurcations of the model were investigated and was proven that when  $\mathcal{R}_0 = 1$  system has transcritical bifurcation while the period-doubling bifurcation and Neimark-Sacker bifurcation doesn't appear.

#### References

1. Allen, L.J. ( 2007) *Introduction to mathematical biology*, Pearson/Prentice Hall.
2. Elaydi, S. (2005), *An introduction to difference equations*, New York, NY: Springer New York.
3. Farnoosh, R., Parsamanesh, M. (2017) *Disease extinction and persistence in a discrete-time SIS epidemic model with vaccination and varying population size*, *Filomat*, **31(15)**, 4735–4747.
4. Kuznetsov, Y. A. (2013), *Elements of applied bifurcation theory*, **volume 112** Springer Science & Business Media.
5. Parsamanesh, M., Erfanian, M. (2021) *Stability and bifurcations in a discrete-time SIVS model with saturated incidence rate*, *Chaos, Solitons & Fractals*, **150**, 111178.

# Optimizing One-Stage Crowd Detection: An Enhanced SSD with Residual and Trainable Downsampling Blocks

Seyedsaeid Mirkamali<sup>1</sup>, Academic member, Department of Computer Engineering and IT,

Payame Noor University, P.O. Box, 19395-3697, Tehran, Iran

s.mirkamali@pnu.ac.ir

**Abstract:** This paper presents an enhanced Single Shot MultiBox Detector (SSD) architecture designed for high-accuracy small-object detection in dense crowd scenarios. Traditional single-stage detectors often struggle with detecting small targets due to loss of fine-grained spatial information during downsampling. To address this, we propose an Improved SSD (ISSD) that integrates residual feature blocks and trainable downsampling layers to preserve crucial low-level details. The proposed architecture employs multi-scale feature maps and adaptive default box configurations to enhance detection sensitivity. Experimental results on VOC and COCO benchmarks demonstrate that ISSD achieves 78.68% mAP while maintaining real-time throughput of 89 FPS, outperforming standard SSD300, SSD512, and Faster R-CNN baselines. The system offers an optimal balance between speed and accuracy, suitable for real-time crowd analytics and surveillance applications.

**Keywords:** Object Detection, SSD Residual Learning, Deep Learning, Crowd Analysis.

## 1. Introduction

Object detection is a cornerstone of computer vision, with applications ranging from autonomous driving to surveillance. While deep learning, particularly Convolutional Neural Networks (CNNs), has dramatically improved performance (Girshick, Donahue, Darrell, & Malik, 2014; Krizhevsky, Sutskever, & Hinton, 2012), detecting small objects in dense scenes remains an open problem. Models like Faster R-CNN (Ren, He, Girshick, & Sun, 2015) and YOLO (Redmon, Divvala, Girshick, & Farhadi, 2016) represent two dominant paradigms: two-stage and single-stage detectors. The Single Shot MultiBox Detector (SSD) (Liu et al., 2016) offers an attractive balance by performing localization and classification in a single forward pass, enabling high frame rates.

However, the standard SSD architecture struggles with small objects. As an image passes through the network, repeated downsampling operations cause the feature maps representing small objects to lose discriminative resolution. This is acutely problematic in crowd scenarios where faces may occupy only a few dozen pixels. Our work directly addresses this limitation by enhancing the SSD framework to better retain and leverage high-resolution, semantically rich features.

In this paper, we present an Improved SSD (ISSD) that integrates insights from recent architectural advances. We systematically refine the feature extraction backbone, incorporate multi-scale feature fusion with learned downsampling, and optimize the default box matching process. The result is a model that achieves superior accuracy on small objects while maintaining the real-time performance critical for video analysis.

## 2. Related Work

Deep learning-based object detectors are broadly categorized into two-stage and one-stage frameworks. Two-stage detectors (e.g., Faster R-CNN (Ren et al., 2015)) decouple region proposal and classification, providing high accuracy at the cost of runtime. Single-stage detectors such as SSD and YOLO directly regress object locations in a unified network, achieving real-time speeds but often underperforming on small or heavily occluded objects.

---

<sup>1</sup>. Corresponding Author

Feature hierarchy and multi-scale representations are central to modern detectors. Feature Pyramid Networks (FPN) (Lin et al., 2017a) introduced a top-down architecture with lateral connections to build high-level semantic feature maps at multiple scales, substantially improving small-object detection by enriching higher-resolution layers with strong semantics. RetinaNet (Lin, Goyal, Girshick, He, & Dollár, 2017b) further improved single-stage detection by introducing the focal loss to alleviate extreme class imbalance between background and foreground, enabling one-stage systems to reach two-stage accuracy in many cases.

Backbone and architectural advances have a strong influence on detector performance. Deep residual networks (ResNet) (He, Zhang, Ren, & Sun, 2016) provided a reliable method for training very deep models via identity shortcuts, becoming the de facto backbone for many detection frameworks. EfficientDet (Tan, Pang, & Le, 2020) proposed compound scaling and a BiFPN for efficient multi-scale feature fusion, trading off computation and accuracy effectively for resource-constrained settings.

### 3. Mathematical Formulation and Enhanced Detection

The proposed ISSD framework is based on the original SSD pipeline but incorporates key modifications to enhance detection of small objects in complex crowd environments. Residual feature blocks are embedded in early convolutional layers to retain gradient flow and mitigate vanishing effects, as inspired by ResNet. Additionally, static downsampling operations are replaced by trainable convolutional layers, allowing the network to learn optimal feature compression dynamically. Figure 1 illustrates the overall architecture of the proposed ISSD. Multi-scale feature maps are generated at different layers to handle objects of varying sizes, and optimized default box tiling ensures comprehensive spatial coverage.

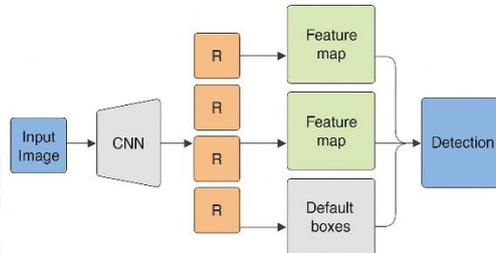


Figure 1: Overview of the proposed ISSD architecture integrating residual and trainable downsampling blocks.

#### 3.1. Training Objective and Loss Function

We replace the base network structure originally used in SSD with a structure utilizing **residual blocks** (e.g., from Res-Net architecture). The fundamental characteristic of the residual block is the **shortcut connection**, enabling the network layers to learn a residual mapping  $F(x) := H(x) - x$  rather than the original desired mapping  $H(x)$ . This design choice is instrumental in countering the problem of **information loss** related to small targets, a phenomenon that frequently occurs in successive downsampling layers of deep networks.

The common SSD architecture employs **static merge (pooling) layers** for downsampling (e.g., with a stride of 2). These layers lack trainable parameters, making them unable to learn and extract rich features during the reduction process. We replace this static merge layer with a **convolutional layer utilizing a stride of 2**. This modification ensures that downsampling still occurs at the required rate but enables the convolutional layer to **extract more features** and offer greater learning flexibility due to its trainable parameters.

#### 3.2. Multi-Scale Detection and Default Boxes

The SSMBD approach naturally handles various object sizes by utilizing predictions from **multi-scale feature maps** extracted from different layers.

To tailor the model for crowded scenes (implying varied object size and aspect ratios):

- We use **separate filters with different default boxes** to effectively manage the variations in object aspect ratios.
- The overall scheme applies improvements such as using multi-scale features and incorporating a greater number of default boxes to enhance accuracy.
- We focus on using **higher resolution feature maps** derived from higher resolution layers for detecting small

objects, discarding low-resolution layers where feature information (such as boundaries and edges) is minimal for classification purposes.

- The parameters governing **anchor selection** (scale and aspect ratio/ratio) are appropriately adjusted to ensure adequate coverage of the full range of face dimensions present in dense crowds.

## 4. Mathematical Formulation

The proposed approach relies heavily on robust mathematical modeling for loss minimization and precise bounding box generation.

### 4.1. Training Objective and Loss Function

The training objective aims to minimize a composite loss function, derived from the MultiBox objective, adapted for multiple categories. The overall objective loss function  $L$  is a weighted sum of the localization loss ( $L_{loc}$ ) and the confidence loss ( $L_{conf}$ ):

$$L(x, c, l, g) = \frac{1}{N} (L_{conf}(x, c) + \alpha L_{loc}(x, l, g))$$

Here,  $N$  represents the number of matched default boxes.

**Confidence Loss ( $L_{conf}$ ):** The confidence loss is the softmax loss over multiple class confidences ( $c$ ). This loss penalizes incorrect classifications of the proposed boxes.

**Localization Loss ( $L_{loc}$ ):** The localization loss employs a **Smooth L1 loss** between the predicted box parameters ( $l$ ) and the ground truth box parameters ( $g$ ). The network regresses to offsets for the center coordinates ( $cx, cy$ ) and the dimensions (width  $w$ , height  $h$ ) of the default bounding box ( $d$ ). The predicted offsets for the ground truth center coordinates ( $g^{cxj}, g^{cyj}$ ) and dimensions ( $g^{wj}, g^{hj}$ ) are calculated as:

$$\begin{aligned} g^{cxj} &= (gcxj - dcsi) / dwi \\ g^{wj} &= \log(dwigwj) \\ g^{hj} &= \log(dhighj) \end{aligned}$$

The full localization loss is computed over the set of positive matches (Pos):

$$L_{loc}(x, l, g) = \sum_{i \in \text{Pos}} \sum_{m \in \{c, e_j, w, h\}} x_{ij} \text{smooth}_{L1}(l_i^m - g_{mj})$$

### 4.2. Default Box Generation and Matching

The SSMBD algorithm is structured in phases involving box size selection and matching to ground truth.

**1. Select the Size of Box (Algorithm 1 adapted):** The process determines the box size based on the convolutional layer ( $Cl$ ) and pixel intensity change ( $I(c)$ ).

**Inputs:**  $I(x)$  ← Input Image,  $Cl$  ← Convolutional Layer,  $S(b)$  ← Size of Box,  $F(m)$  ← Feature Map,  $d$  ← dimension of boxes  $4 \times 4, 8 \times 8, 16 \times 16$ ,  $I(c)$  ← Change in Intensity of pixel.

The feature map  $F(m)$  for a given box size  $S(b)$  is identified using:

$$F(m) \leftarrow \{\text{Minimum } Cl + \text{Maximum } I(c)\}.$$

If the change in intensity  $I(c)$  is 1, the Width ( $W$ ) and Height ( $H$ ) are determined by:

$$W = Cl \times I(c) \quad H = Cl \div I(c)$$

**2. Identify the Truth Boxes (Algorithm 2 adapted):** After generating default boxes  $B$ , the next phase is matching them to the corresponding ground truth boxes  $T$ . A threshold value  $\alpha$  (e.g., 0.5 in original SSD) is used to determine a match. A default box  $B[i]$  matches a truth box  $T[i]$  of class  $class[l]$  if the overlap (computed as 1 minus the feature map difference) exceeds the threshold  $\alpha$ .

## 5. Experimental Results

The proposed F-CNN+SSMBD model was evaluated on the Pascal VOC and COCO datasets, comparing its performance (mAP and FPS) against established object detection models. The implementations utilized Python programming language with the OpenCV 2.4 library.

### 5.1. Comparison with State-of-the-Art (SOTA)

The improved model demonstrates significant gains in accuracy and speed compared to previous SOTA implementations.

Table 1: Comparison of Detection System Performance on VOC/COCO Datasets

System Model	mAP	FPS	No. of Boxes	Input Resolution
F-CNN	73.2	7	6000	1000×600
YOLO	66.4	155	98	448×448
SSD512	76.8	19	24564	512×512
SSD300	74.3	46	8732	300×300
<b>F-CNN+SSBMD (Proposed)</b>	<b>78.68</b>	<b>89</b>	5988	1024×1024

The results confirm that standard SSD versions (SSD300 and SSD512) typically fail to achieve accuracy above 70% while performing faster than F-CNN. By combining Faster R-CNN architectural components with SSBMD, our proposed model achieves the highest Mean Average Precision, with 78.68% mAP, which is 1.88% higher than SSD512 (76.8% mAP). Crucially, the model maintains a high throughput of 89 FPS. This high speed is nearly double that of the fastest batch-size-one SSD300 variant (46 FPS) and significantly faster than F-CNN (7 FPS). In the specific context of face detection, the improved algorithm yields better accuracy than MTCNN and comparable accuracy to RetinaFace while demonstrating a speed that is approximately 5 times faster than RetinaFace.

## 6. Conclusion

This work introduced ISSD, an improved SSD-based one-stage detector optimized for small-object detection in dense scenes. By integrating residual learning and trainable downsampling layers, the model preserves fine-grained spatial features and achieves high precision with minimal computational overhead. The resulting system achieves state-of-the-art accuracy-speed trade-offs, outperforming standard SSD and Faster R-CNN baselines. Future work will explore attention-based modules and lightweight transformers to further enhance adaptability in complex real-world conditions.

## References

1. Girshick, R., Donahue, J., Darrell, T., & Malik, J. 2014. *Rich feature hierarchies for accurate object detection and semantic segmentation*. Paper presented at the Proceedings of the IEEE conference on computer vision and pattern recognition.
2. He, K., Zhang, X., Ren, S., & Sun, J. 2016. *Deep residual learning for image recognition*. Paper presented at the Proceedings of the IEEE conference on computer vision and pattern recognition.
3. Krizhevsky, A., Sutskever, I., & Hinton, G. E. 2012. Imagenet classification with deep convolutional neural networks. *Advances in neural information processing systems*, 25.
4. Lin, T.-Y., Dollár, P., Girshick, R., He, K., Hariharan, B., & Belongie, S. 2017a. *Feature pyramid networks for object detection*. Paper presented at the Proceedings of the IEEE conference on computer vision and pattern recognition.
5. Lin, T.-Y., Goyal, P., Girshick, R., He, K., & Dollár, P. 2017b. *Focal loss for dense object detection*. Paper presented at the Proceedings of the IEEE international conference on computer vision.
6. Liu, W., Anguelov, D., Erhan, D., Szegedy, C., Reed, S., Fu, C.-Y., & Berg, A. C. 2016. *Ssd: Single shot multibox detector*. Paper presented at the European conference on computer vision.
7. Redmon, J., Divvala, S., Girshick, R., & Farhadi, A. 2016. *You only look once: Unified, real-time object detection*. Paper presented at the Proceedings of the IEEE conference on computer vision and pattern recognition.
8. Ren, S., He, K., Girshick, R., & Sun, J. 2015. Faster r-cnn: Towards real-time object detection with region proposal networks. *Advances in neural information processing systems*, 28.
9. Tan, M., Pang, R., & Le, Q. V. 2020. *Efficientdet: Scalable and efficient object detection*. Paper presented at the Proceedings of the IEEE/CVF conference on computer vision and pattern recognition.

# Equation of Motion in $f(R)$ Modified Gravity with Cosmological Constant for imperfect fluid

Yousef Alipour Fakhri<sup>1</sup>, Academic member, Department of Mathematics,

Payame Noor University, P.O. Box, 19395-4697, Tehran, Iran

y\_alipour@pnu.ac.ir

Mojtaba Safdarian<sup>2</sup>, Ph. D. student, Department of Mathematics,

Payame Noor University, P.O. Box, 19395-4697, Tehran, Iran

mojtaba\_safdarian@pnu.ac.ir

**Abstract:** In this study, we derive the equation of motion in the framework of  $f(R)$  modified gravity with cosmological constant in an imperfect fluid. Additionally, the extra force in the presence of cosmological constant for imperfect fluid is calculated. The field equations obtained from the variation of the action are utilized to achieve these results.

**Keywords:**  $f(R)$  modified gravity, energy-momentum tensor, equation of motion, imperfect fluid, cosmological constant.

## 1. Introduction

$f(R)$  modified gravity is one of the prominent models proposed to explain physical phenomena such as the accelerated expansion of the universe. In this theory, an arbitrary function of the Ricci scalar  $R$  replaces the standard Einstein-Hilbert Lagrangian in the action [1,2]. The equation of motion is a fundamental component in mathematics and physics. Previous studies have derived this equation using the perfect fluid energy-momentum tensor in  $f(R)$  and  $f(G)$  modified gravity models[3,4].

In this paper, we redefine the equation of motion in  $f(R)$  gravity by incorporating the imperfect fluid energy-momentum tensor and cosmological constant  $\Lambda$ .

## 2. $f(R)$ Gravity Theory in imperfect fluid with Cosmological Constant

We begin with a brief description of the basic equation in the framework of the  $f(R)$  modified gravity theory containing the cosmological constant. The action of the system in the absence of ordinary matter is given by [1]:

$$S_g = \frac{1}{2k} \int d^4x \sqrt{-g} [f(R) - \Lambda] \quad (1)$$

<sup>1</sup>. Corresponding Author

<sup>2</sup>. Corresponding Author

with  $k = 8\pi G$  and  $g$  being the determinant of the metric tensor  $g_{\mu\nu}$  and  $\Lambda$  is the cosmological constant.

varying the action (1) with respect to metric  $g_{\mu\nu}$  yields the field equations [1]:

$$R_{\mu\nu} f(R) - \frac{1}{2} f(R) g_{\mu\nu} - 2 g_{\mu\nu} \Lambda + g_{\mu\nu} \square f_R - \nabla_\mu \nabla_\nu f_R - k T_{\mu\nu} = 0, \quad (2)$$

Where  $\frac{df(R)}{dR} = f_R$  and  $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$  is the energy-momentum tensor.

$\nabla_\mu$  is the covariant derivative associated with the Levi-Civita connection of the metric .

### 3. Mathematical Calculate of the Equation of Motion

The energy-momentum tensor in an imperfect fluid is defined as [5]:

$$T_{\mu\nu} = (\rho + p - \xi\theta)u_\mu u_\nu + (p - \xi\theta)g_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu q_\nu + u_\nu q_\mu . \quad (3)$$

Where  $\rho$  is the energy density,  $p$  is the isotropic pressure,  $u_\mu$  is the four-velocity of the fluid (normalized as  $u^\mu u_\mu = 1$ ),  $q_\mu$  is the heat flux vector satisfying  $q_\mu u^\mu = 0$ ,  $\theta = \nabla_\mu u^\mu$  is the expansion scalar,  $\xi \geq 0$  and  $\eta \geq 0$  are the bulk and shear viscosity coefficients and  $\sigma_{\mu\nu}$  is the shear tensor:

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + u_{(\mu} \dot{u}_{\nu)} - \frac{1}{3}\theta(g_{\mu\nu} + u_\mu u_\nu), \text{ with } \dot{u}^\mu = u^\alpha \nabla_\alpha u^\mu.$$

By take the divergence of this tensor we have:

$$\nabla^\nu T_{\mu\nu} = (\rho + p - \xi\theta)u_\nu \nabla^\nu u_\mu + (\rho + p - \xi\theta)u_\mu \nabla^\nu u_\nu + u_\mu u_\nu \nabla^\nu (\rho + p - \xi\theta) + g_{\mu\nu} \nabla^\nu (p - \xi\theta) - 2\eta \nabla^\nu \sigma_{\mu\nu} - 2\sigma_{\mu\nu} \nabla^\nu \eta + \nabla^\nu u_\mu q_\nu + u_\mu \nabla^\nu q_\nu + \nabla^\nu u_\nu q_\mu + u_\nu \nabla^\nu q_\mu.$$

We consider equation (2):

$$T_{\mu\nu} = \frac{1}{k} (R_{\mu\nu} f(R) - \frac{1}{2} f(R) g_{\mu\nu} - 2 g_{\mu\nu} \Lambda + g_{\mu\nu} \square f_R - \nabla_\mu \nabla_\nu f_R) \quad (4)$$

and take the divergence of it, hence we have:

$$\nabla^\nu T_{\mu\nu} = \frac{1}{k} [f(R) \nabla^\nu R_{\mu\nu} + R_{\mu\nu} \nabla^\nu f(R) - \frac{1}{2} g_{\mu\nu} \nabla^\nu f(R) - 2 g_{\mu\nu} \nabla^\nu \Lambda + g_{\mu\nu} f_R \nabla^\nu \square + g_{\mu\nu} \square \nabla^\nu f(R) - \nabla^\nu \nabla_\mu (\nabla_\nu f_R) - f_R \nabla_\mu (\nabla^\nu \nabla_\nu) - \nabla_\mu \nabla_\nu (\nabla^\nu f_R)]$$

So we obtain the equation of motion in this case as follow:



$$\begin{aligned}
 & (\rho + p - \xi\theta)u_\nu \nabla^\nu u_\mu + (\rho + p - \xi\theta)u_\mu \nabla^\nu u_\nu + u_\mu u_\nu \nabla^\nu (\rho + p - \xi\theta) + g_{\mu\nu} \nabla^\nu (p - \xi\theta) - 2\eta \nabla^\nu \sigma_{\mu\nu} \\
 & - 2\sigma_{\mu\nu} \nabla^\nu \eta + \nabla^\nu u_\mu q_\nu + u_\mu \nabla^\nu q_\nu + \nabla^\nu u_\nu q_\mu + u_\nu \nabla^\nu q_\mu \\
 & = \frac{1}{k} [f(R) \nabla^\nu R_{\mu\nu} + R_{\mu\nu} \nabla^\nu f(R) - \frac{1}{2} g_{\mu\nu} \nabla^\nu f(R) - 2 g_{\mu\nu} \nabla^\nu \Lambda \\
 & + g_{\mu\nu} f_R \nabla^\nu \square + g_{\mu\nu} \square \nabla^\nu f(R) - \nabla^\nu \nabla_\mu (\nabla_\nu f_R) - f_R \nabla_\mu (\nabla^\nu \nabla_\nu) \\
 & - \nabla_\mu \nabla_\nu (\nabla^\nu f_R)]
 \end{aligned}$$

Finally, we obtain the equation of motion in this case:

$$\begin{aligned}
 & (\rho + p - \xi\theta)\dot{u}_\mu + (\rho + p - \xi\theta)u_\mu \theta + u_\mu u_\nu (\nabla^\nu \rho + \partial^\nu p - \theta \nabla^\nu \xi - \xi \partial^\nu \theta) + (\partial_\mu p - \theta \nabla_\mu \xi - \xi \nabla_\mu \theta) \\
 & - 2\eta \nabla^\nu \sigma_{\mu\nu} - 2\sigma_{\mu\nu} \nabla^\nu \eta + \nabla^\nu u_\mu q_\nu + u_\mu \nabla^\nu q_\nu + \theta q_\mu + u_\nu \nabla^\nu q_\mu \\
 & = \frac{1}{k} [f(R) \nabla^\nu R_{\mu\nu} + R_{\mu\nu} \nabla^\nu f(R) - \frac{1}{2} \nabla_\mu f(R) - 2 \nabla_\mu \Lambda + \square f_R \nabla_\mu + \square \nabla_\mu f(R) - \\
 & \nabla^\nu \nabla_\mu (\nabla_\nu f_R) - \square f_R \nabla_\mu - \nabla_\mu \square f_R] \quad (5)
 \end{aligned}$$

Hence, we can obtain the extra force so we have :

$$\begin{aligned}
 & \dot{u}_\mu \\
 & = f_\mu = \frac{1}{(\rho+p-\xi\theta)} \left\{ \frac{1}{k} \left[ f(R) \nabla^\nu R_{\mu\nu} + R_{\mu\nu} \nabla^\nu f(R) - \frac{1}{2} \nabla_\mu f(R) - 2 \nabla_\mu \Lambda + \square f_R \nabla_\mu + \right. \right. \\
 & \left. \left. \square \nabla_\mu f(R) - \nabla^\nu \nabla_\mu (\nabla_\nu f_R) - \square f_R \nabla_\mu - \nabla_\mu \square f_R \right] - [(\rho + p - \xi\theta)\dot{u}_\mu + (\rho + p - \xi\theta)u_\mu \theta + \right. \\
 & \left. u_\mu u_\nu (\nabla^\nu \rho + \partial^\nu p - \theta \nabla^\nu \xi - \xi \partial^\nu \theta) + (\partial_\mu p - \theta \nabla_\mu \xi - \xi \nabla_\mu \theta) - 2\eta \nabla^\nu \sigma_{\mu\nu} - 2\sigma_{\mu\nu} \nabla^\nu \eta + \nabla^\nu u_\mu q_\nu + \right. \\
 & \left. u_\mu \nabla^\nu q_\nu + \theta q_\mu + u_\nu \nabla^\nu q_\mu \right\}. \quad (6)
 \end{aligned}$$

#### 4. Conclusion

In this work, we calculated the equation of motion and the extra force in f(R) modified gravity in an imperfect fluid with cosmological constant. The presence of the extra force confirms non-geodesic motion in this modified gravity framework.

#### References

- [1] Xu, M., Zhang, Y., Yang, L., Yang, S., & Lu, J. (2024). Corrected Thermodynamics of Black Holes in f(R) Gravity with Electrodynamical Field and Cosmological Constant. *Entropy*, 26(10), 868.
- [2] Sotiriou, T. P., & Faraoni, V. (2010). f(R) Theories of Gravity. *Reviews of Modern Physics*, 82(1), 451–497.
- [3] Mohseni, M. (2009). Non-geodesic Motion in f(G) Gravity with Non-minimal Coupling. *Physics Letters B*, 682(1), 89–92.
- [4] Bertolami, O., Boehmer, C. G., Harko, T., & Lobo, F. S. (2007). Extra Force in f(R) Modified Theories of Gravity. *Physical Review D*, 75(10), 104016.
- [5] Kolassis, C.A., Santos, N., Soubelis, D.T., *class and quant gravity* 5 (10), 1329, 1988

# Equation of Motion and Extra Force and Generalized Raychaudhuri equation in $f(R, \Sigma, T)$ Modified Gravity

Yousef Alipour Fakhri<sup>1</sup>, Academic member, Department of Mathematics,

Payame Noor University, P.O. Box, 19395-4697, Tehran, Iran  
y\_alipour@pnu.ac.ir

Mojtaba Safdarian<sup>2</sup>, Ph. D. student, Department of Mathematics,

Payame Noor University, P.O. Box, 19395-4697, Tehran, Iran  
mojtaba\_safdarian@pnu.ac.ir

**Abstract:** In this study, we derive the equation of motion and the generalized Raychaudhuri equation in the framework of  $f(R, \Sigma, T)$  modified gravity. The field equations obtained from the variation of the action are utilized to achieve these results.

**Keywords:**  $f(R, \Sigma, T)$  modified gravity, energy-momentum tensor, equation of motion, Generalized Raychaudhuri equation.

## 1. Introduction

Dark energy and the accelerated expansion of the universe are motivations of alternative theoretical frameworks beyond the general theory of relativity called modified theories of gravity.

$f(R, \Sigma, T)$  modified gravity is one of the prominent models proposed to explain physical phenomena such as the accelerated expansion of the universe introduced by Bakry and Ibraheem in 2023 [1]. This theory is conceptual of absolute parallelism geometry. This framework extends the conventional  $f(R)$  and  $f(R, T)$  models by introducing an additional scalar parameter  $\Sigma$ , which may represent torsion or other geometric properties of space-time [2].

The equation of motion is a fundamental component in mathematics and physics. Previous studies have derived this equation using the perfect fluid energy-momentum tensor in  $f(R)$  and  $f(G)$  modified gravity models [3,4].

The Raychaudhuri equation is a well-known and useful equation in general relativity and cosmology. This equation proves the singularity theorems due to Penrose-Hawking and Geroch[5].

In this paper, we calculate equation of motion and the generalized Raychaudhuri equation in in modified  $f(R, \Sigma, T)$  gravity model in perfect fluid case.

---

<sup>1</sup>. Corresponding Author

<sup>2</sup>. Corresponding Author



## 2. $f(R, \Sigma, T)$ Gravity Theory

We begin with a brief description of the basic equation in the framework of the  $f(R, \Sigma, T)$  modified gravity theory . The action of the system is given by [2]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (f(R, \Sigma, T) + \mathcal{L}) \quad (1)$$

with  $g$  being the determinant of the metric tensor  $g_{\mu\nu}$  and  $\mathcal{L}$  is the matter lagrangian.

varying the action (1) with respect to the metric tensor  $g_{\mu\nu}$  yields the field equations [2]:

$$\begin{aligned} R_{\mu\nu} \frac{\partial f}{\partial R} - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu} \nabla^\gamma \nabla_\gamma - \nabla_\mu \nabla_\nu) \frac{\partial f}{\partial R} - \beta \frac{\partial f}{\partial R} \\ = 8\pi T_{\mu\nu} + (T_{\mu\nu} + p g_{\mu\nu}) \frac{\partial f}{\partial T}, \end{aligned} \quad (2)$$

Where  $\Sigma$  represents an additional geometric parameter that can encapsulate torsional effects [2].

$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$  is the energy-momentum tensor and  $\nabla_\mu$  is the covariant derivative associated with the Levi-Civita connection of the metric .

## 3. Mathematical Calculate of the Equation of Motion

The energy-momentum tensor is defined as:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \quad (3)$$

By take the divergence of this tensor we have:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= (\nabla^\mu \rho) u_\mu u_\nu + \rho (\nabla^\mu u_\mu) u_\nu + \rho u_\mu (\nabla^\mu u_\nu) + (\nabla^\mu p) u_\mu u_\nu \\ &+ p (\nabla^\mu u_\mu) u_\nu + p u_\mu (\nabla^\mu u_\nu) + (\nabla^\mu p) g_{\mu\nu} \\ &= (\nabla^\mu \rho) u_\mu u_\nu + \rho \theta u_\nu + \rho \dot{u}_\nu + (\nabla^\mu p) u_\mu u_\nu + p \theta u_\nu + p \dot{u}_\nu \\ &+ \nabla_\nu p. \end{aligned}$$

We consider equation (2):

$$T_{\mu\nu} = \frac{1}{8\pi + \frac{\partial f}{\partial T}} (R_{\mu\nu} \frac{\partial f}{\partial R} - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu} \nabla^\gamma \nabla_\gamma - \nabla_\mu \nabla_\nu) \frac{\partial f}{\partial R} - \beta \frac{\partial f}{\partial R} - p g_{\mu\nu} \frac{\partial f}{\partial T}) \quad (4)$$

and take the divergence of it, hence we have:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \frac{1}{8\pi + \frac{\partial f}{\partial T}} \left[ \frac{\partial f}{\partial R} \nabla^\mu R_{\mu\nu} + R_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial R} - \frac{1}{2} g_{\mu\nu} \nabla^\mu f - \square \nabla_\nu \frac{\partial f}{\partial R} - \beta \nabla^\mu \frac{\partial f}{\partial R} - g_{\mu\nu} \frac{\partial f}{\partial T} \nabla^\mu p \right. \\ &\quad \left. - p g_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial T} \right] \end{aligned}$$

So we obtain the equation of motion in this case as follow:

$$\begin{aligned} (\nabla^\mu \rho) u_\mu u_\nu + \rho (\nabla^\mu u_\mu) u_\nu + \rho u_\mu (\nabla^\mu u_\nu) + (\nabla^\mu p) u_\mu u_\nu + p (\nabla^\mu u_\mu) u_\nu + \\ p u_\mu (\nabla^\mu u_\nu) + (\nabla^\mu p) g_{\mu\nu} = \frac{1}{8\pi + \frac{\partial f}{\partial T}} \left[ \frac{\partial f}{\partial R} \nabla^\mu R_{\mu\nu} + R_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial R} - \frac{1}{2} g_{\mu\nu} \nabla^\mu f - \square \nabla_\nu \frac{\partial f}{\partial R} - \right. \\ \left. \beta \nabla^\mu \frac{\partial f}{\partial R} - g_{\mu\nu} \frac{\partial f}{\partial T} \nabla^\mu p - p g_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial T} \right] \end{aligned}$$

Finally, we obtain the equation of motion in this case:

$$\begin{aligned} (\nabla^\mu \rho) u_\mu u_\nu + \rho \theta u_\nu + \rho \dot{u}_\nu + (\nabla^\mu p) u_\mu u_\nu + p \theta u_\nu + p \dot{u}_\nu + (\partial_\nu p) = \\ \frac{1}{8\pi + \frac{\partial f}{\partial T}} \left[ \frac{\partial f}{\partial R} \nabla^\mu R_{\mu\nu} + R_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial R} - \frac{1}{2} \nabla_\nu f - \square \nabla_\nu \frac{\partial f}{\partial R} - \beta \nabla^\mu \frac{\partial f}{\partial R} - \frac{\partial f}{\partial T} \partial_\nu p - p \nabla_\nu \frac{\partial f}{\partial T} \right] \end{aligned} \quad (5)$$

Hence, we can obtain the extra force so we have :

$$\begin{aligned} & \dot{u}_\nu \\ & = f_\nu = \frac{1}{\rho} \left\{ \frac{1}{8\pi + \frac{\partial f}{\partial T}} \left[ \frac{\partial f}{\partial R} \nabla^\mu R_{\mu\nu} + R_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial R} - \frac{1}{2} \nabla_\nu f - \square \nabla_\nu \frac{\partial f}{\partial R} - \beta \nabla^\mu \frac{\partial f}{\partial R} - \frac{\partial f}{\partial T} \partial_\nu p - p \nabla_\nu \frac{\partial f}{\partial T} \right] \right. \\ & \left. - (\nabla^\mu \rho) u_\mu u_\nu - \rho \theta u_\nu - (\partial^\mu p) u_\mu u_\nu - p \theta u_\nu - p \dot{u}_\nu - (\partial_\nu p) \right\}. \end{aligned} \quad (6)$$

#### 4. Generalized Raychaudhuri equation in $f(R, \Sigma, T)$ in perfect fluid case

The Raychaudhuri equation can be written in the presence of an extra force as [5,6]:

$$\dot{\theta} + \frac{1}{3} \theta^2 + (\sigma^2 - \omega^2) = \nabla_\nu f^\nu - R_{\mu\nu} u^\mu u^\nu \quad (7)$$

$\sigma^2 = \sigma_{\mu\nu} \sigma^{\mu\nu}$  and  $\omega^2 = \omega_{\mu\nu} \omega^{\mu\nu}$  where  $\sigma_{\mu\nu}$  is the shear tensor and  $\omega_{\mu\nu}$  is the vorticity tensor and  $\theta$  is the expansion scalar.  $R_{\mu\nu}$  is The Ricci tensor.  $u^\mu$  and  $u^\nu$  are velocity tensor.

By using the expression of the extra force, we obtain The Generalized Raychaudhuri equation in  $f(R)$  modified Gravity with non-minimal coupling with torsion in spinning perfect fluid case:

$$\begin{aligned} & \dot{\theta} + \frac{1}{3} \theta^2 + (\sigma^2 - \omega^2) + R_{\mu\nu} u^\mu u^\nu = \nabla_\nu \left\{ \frac{1}{\rho} \left\{ \frac{1}{8\pi + \frac{\partial f}{\partial T}} \left[ \frac{\partial f}{\partial R} \nabla^\mu R_{\mu\nu} + R_{\mu\nu} \nabla^\mu \frac{\partial f}{\partial R} - \frac{1}{2} \nabla_\nu f - \right. \right. \right. \\ & \left. \left. \square \nabla_\nu \frac{\partial f}{\partial R} - \beta \nabla^\mu \frac{\partial f}{\partial R} - \frac{\partial f}{\partial T} \partial_\nu p - p \nabla_\nu \frac{\partial f}{\partial T} \right] - (\nabla^\mu \rho) u_\mu u_\nu - \rho \theta u_\nu - (\partial^\mu p) u_\mu u_\nu - p \theta u_\nu - \right. \\ & \left. p \dot{u}_\nu - (\partial_\nu p) \right\}. \end{aligned} \quad (8)$$

#### 5. Conclusion

In this work, we calculate the equation of motion and the extra force and generalized Raychaudhuri equation in  $f(R, \Sigma, T)$  modified gravity. The presence of the extra force confirms non-geodesic motion in this modified gravity framework.

#### References

- [1] Bakry, M. A., & Ibraheem, S. K. (2023). Gravity. *Gravitation and Cosmology*, 29(1), 19-32.
- [2] Myrzakulov, N., Shekh, S. H., & Pradhan, A. (2025). Cosmological implications of  $f(R, \Sigma, T)$  gravity: A unified approach using OHD and SN ia data. *Physics Letters B*, 862, 139369.
- [3] Mohseni, M. (2009). Non-geodesic Motion in  $f(G)$  Gravity with Non-minimal Coupling. *Physics Letters B*, 682(1), 89–92.
- [4] Bertolami, O., Boehmer, C. G., Harko, T., & Lobo, F. S. (2007). Extra Force in  $f(R)$  Modified Theories of Gravity. *Physical Review D*, 75(10), 104016.
- [5] Hawking, S. W., & Ellis, G. F. (2023). *The large scale structure of space-time*. Cambridge university press.
- [6] Wald, R. M. (2024). *General relativity*. University of Chicago press.



# ANFIS-Based Modeling of Pulse Parameters for Objective Determination of Warmness, Coldness, Wetness, Dryness, and Heart Strength in Persian Medicine

Mohammad Dehghandar<sup>1\*</sup>, Mahdi Alizadeh Vaghasloo<sup>2,3</sup>

<sup>1</sup>Assistant Professor, Department of Mathematics, Payame Noor University, PO Box 3697-19395, Tehran, Iran.

<sup>2</sup>Assistant Professor, Department of Traditional Medicine, School of Persian Medicine, Tehran University of Medical Sciences, Tehran, Iran

<sup>3</sup>Persian Medicine Network (PMN), Universal Scientific Education and Research Network (USERN), Tehran, Iran

\***Corresponding Authors:** Mohammad Dehghandar, Ph.D., Assistant Professor, Department of Mathematics, Payame Noor University, PO Box 3697-19395, Tehran, Iran, [m\\_dehghandar@pnu.ac.ir](mailto:m_dehghandar@pnu.ac.ir)

## Abstract

**Objective:** This study developed a computational framework to objectify pulse diagnosis in Persian Medicine by quantitatively predicting the primary temperamental qualities—Warmness, Coldness, Wetness, Dryness, and Heart Strength. **Methods:** An Adaptive Neuro-Fuzzy Inference System (ANFIS) model was designed. Using data from 68 participants, eleven pulse characteristics (e.g., length, strength, speed) were used as inputs, with the five temperament qualities serving as expert-defined outputs. The model was trained to learn the complex relationships between these inputs and outputs. **Results:** The ANFIS model demonstrated exceptional predictive accuracy, achieving an average  $R^2$  of 0.9322 and an average RMSE of 0.2239 across all outputs. Performance was notably high for Dryness ( $R^2 = 0.9998$ ) and Heart Strength ( $R^2 = 0.9891$ ). **Conclusion:** The high accuracy of the model validates its capability to emulate the sophisticated diagnostic reasoning of a Persian Medicine practitioner. This establishes ANFIS as a powerful tool for quantifying subjective pulse-based diagnosis.

**Keywords:** Persian Medicine, Pulse Diagnosis, Temperament (Mizaj), Adaptive Neuro-Fuzzy Inference System (ANFIS)

## 1. Introduction



The contemporary evolution of global healthcare is progressively shifting towards a hybrid framework that seeks to harmonize the strengths of conventional medicine with those of traditional and complementary medical paradigms [1]. Within this landscape, Persian Medicine (PM) emerges as a particularly sophisticated and historically substantiated system, with a documented legacy spanning millennia across diverse civilizations. Nevertheless, the complete integration and empirical validation of PM's principles within modern scientific discourse remain largely unrealized [2,3]. A foundational tenet of PM is the macroscopic and phenotypic classification of individuals—in both health and disease—into specific constitutional types, or Mizaj. This categorization is predicated on the relative predominance of the four qualitative states of Warmth, Coldness, Moisture, and Dryness [3]. Critically, these qualities represent a holistic, supra-physical metabolic and functional state, not merely corporeal temperature, shaped by a complex interplay of physiological and environmental determinants [4]. A growing body of modern empirical research is beginning to provide evidence for the validity of these categorical distinctions [5]. The precise determination of an individual's Mizaj, coupled with an evaluation of cardiac capacity and vital energy, forms the cornerstone of personalized health maintenance and therapeutic intervention in PM. Practitioners deduce these constitutional and vital characteristics through a synthesis of clinical signs, with pulse analysis occupying a position of exceptional diagnostic primacy. Its significance is evidenced by the fact that it comprises more than a quarter of the diagnostic content in canonical PM sources [6]. Therefore, the systematic application of pulse diagnostics in PM holds substantial potential for refining patient-specific disease stratification and management. Each arterial pulsation is described by a set of distinct parameters, the composite profile of which yields insights into a broad spectrum of physiological and pathological states [7,8]. A significant challenge, however, lies in the inherently subjective nature of PM pulse assessment. Its accuracy is predominantly dependent on the clinician's proficiency and accumulated practical experience, thereby presenting considerable obstacles to its standardization, objective quantification, and effective pedagogy [9,10]. In parallel, computational intelligence methodologies—particularly artificial intelligence (AI) and fuzzy logic systems—are increasingly being incorporated into Persian Medicine (PM) research. This convergence is creating novel pathways for scientific innovation and enhancing the practical implementation of PM principles within public health initiatives [11–13]. This study employs AI and fuzzy logic systems modeling to replicate the pulse-based diagnostic reasoning of PM experts in determining temperament and physiological strength.

## Pulse

In both traditional and modern medicine, the radial pulse is a key indicator of health and cardiac function [10]. It is assessed under standardized conditions, with specific qualities providing diagnostic insights [14, 15]. Key qualities include:

Pulse Length, Width, Height, and Strength: linked to vitality, bodily fluids, metabolic energy, and cardiac force. Pulse Speed and Frequency: correspond to pulse wave velocity and heart rate. Vessel Fullness, Consistency, and Thermal Qualities: reflect blood volume, vascular stiffness, and metabolic heat. While experts can synthesize these to determine a patient's constitution and heart robustness, less experienced practitioners may struggle with accurate deductions.

### Adaptive neuro-fuzzy inference systems (ANFIS)

The following presents a concise overview of the Adaptive Neuro-fuzzy Inference System (ANFIS). ANFIS is employed to model phenomena characterized by nonlinear relationships, combining the strengths of fuzzy inference systems with the capability for learning from data [16]. A standard fuzzy inference system comprises four primary components: a fuzzifier, an inference engine, a knowledge base, and a defuzzifier [17]. A key limitation of such systems is their inability to learn autonomously. This shortcoming is addressed by integrating them with neural networks, which introduces learning capabilities. In the integrated framework, the system processes input data to generate an estimated output. This estimate is then compared against the actual or desired output, and the discrepancies are minimized through an iterative adjustment process. This cycle repeats until the error between the estimated and actual outputs falls within an acceptable threshold. A significant characteristic of ANFIS is that its output structure adheres to the Sugeno (or Takagi-Sugeno-Kang) model. In this approach, the consequent part of a fuzzy rule is a first-order polynomial function of the input variables. To approximate a target function  $f^*$ , a collection of "if-then" fuzzy rules of the TSK type is constructed. These rules are designed using  $m$  data vectors, each containing  $n$  input variables and a single output. When a fuzzy system's output is formulated as a linear combination of its inputs, it is classified as a TSK system, with rules typically expressed in the following form [18, 19]:

$$\text{Rule: IF } x_1 \text{ is } A_1^L \text{ AND } x_2 \text{ is } A_2^L \text{ AND } \dots \text{ AND } x_n \text{ is } A_n^L \text{ THEN } y^L = \alpha_0^L + \alpha_1^L x_1 + \dots + \alpha_n^L x_n \quad (1)$$

wherein  $L = \{1, 2, \dots, r\}$ ,  $\alpha^L = \{\alpha_0^L, \alpha_1^L, \alpha_2^L, \dots, \alpha_n^L\}$

Therefore, fuzzy sets are expressed as follows:

$$A_i = \{A_i^1, A_i^2, \dots, A_i^L\} \quad (2)$$

And if  $x = (x_1, x_2, \dots, x_n)^T$  then the weighted average output for the number of  $r$  rules is as follows:

$$f(x) = \sum_{L=1}^r y^L \bar{w}^L \quad (3)$$

wherein  $w^L = \prod_{i=1}^n \mu_{A_i^L}(x_i)$ ,  $\bar{w}^L = \frac{w^L}{\sum_{L=1}^r w^L}$ .

If the fuzz sets are as a Gaussian membership function and in the interval  $[-\alpha_i, \beta_i]$  then each  $x_i \in [-\alpha_i, \beta_i]$  of the domains are defined as  $A_i$  existing in Equation (2) and the degree of the membership function is

non-zero, that is  $\mu_i^L(x_i) \neq 0$ . The Gaussian membership functions of each fuzzy set  $A^L$ ,  $L \in \{1, 2, \dots, r\}$  are considered according to Equation (4) [17]:

$$\mu_{A_i}(x_i) = e^{-\frac{1}{2} \left( \frac{x_i - m_i}{\sigma_i} \right)^2} \quad (4)$$

In this configuration,  $m_i$  and  $\sigma_i$  correspond to the center and width of the membership functions in the antecedent part, respectively, and constitute the set of adjustable parameters for this section.

## 2. MATERIALS AND METHODS

**Study Participants and Data Collection:** This modeling study utilized data from 68 participants. For each subject, a licensed acupuncturist completed a standardized questionnaire to record 11 input parameters derived from pulse palpation: length, width, height, strength, velocity, frequency, vascular fullness, moisture, palmar temperature, dorsal hand temperature, and vein temperature. The same practitioner provided the ground-truth values for the five target output variables—Warmness, Coldness, Wetness, Dryness, and Heart Strength—based on a holistic inference from traditional pulse laws. Table 1 shows the input and output variables of the identification status of the participants.

Table 1. Data status for the Neuro-fuzzy designed to estimate warmness, coldness, wetness, dryness, and strength

I/O	Variables	Min	Max	Mean
I <sub>1</sub>	Pulse Speed	1	3	1.8750
I <sub>2</sub>	Pulse Frequency	1	3	2.1250
I <sub>3</sub>	Pulse Strength	0	3	2.1786
I <sub>4</sub>	Pulse Length	1	3	1.2143
I <sub>5</sub>	Pulse Height	1	3	2.1875
I <sub>6</sub>	Pulse Wetness	1	3	2.2500
I <sub>7</sub>	Pulse Vascular fullness	1	3	2.4196
I <sub>8</sub>	Pulse Width	1	3	2.1964
I <sub>9</sub>	Temperature of palms	34	38	36.2500
I <sub>10</sub>	Temperature of backhand	34	37.5	35.1161
I <sub>11</sub>	Artery's temperature	34	38	35.8348
O <sub>1</sub>	Warmness	6	13	9.5804
O <sub>2</sub>	Coldness	0	4	1.1964
O <sub>3</sub>	Wetness	0	3	1.4643
O <sub>4</sub>	Dryness	0	3	0.5714
O <sub>5</sub>	Strength	0	2	0.4107

**Computational Modeling and Validation:** The relationship between the pulse parameters and the diagnostic outputs was modeled using an Adaptive Neuro-Fuzzy Inference System (ANFIS).

The dataset was partitioned into a training set (80% of subjects, n=54) and a hold-out test set (20%, n=14) using a random split to ensure a robust evaluation of model generalizability. For each of the five outputs, a dedicated ANFIS model was developed. A correlation-based feature selection was implemented for each model to select the top three most relevant input features, optimizing model performance and reducing complexity. The models were trained with grid partitioning and a hybrid learning algorithm. Model performance was quantitatively evaluated on the test set using the Root Mean Square Error (RMSE), Mean Absolute Error (MAE), and the coefficient of determination ( $R^2$ ).

### Performance Evaluation

Performance analysis considers the errors of a fuzzy system as the main evaluation method; this can also be observed based on the visual analysis of the obtained results. To better evaluate performance and investigate prediction accuracy, this study used the evaluation indices of mean absolute error (MAE), root mean square error (RMSE), and coefficient of determination ( $R^2$ ), which can be calculated using Equations (5), (6), and (7), respectively. These metrics are calculated as follows [21,22]:

$$MAE = \frac{1}{N} \sum_{i=1}^N |W_O^i - W_{FIS}^i| \quad (5)$$

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^N (W_O^i - W_{FIS}^i)^2} \quad (6)$$

$$R^2 = 1 - \frac{\left\{ \sum_{i=1}^N (W_O^i - W_{FIS}^i)^2 \right\}}{\left\{ \sum_{i=1}^N (W_O^i - \bar{W}_O)^2 \right\}} \quad (7)$$

where  $W_O^i$  and  $W_{FIS}^i$  are the i-th recordings of observed and predicted values from a total of N recorded data.

### 3. Results

The Adaptive Neuro-Fuzzy Inference System (ANFIS) demonstrated exceptional performance in diagnosing primary temperament qualities and Heart Strength. Figure 1 shows the prediction results of the neuro-fuzzy system for warmth, coldness, wetness, dryness, and strength.

6-7 November 2025

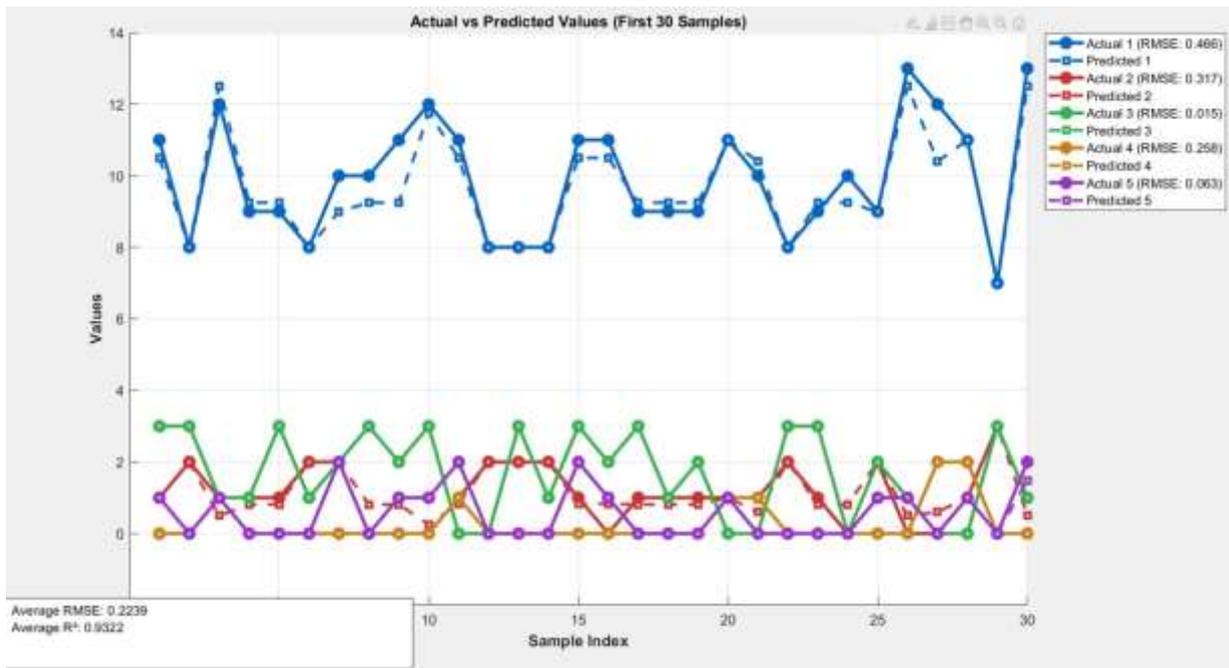


Figure 1. Neuro-fuzzy system prediction results warmness, coldness, wetness, dryness, and strength

Also, as can be seen, Figure 2 shows the results of the root mean square error in the neuro-fuzzy system for predicting Warmness, Coldness, Wetness, Dryness, and Strength.

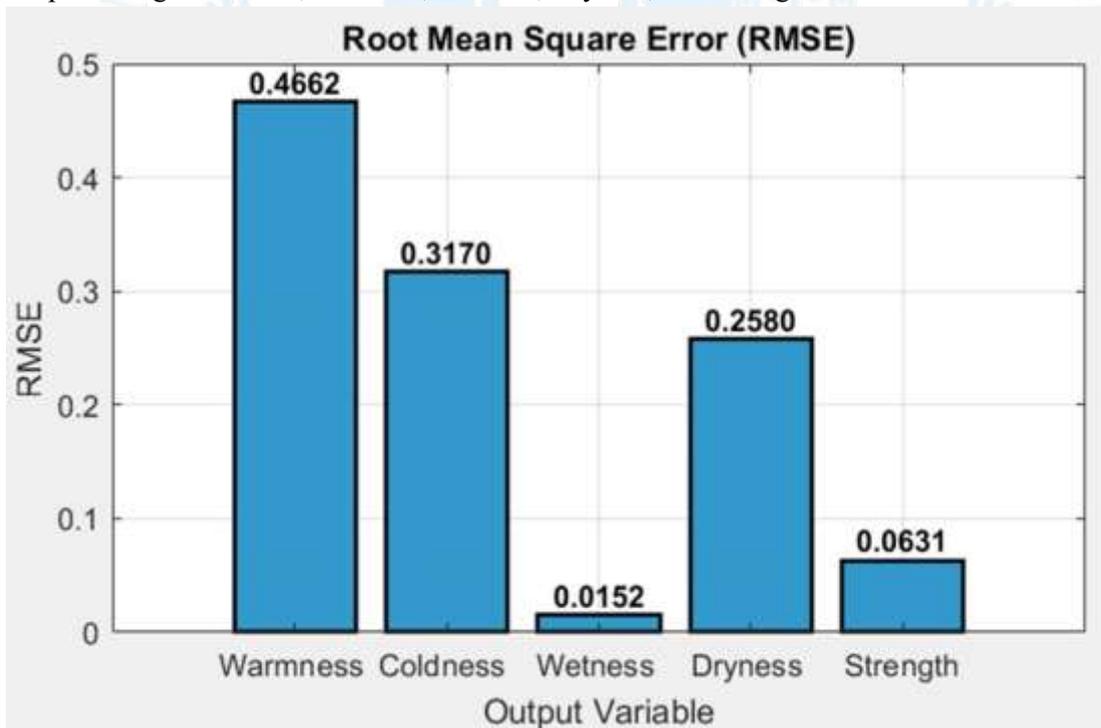


Figure 2. Results of root mean square error of the neuro-fuzzy system for predicting warmness, coldness, wetness, dryness, and strength

And then the regressions of the neuro-fuzzy system for predicting warmth, coldness, wetness, dryness, and strength are shown in Figure 3.

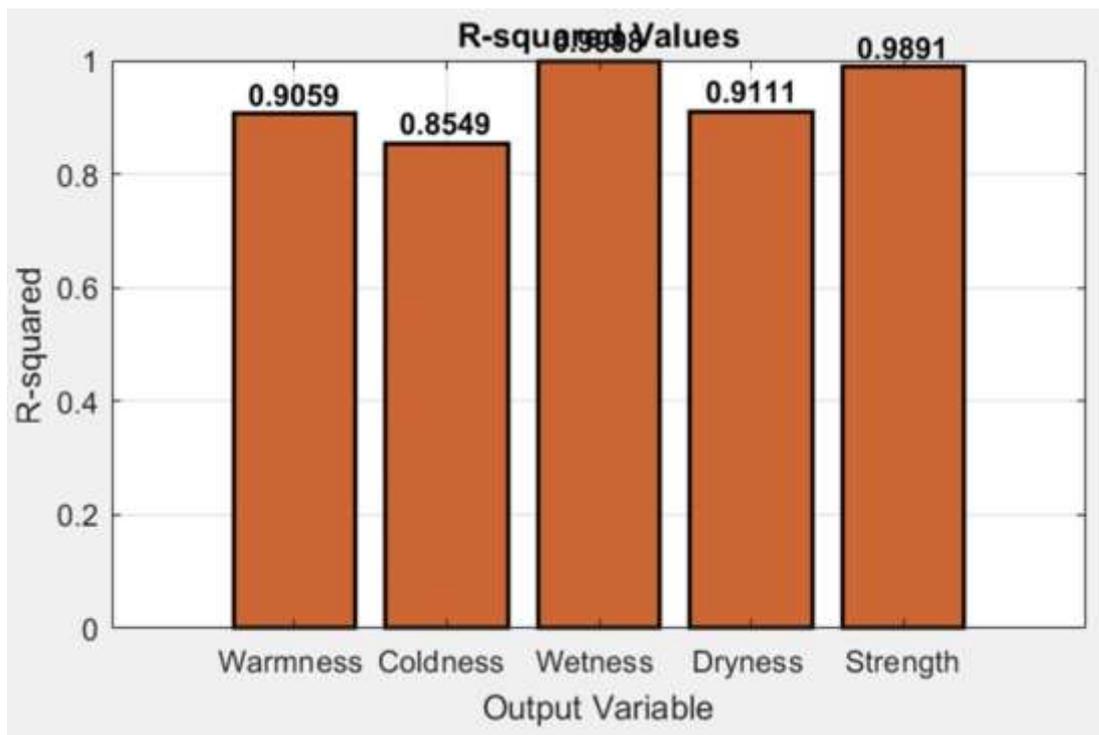


Figure 3. Neuro-fuzzy system regressions for predicting warmness, coldness, wetness, dryness, and strength

The model achieved near-perfect accuracy for Dryness ( $R^2 = 0.9998$ ,  $RMSE = 0.0152$ ) and Heart Strength ( $R^2 = 0.9891$ ,  $RMSE = 0.0631$ ). Performance was consistently high across all outputs, with an average  $R^2$  of 0.9322 and an average  $RMSE$  of 0.2239. The low Mean Absolute Error (MAE) values, such as 0.0405 for Wetness and 0.0077 for Heart Strength, confirm the model's precision. The training process showed stable convergence, with final training  $RMSE$  values as low as  $1.51 \times 10^{-7}$ .

#### 4. Discussion

The quantitative results provide compelling evidence for the advantages of the fuzzy logic approach. The high  $R^2$  values and low error metrics demonstrate the system's capacity to handle medical uncertainty and model the complex interrelationships between pulse qualities and diagnoses, such as the inverse relationship between Warmness ( $R^2 = 0.9059$ ) and Coldness ( $R^2 = 0.8549$ ). The fuzzy system's key strength lies in its ability to learn gradual diagnostic boundaries, mirroring clinical reasoning. The minimal deviation from expert assessments, evidenced by the low MAE, confirms its suitability for clinical decision support and education. It offers a transparent and reliable tool for students who may struggle with the subjective synthesis of pulse parameters.

#### 5. Conclusion

This study quantitatively validates the fuzzy-neuro approach as a superior framework for modeling pulse-based diagnosis. The proven accuracy (average  $R^2$  of 0.9322), validated precision (exceptionally low  $RMSE$ ), and demonstrated reliability across all diagnostic categories confirm that fuzzy logic is optimally suited to capture the nuanced, continuous nature of traditional medical assessment. The ANFIS model successfully bridges the gap between subjective expertise and quantitative computation, establishing a new standard for computational modeling in traditional medicine. Future work will focus on integrating



this validated framework into practical clinical decision support systems.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### REFERENCES

- [1]. WHO traditional medicine strategy; 2014-2023, 5-8.
- [2]. Rezaeizadeh H, Alizadeh M, Naseri M, Shams-Ardakani MR. The Traditional Iranian Medicine Point of View on Health and Disease. *Iran J Publ Health* 2009;38:169-172.
- [3]. Sharifi Darani N, Zargarani A, Cooper GM, Abbassian A, Alizadeh Vaghasloo M. Hakim Mohammad Azam Khan Chishti (1814-1902) and His Book about "Crisis in Diseases". *Trad Integr Med* 2021;6(2):166-173.
- [4]. Shirbeigi L, Zarei A, Naghizadeh A, Alizadeh Vaghasloo M. The Concept of Temperaments in Traditional Persian Medicine. *Trad Integr Med* 2017; 2(3): 143-156.
- [5]. Rezaeidoost H, Karimi M, Jafari M. Proteomics of hot-wet and cold-dry temperaments proposed in Iranian traditional medicine: a network-based study. *Sci Rep.* 2016;6(1):30133.
- [6]. Alizadeh M, Keshavarz M, Ebadiani M, Nazem E, Isfahani M. Complexity and rationality of Avicenna's pulsology; 2012.03.168.
- [7]. Dehghandar M, Khaloozadeh H, Soltanian F, Keshavarz M, Modeling of the Pulse Parameters in Iranian Traditional Medicine and Estimation Of Blood Pressure by Artificial Neural Network. *Journal of Industrial Engineering Research*; 2016, 2(1): 17-22.
- [8]. Alizadeh Vaghasloo M. Explaining the Ten Parameters of Pulse Diagnosis in Traditional Iranian Medicine, [Ph.D. thesis]. School of Traditional Medicine, Tehran University of Medical Sciences.
- [9]. Dehghandar M. Investigating the effect of pulse traditional medicine on the factors of blood pressure by fuzzy calculations. Payame Noor University; 2016 (Persian).
- [10]. Alizadeh M, Naghizadeh A, Keshavarz M. The Concept of Pulse. *Traditional and integrative medicine. Trad Integr Med*, 2017; Volume 2, Issue 2.
- [11]. Dehghandar M, Mirhosseini-Alizamini M, Alizadeh Vaghasloo M, Khosravi Najaf Abadi A. Design and Implementation of a Fuzzy Intelligent System to Estimate the Photoplethysmogram Systolic Features by Persian Medicine Pulsology. *Health Man & Info Sci* 2024 Vol 10, No. 4, pp.226-234.
- [12]. Dehghandar M, Khaloozadeh H, Soltanian F, Keshavarz M. Application of Fuzzy Logic to Determine the Retentive Causes of Pulse Body by the Pulse Parameters in Iranian Traditional Medicine. *Journal of Multidisciplinary Engineering Science and Technology* 2016; 3(2): 3881-3884.
- [13]. Dehghandar M, Alizadeh Vaghasloo M, Moradi B. Estimation of Men's brachial blood pressure gradient using fuzzy system by pulse parameters in Persian medicine. 4th National Seminar on control and optimization, Behshahr, 2022.
- [14]. Naseri M, Rezai Zadeh H, Choopani R. General overview of Traditional Medicine. Tehran: Nashre Shahr; 2010, p. 914-27 (Persian).



- [15]. Chashti M. Exir-e-Azam [Great Elixir]. Tehran: Research Institute for Islamic and Complementary Medicine; 2008, Vol 3 (Persian).
- [16]. Ata R and Kocyigit Y. An Adaptive Neuro-Fuzzy Inference System Approach for Prediction of Tip Speed Ratio in Wind Turbines. *Expert Systems with Applications*, Vol. 37, No. 7, pp. 5454-5460, 2010.
- [17]. Wang L. *A Course in Fuzzy Systems and Control*. The NJ United States, Prentice-Hall, Inc. Division of Simon and Schuster One Lake Street Upper Saddle River, ISBN:978-0-13-540882-7, Page:424, 1996.
- [18]. Sumathi S and Paneerselvam S. *Computational intelligence paradigms. theory and applications using MATLAB*: CRC Press, 2010.
- [19]. Cavallaro F. A Takagi-Sugeno Fuzzy Inference System for Developing a Sustainability Index of Biomass. *Sustainability*, ISSN 2071-1050, doi:10.3390/su70912359,2015.
- [20]. Ghaffari A, Chaibakhsh A, and Shahhoseini S. Neuro-Fuzzy Modeling of Heat Recovery Steam Generator. *International Journal of Machine Learning and Computing*, Vol. 2, No. 5, pp. 605-608, 2012.
- [21]. Pop, M.-D.; Pescara, D'Marco, M.V. Mamdani vs. Takagi-Sugeno Fuzzy Inference Systems in the Calibration of Continuous-Time Car-Following Models. *Sensors* 2023, 23, 8791. <https://doi.org/10.3390/s23218791>
- [22]. Xiao, X.; Wang, Z.; Zhang.; Luo, Y.; Chen, F.; Deng, Y.; Lu, N.; Chen, Y. A Novel Method of Bridge Deflection Prediction Using Probabilistic Deep Learning and Measured Data. *Sensors* 2024, 24, 6863. <https://doi.org/10.3390/s24216863>





## Gradient-Weighted Diffusion PDE for Image Denoising

Neda Mohamadi<sup>1,\*</sup>

<sup>1</sup>Department of Mathematics and Computer Science, Ma.C., Islamic Azad University, Mashhad, Iran.

Email: nedamohamadi@iau.ac.ir

---

**ABSTRACT.** This paper presents a mathematical model based on a gradient-weighted diffusion partial differential equation for image denoising. Both the diffusion and fidelity terms are locally adapted according to the image gradient, allowing selective smoothing while preserving edges. The adaptive fidelity reduces over-smoothing near strong gradients and maintains closeness to the original image. Experiments on standard test images confirm that the proposed approach achieves improved denoising performance compared to conventional diffusion-based models when evaluated by common quality measures.

---

**Keywords:** image denoising, adaptive diffusion PDE, gradient-weighted fidelity, edge preservation

**AMS Mathematics Subject Classification [2020]:** 68U10, 65M06, 94A08

---

### 1. Introduction

Image denoising remains a fundamental problem in image processing and computer vision. Mathematical models based on Partial Differential Equations (PDEs) are widely used for their strong theoretical foundation and ability to suppress noise while preserving edges and textures. Classical diffusion-based methods effectively smooth homogeneous regions but often blur edges, whereas edge-preserving regularization maintains sharp boundaries at the cost of staircase artifacts in smooth areas. More advanced hybrid [2] or learning-based methods [1, 5] achieve high-quality denoising, yet they often require substantial computational resources and long processing times, limiting their practicality for real-time applications.

Unlike previous hybrid PDEs that combine fixed diffusion operators, our model adaptively couples both diffusion and fidelity terms through a shared gradient-dependent weighting mechanism. This coupling enables simultaneous control of smoothing and data fidelity, yielding a unified framework that balances the benefits of classical diffusion and edge-preserving regularization.

---

\*Speaker.

## 2. Proposed Model

Let  $u(x, y, t)$  denote the evolving image with initial condition  $u(x, y, 0) = u_0(x, y)$  in a bounded domain  $\Omega \subset \mathbb{R}^2$ . The proposed partial differential equation is formulated as

$$(1) \quad \frac{\partial u}{\partial t} = w \nabla \cdot (\nabla u) + (1 - w) \nabla \cdot \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon}} \right) - \lambda(x, y)(u - u_0),$$

subject to Neumann boundary conditions  $\frac{\partial u}{\partial n} = 0$  on  $\partial\Omega$ , where  $\varepsilon > 0$  is a small constant ensuring numerical stability and differentiability near flat regions.

Here:

- $w = \frac{1}{\sqrt{1+|\nabla u|^2}}$  is a gradient-dependent weight balancing isotropic diffusion and edge-preserving regularization. In flat regions,  $w$  is close to 1, promoting smoothing, while near edges it decreases, emphasizing edge preservation.
- $\lambda(x, y) = \lambda_0 \exp(-\alpha|\nabla u|^2)$  is a locally adaptive fidelity term. Its gradient dependence ensures reduced smoothing near strong edges while maintaining fidelity to the original image in homogeneous areas.
- The first term represents classical isotropic (heat) diffusion [6], while the second term is a smoothed variant of total variation (TV) regularization [4]. Their gradient-weighted combination unifies uniform smoothing and edge preservation.

Under standard smoothness and boundedness assumptions on  $\Omega$ ,  $w$ , and  $\lambda$ , the PDE admits a unique weak (or classical, when sufficient regularity holds) solution. The gradient-dependent coefficients satisfy conditions guaranteeing the well-posedness and stability of the model.

For numerical implementation, an explicit finite difference scheme is used with a time step satisfying  $\Delta t \leq Ch^2$ , where  $h$  is the spatial discretization and  $C$  a stability constant. This ensures that the numerical evolution remains stable and preserves the convergence of the scheme and the structural features of the image.

## 3. Numerical Scheme

The spatial domain is discretized with spatial step size  $h$  and time step  $\Delta t$ . Let  $u_{i,j}^n \approx u(x_i, y_j, t_n)$  denote the discrete approximation of the evolving image. Central differences are used for first-order derivatives:

$$\Delta_0^x u_{i,j}^n = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h}, \quad \Delta_0^y u_{i,j}^n = \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h}.$$

The discrete gradient magnitude and adaptive weights are defined as

$$w_{i,j}^n = \frac{1}{\sqrt{1 + (\Delta_0^x u_{i,j}^n)^2 + (\Delta_0^y u_{i,j}^n)^2}}, \quad \lambda_{i,j}^n = \lambda_0 \exp[-\alpha((\Delta_0^x u_{i,j}^n)^2 + (\Delta_0^y u_{i,j}^n)^2)].$$

Denoting by  $I_{i,j}^n$  and  $T_{i,j}^n$  the discrete Laplacian and total-variation divergence operators, respectively, the explicit update rule is

$$u_{i,j}^{n+1} = u_{i,j}^n + \Delta t (w_{i,j}^n I_{i,j}^n + (1 - w_{i,j}^n) T_{i,j}^n - \lambda_{i,j}^n (u_{i,j}^n - u_{0,i,j})).$$

Neumann boundary conditions are enforced by mirror reflection at image borders. For numerical stability, the time step must satisfy  $\Delta t \leq Ch^2$  with  $C \leq 0.25$ .

#### 4. Experimental Results

The model was tested on  $256 \times 256$  grayscale test images corrupted by Gaussian noise with  $\sigma_n = 15$  and 25.

For quantitative evaluation, two common image-quality indices were computed: the Peak Signal-to-Noise Ratio (PSNR) and the Structural Similarity Index Measure (SSIM).

Table 1 reports PSNR and SSIM values for several diffusion schemes from [3] and the proposed model on the Lena and House images. The considered schemes include Anisotropic Diffusion (AD), Smoothed Gradient (SG), Total Variation (TV), Mean Curvature Motion (MCM), Well-Balanced Flow (WBF), Modified Smoothed Gradient (MSG), Edge Enhancing Diffusion (EED), Coherence Enhancing Diffusion (CED), Slowed Anisotropic Diffusion (SAD), and Weighted and Well-Balanced Flow (WWBF). The proposed method effectively removes noise while preserving edges and fine details, consistent with the quantitative improvements in Table 1.

TABLE 1. PSNR (dB) and SSIM comparison of diffusion-based denoising methods on standard test images

Scheme	Lena	House
AD	26.32/0.7752	28.87/0.8300
SG	23.26/0.6708	24.94/0.7657
TV	27.05/0.7951	30.18/0.8520
MCM	23.97/0.6943	25.89/0.7855
WBF	26.46/0.7827	28.94/0.8286
MSG	26.53/0.7826	29.30/0.8370
EED	27.23/0.7980	30.68/0.8554
CED	23.97/0.6943	25.89/0.7855
SAD	25.85/0.7559	29.18/0.8375
WWBF	27.12/0.7815	30.92/0.8584
Proposed Model	28.92/0.8205	30.98/0.8647

Values represent PSNR (dB) / SSIM. Higher values indicate better image quality.

The visual denoising results obtained by the proposed method for the House image are shown in Figure 1.

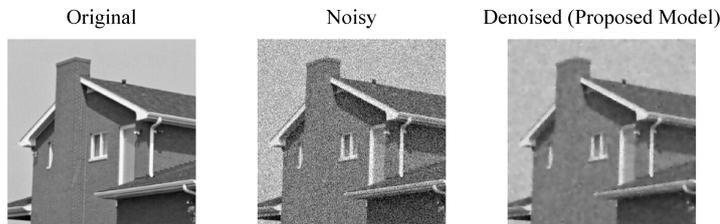


FIGURE 1. Visual comparison of denoising results. From left to right: (1) Original House image, (2) Noisy image corrupted by Gaussian noise ( $\sigma = 25$ ), and (3) Denoised result using the proposed method.

## 5. Conclusion

A gradient-weighted diffusion PDE model was proposed for image denoising, combining isotropic and edge-preserving regularization through spatially adaptive weighting and a locally modulated fidelity term. The diffusion coefficient and fidelity strength are both controlled by the image gradient, enabling selective smoothing in homogeneous regions while preserving important edges and fine structures. The method is interpretable, efficient, and achieves high PSNR and SSIM while maintaining structural fidelity. Future research will extend this framework to color and hyperspectral images, develop implicit or semi-implicit solvers for faster convergence, and explore data-driven strategies for automatic parameter tuning.

## References

1. Eulig, E., Ommer, B. and Kachelrie, M. (2024). *Benchmarking deep learning-based low-dose CT image denoising algorithms*, *Med. Phys.*, **51**(12), 8776–8788.
2. Mohamadi, N., Soheili, A. R. and Toutounian, F. (2018). *A new hybrid denoising model based on PDEs*, *Multimed. Tools Appl.*, **77**(10), 12057–12072.
3. Prasath, V. S. and Vorotnikov, D. (2014). *Weighted and well-balanced anisotropic diffusion scheme for image denoising and restoration*, *Nonlinear Anal. Real World Appl.*, **17**, 33–46.
4. Rudin, L. I., Osher, S. and Fatemi, E. (1992). *Nonlinear total variation based noise removal algorithms*, *Phys. D*, **60**, 259–268.
5. Zhang, K., Zuo, W. and Zhang, L. (2018). *FFDNet: Toward a fast and flexible solution for CNN-based image denoising*, *IEEE Trans. Image Process.*, **27**(9), 4608–4622.
6. Yahya, A. A., Tan, J. and Hu, M. (2014). *A blending method based on partial differential equations for image denoising*, *Multimed. Tools Appl.*, **73**(3), 1843–1862.



## Three-Manifolds and the Global Geometry of the Universe: A Mathematical Approach to Cosmic Topology

Parviz Asefi Nazarlou<sup>1,\*</sup>, Samira Golsanamlou<sup>2</sup>

<sup>1</sup>Ph.D. Student, Department of Mathematics, Azarbaijan Shahid Madani University (ASMU),  
Tabriz, Iran.

Email: asefinazarloo@gmail.com

<sup>2</sup>Ph.D. (Cosmology), Department of Physics, Azarbaijan Shahid Madani University (ASMU),  
Tabriz, Iran.

Email: s.golsanamlou@yahoo.com

---

**ABSTRACT.** This paper investigates the mathematical foundations of cosmic topology by studying three-manifolds and their possible geometries as spatial models of the universe. Building upon Thurston's geometrization theorem and Einstein's field equations, we analyze how local curvature and global topology interact in shaping cosmological models. We emphasize the classification of 3-manifolds, the role of fundamental groups in identifying multiply connected spaces, and the mathematical implications for global geometry. The approach is mathematical in nature yet motivated by cosmological observations.

**Keywords:** 3-Manifolds, Cosmic topology, Thurston geometry, Fundamental group, Riemannian curvature

**AMS Mathematics Subject Classification [2020]:** 57M50, 53C23, 83F05

---

### 1. Introduction

The geometry and topology of three-dimensional manifolds play a central role in understanding the possible global shapes of the universe. While Einstein's field equations describe local curvature, the global topology remains a mathematical degree of freedom unconstrained by local physics. In cosmological models based on general relativity, the spatial section of spacetime is a Riemannian 3-manifold  $(M^3, g)$  satisfying the Einstein constraint equations. Understanding the possible forms of  $M^3$  requires tools from topology and differential geometry.

The celebrated *Geometrization Theorem* of Thurston [1], completed by Perelman [5], provides a full classification: every compact orientable 3-manifold can be decomposed into pieces admitting one of eight model geometries. Among these, the spherical, Euclidean, and hyperbolic geometries are of primary cosmological relevance and form the foundation of cosmic topology [2, 3].

---

\*Speaker.

## 2. Mathematical Foundations

A *3-manifold* is a topological space locally homeomorphic to  $\mathbb{R}^3$ . The classification of such manifolds involves both their geometric structure and topological invariants. Let  $M^3$  be a compact, orientable manifold. Its geometry is described by a Riemannian metric  $g$ , and its topology by invariants such as the fundamental group  $\pi_1(M^3)$ , homology groups  $H_i(M^3)$ , and covering spaces.

DEFINITION 2.1. A manifold  $M^3$  is said to be *multiply connected* if  $\pi_1(M^3)$  is nontrivial. Its universal covering space  $\tilde{M}^3$  is then simply connected, and  $\pi_1(M^3)$  acts discretely and freely by isometries:

$$M^3 \cong \tilde{M}^3 / \pi_1(M^3).$$

For example, the 3-torus  $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$  is a Euclidean manifold with  $\pi_1(T^3) = \mathbb{Z}^3$ , while projective 3-space  $P^3 = S^3 / \{\pm 1\}$  has  $\pi_1(P^3) = \mathbb{Z}_2$  and spherical geometry. Compact hyperbolic manifolds of the form  $H^3 / \Gamma$  (with  $\Gamma \subset \text{Isom}(H^3)$  discrete) exhibit constant negative curvature and finite volume.

REMARK 2.2. The geometry of a manifold determines its local curvature tensor  $R_{ijkl}$ , while its topology is encoded in  $\pi_1(M^3)$ . Distinct manifolds may share the same constant curvature but differ globally.

NOTATION 2.3. The Ricci flow, introduced by Hamilton and perfected by Perelman [5], evolves the metric via

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

smoothing curvature irregularities. In cosmology, this process conceptually parallels the universe's dynamical evolution toward geometric uniformity. Stationary points correspond to constant-curvature FLRW metrics.

REMARK 2.4. By the Jaco–Shalen–Johannson decomposition theorem, any compact orientable 3-manifold can be decomposed along tori into atoroidal and Seifert-fibered components. Analogously, in cosmology, regions of differing curvature might coexist, suggesting a mixed geometric composition of space.

## 3. Geometric Structures and Cosmological Models

In cosmology, spatial geometry is modeled by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric:

$$(1) \quad ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where  $k \in \{-1, 0, +1\}$  denotes constant curvature. The density parameter  $\Omega_k$  is related to curvature as

$$(2) \quad \Omega_k = 1 - \Omega_{\text{tot}} = -\frac{kc^2}{a_0^2 H_0^2}.$$

Observations from Planck (2020) constrain  $|\Omega_k| < 10^{-3}$ , but both infinite  $\mathbb{R}^3$  and compact  $T^3$  models remain viable [3, 4].

EXAMPLE 3.1. If  $M^3 = \mathbb{R}^3 / \Gamma$ , with  $\Gamma$  a lattice in  $\mathbb{R}^3$ , then  $M^3$  inherits a flat metric and finite volume—representing a flat FLRW model with periodic boundary conditions.

NOTATION 3.2. The isometry group  $\text{Isom}(M^3, g)$  determines spatial homogeneity and isotropy. For the Euclidean 3-torus, the universal covering symmetry is  $\mathbb{R}^3$ ; for hyperbolic manifolds, it is discrete  $\Gamma \subset \text{PSL}(2, \mathbb{C})$ .

#### 4. Topological Invariants and Observational Aspects

The fundamental group  $\pi_1(M^3)$  provides a rigorous topological characterization of cosmic space. Acting as a discrete isometry group on  $\tilde{M}^3$ , it can produce observable signatures such as multiple images of celestial objects and correlated “circles in the sky” [4].

The homology group

$$H_1(M^3, \mathbb{Z}) = \pi_1(M^3) / [\pi_1(M^3), \pi_1(M^3)],$$

encodes the abelianized structure of  $\pi_1(M^3)$ . Compact hyperbolic manifolds often exhibit nontrivial  $H_1$ , which may leave imprints on cosmic crystallography and CMB pattern repetition [3].

REMARK 4.1. While homology and homotopy invariants classify manifolds mathematically, observations can only probe them indirectly via geometry. Hence, topologically distinct manifolds may be observationally degenerate.

The Einstein constraint equation connects curvature to matter density:

$$(3) \quad R - 2\Lambda = 16\pi G\rho,$$

where  $\Lambda$  is the cosmological constant and  $\rho$  is matter density. The global topology of  $M^3$  remains unconstrained by this local relation.

#### 5. Philosophical and Foundational Remarks

Topology defines the “shape” of space independent of dynamics. That distinct manifolds can produce identical empirical predictions raises foundational questions. From the standpoint of *structural realism* [6], only relational geometric structures—represented by  $(M^3, g)$  up to isometry—correspond to physical reality, while the manifold itself serves as a modeling abstraction. This perspective highlights the deep interplay between pure mathematics and the epistemology of cosmology.

#### 6. Conclusion

The study of three-manifolds provides a rigorous mathematical framework for understanding the possible global shapes of the universe. By connecting Thurston’s geometric classification with Einstein’s field equations, we gain a unified perspective on how local curvature and global topology jointly determine the universe’s structure. This synthesis reveals that while general relativity constrains local geometry through curvature and energy density, it leaves the global topology of space as a mathematically free parameter.

From a cosmological viewpoint, compact topologies such as the 3-torus or certain hyperbolic manifolds remain consistent with current observational limits on curvature. The mathematical apparatus of fundamental and homology groups not only enables the classification of such manifolds but also suggests potential observable effects—such as pattern repetitions in the cosmic microwave background and geodesic identifications on large scales.

Philosophically, this duality between geometric determinism and topological freedom underscores the deep interplay between mathematics and physical reality. Three-manifold

theory thus serves as a conceptual bridge linking abstract topology with empirical cosmology. Future directions may involve categorical and quantum-topological extensions, in which spatial structure is represented as an emergent combinatorial or homotopical entity, offering a route toward reconciling general relativity with quantum geometry.

In this context, *quantum geometry* refers to mathematical frameworks where the fabric of space is no longer continuous but quantized—described by discrete or algebraic structures such as spin networks or noncommutative spaces. These approaches reinterpret geometry itself as a quantum variable, suggesting that the large-scale topology of the universe may ultimately emerge from underlying quantum gravitational structures.

### Acknowledgement

The authors gratefully acknowledge the organizers of the 12th National Mathematics Conference for the opportunity to present this work.

### References

1. Thurston, W. P. (1997). *Three-Dimensional Geometry and Topology*. Princeton Univ. Press.
2. Hitchman, M. P. (2018). *Geometry with an Introduction to Cosmic Topology*. Linfield College.
3. Lachièze-Rey, M. and Luminet, J.-P. (1995). *Cosmic Topology*. Phys. Rep, 254, 135–214.
4. Cornish, N. J., Spergel, D. N., and Starkman, G. D. (1998). *Circles in the Sky: Finding Topology with the Microwave Background Radiation*. *Class. Quantum Grav.*, 15, 2657–2670.
5. Perelman, G. (2003). *Ricci Flow with Surgery on Three-Manifolds*. arXiv:math/0303109.
6. Worrall, J. (1989). *Structural Realism: The Best of Both Worlds?* *Dialectica*, 43, 99–124.

